INFINITE DIVISIBILITY OF PROBABILITY DISTRIBUTIONS ON THE REAL LINE

FRED W. STEUTEL
University of Technology
Eindhoven, The Netherlands

KLAAS VAN HARN
Free University
Amsterdam, The Netherlands

MARCEL DEKKER, INC. NEW YORK • BASEL

Cover photograph by Gert van de Kraats

Library of Congress Cataloging-in-Publication Data
A catalog record for this book is available from the Library of Congress.

ISBN: 0-8247-0724-9

This book is printed on acid-free paper.

Headquarters
Marcel Dekker, Inc., 270 Madison Avenue, New York, NY 10016, U.S.A.
tel: 212-696-9000; fax: 212-685-4540

Distribution and Customer Service
Marcel Dekker, Inc., Cimarron Road, Monticello, New York 12701, U.S.A.
tel: 800-228-1160; fax: 845-796-1772

Eastern Hemisphere Distribution
Marcel Dekker AG, Hutgasse 4, Postfach 812, CH-4001 Basel, Switzerland
tel: 41-61-260-6300; fax: 41-61-260-6333

World Wide Web
http://www.dekker.com

The publisher offers discounts on this book when ordered in bulk quantities. For more information, write to Special Sales/Professional Marketing at the headquarters address above.

Current printing (last digit):

10 9 8 7 6 5 4 3 2 1

PRINTED IN THE UNITED STATES OF AMERICA

MONOGRAPHS AND TEXTBOOKS IN PURE AND APPLIED MATHEMATICS

1. *K. Yano,* Integral Formulas in Riemannian Geometry (1970)
2. *S. Kobayashi,* Hyperbolic Manifolds and Holomorphic Mappings (1970)
3. *V. S. Vladimirov,* Equations of Mathematical Physics (A. Jeffrey, ed.; A. Littlewood, trans.) (1970)
4. *B. N. Pshenichnyi,* Necessary Conditions for an Extremum (L. Neustadt, translation ed.; K. Makowski, trans.) (1971)
5. *L. Narici et al.,* Functional Analysis and Valuation Theory (1971)
6. *S. S. Passman,* Infinite Group Rings (1971)
7. *L. Dornhoff,* Group Representation Theory. Part A: Ordinary Representation Theory. Part B: Modular Representation Theory (1971, 1972)
8. *W. Boothby and G. L. Weiss, eds.,* Symmetric Spaces (1972)
9. *Y. Matsushima,* Differentiable Manifolds (E. T. Kobayashi, trans.) (1972)
10. *L. E. Ward, Jr.,* Topology (1972)
11. *A. Babakhanian,* Cohomological Methods in Group Theory (1972)
12. *R. Gilmer,* Multiplicative Ideal Theory (1972)
13. *J. Yeh,* Stochastic Processes and the Wiener Integral (1973)
14. *J. Barros-Neto,* Introduction to the Theory of Distributions (1973)
15. *R. Larsen,* Functional Analysis (1973)
16. *K. Yano and S. Ishihara,* Tangent and Cotangent Bundles (1973)
17. *C. Procesi,* Rings with Polynomial Identities (1973)
18. *R. Hermann,* Geometry, Physics, and Systems (1973)
19. *N. R. Wallach,* Harmonic Analysis on Homogeneous Spaces (1973)
20. *J. Dieudonné,* Introduction to the Theory of Formal Groups (1973)
21. *I. Vaisman,* Cohomology and Differential Forms (1973)
22. *B.-Y. Chen,* Geometry of Submanifolds (1973)
23. *M. Marcus,* Finite Dimensional Multilinear Algebra (in two parts) (1973, 1975)
24. *R. Larsen,* Banach Algebras (1973)
25. *R. O. Kujala and A. L. Vitter, eds.,* Value Distribution Theory: Part A; Part B: Deficit and Bezout Estimates by Wilhelm Stoll (1973)
26. *K. B. Stolarsky,* Algebraic Numbers and Diophantine Approximation (1974)
27. *A. R. Magid,* The Separable Galois Theory of Commutative Rings (1974)
28. *B. R. McDonald,* Finite Rings with Identity (1974)
29. *J. Satake,* Linear Algebra (S. Koh et al., trans.) (1975)
30. *J. S. Golan,* Localization of Noncommutative Rings (1975)
31. *G. Klambauer,* Mathematical Analysis (1975)
32. *M. K. Agoston,* Algebraic Topology (1976)
33. *K. R. Goodearl,* Ring Theory (1976)
34. *L. E. Mansfield,* Linear Algebra with Geometric Applications (1976)
35. *N. J. Pullman,* Matrix Theory and Its Applications (1976)
36. *B. R. McDonald,* Geometric Algebra Over Local Rings (1976)
37. *C. W. Groetsch,* Generalized Inverses of Linear Operators (1977)
38. *J. E. Kuczkowski and J. L. Gersting,* Abstract Algebra (1977)
39. *C. O. Christenson and W. L. Voxman,* Aspects of Topology (1977)
40. *M. Nagata,* Field Theory (1977)
41. *R. L. Long,* Algebraic Number Theory (1977)
42. *W. F. Pfeffer,* Integrals and Measures (1977)
43. *R. L. Wheeden and A. Zygmund,* Measure and Integral (1977)
44. *J. H. Curtiss,* Introduction to Functions of a Complex Variable (1978)
45. *K. Hrbacek and T. Jech,* Introduction to Set Theory (1978)
46. *W. S. Massey,* Homology and Cohomology Theory (1978)
47. *M. Marcus,* Introduction to Modern Algebra (1978)
48. *E. C. Young,* Vector and Tensor Analysis (1978)
49. *S. B. Nadler, Jr.,* Hyperspaces of Sets (1978)
50. *S. K. Segal,* Topics in Group Kings (1978)
51. *A. C. M. van Rooij,* Non-Archimedean Functional Analysis (1978)
52. *L. Corwin and R. Szczarba,* Calculus in Vector Spaces (1979)
53. *C. Sadosky,* Interpolation of Operators and Singular Integrals (1979)
54. *J. Cronin,* Differential Equations (1980)
55. *C. W. Groetsch,* Elements of Applicable Functional Analysis (1980)

56. *I. Vaisman,* Foundations of Three-Dimensional Euclidean Geometry (1980)
57. *H. I. Freedan,* Deterministic Mathematical Models in Population Ecology (1980)
58. *S. B. Chae,* Lebesgue Integration (1980)
59. *C. S. Rees et al.,* Theory and Applications of Fourier Analysis (1981)
60. *L. Nachbin,* Introduction to Functional Analysis (R. M. Aron, trans.) (1981)
61. *G. Orzech and M. Orzech,* Plane Algebraic Curves (1981)
62. *R. Johnsonbaugh and W. E. Pfaffenberger,* Foundations of Mathematical Analysis (1981)
63. *W. L. Voxman and R. H. Goetschel,* Advanced Calculus (1981)
64. *L. J. Corwin and R. H. Szczarba,* Multivariable Calculus (1982)
65. *V. I. Istrăţescu,* Introduction to Linear Operator Theory (1981)
66. *R. D. Järvinen,* Finite and Infinite Dimensional Linear Spaces (1981)
67. *J. K. Beem and P. E. Ehrlich,* Global Lorentzian Geometry (1981)
68. *D. L. Armacost,* The Structure of Locally Compact Abelian Groups (1981)
69. *J. W. Brewer and M. K. Smith, eds.,* Emmy Noether: A Tribute (1981)
70. *K. H. Kim,* Boolean Matrix Theory and Applications (1982)
71. *T. W. Wieting,* The Mathematical Theory of Chromatic Plane Ornaments (1982)
72. *D. B.Gauld,* Differential Topology (1982)
73. *R. L. Faber,* Foundations of Euclidean and Non-Euclidean Geometry (1983)
74. *M. Carmeli,* Statistical Theory and Random Matrices (1983)
75. *J. H. Carruth et al.,* The Theory of Topological Semigroups (1983)
76. *R. L. Faber,* Differential Geometry and Relativity Theory (1983)
77. *S. Barnett,* Polynomials and Linear Control Systems (1983)
78. *G. Karpilovsky,* Commutative Group Algebras (1983)
79. *F. Van Oystaeyen and A. Verschoren,* Relative Invariants of Rings (1983)
80. *I. Vaisman,* A First Course in Differential Geometry (1984)
81. *G. W. Swan,* Applications of Optimal Control Theory in Biomedicine (1984)
82. *T. Petrie and J. D. Randall,* Transformation Groups on Manifolds (1984)
83. *K. Goebel and S. Reich,* Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings (1984)
84. *T. Albu and C. Năstăsescu,* Relative Finiteness in Module Theory (1984)
85. *K. Hrbacek and T. Jech,* Introduction to Set Theory: Second Edition (1984)
86. *F. Van Oystaeyen and A. Verschoren,* Relative Invariants of Rings (1984)
87. *B. R. McDonald,* Linear Algebra Over Commutative Rings (1984)
88. *M. Namba,* Geometry of Projective Algebraic Curves (1984)
89. *G. F. Webb,* Theory of Nonlinear Age-Dependent Population Dynamics (1985)
90. *M. R. Bremner et al.,* Tables of Dominant Weight Multiplicities for Representations of Simple Lie Algebras (1985)
91. *A. E. Fekete,* Real Linear Algebra (1985)
92. *S. B. Chae,* Holomorphy and Calculus in Normed Spaces (1985)
93. *A. J. Jerri,* Introduction to Integral Equations with Applications (1985)
94. *G. Karpilovsky,* Projective Representations of Finite Groups (1985)
95. *L. Narici and E. Beckenstein,* Topological Vector Spaces (1985)
96. *J. Weeks,* The Shape of Space (1985)
97. *P. R. Gribik and K. O. Kortanek,* Extremal Methods of Operations Research (1985)
98. *J.-A. Chao and W. A. Woyczynski, eds.,* Probability Theory and Harmonic Analysis (1986)
99. *G. D. Crown et al.,* Abstract Algebra (1986)
100. *J. H. Carruth et al.,* The Theory of Topological Semigroups, Volume 2 (1986)
101. *R. S. Doran and V. A. Belfi,* Characterizations of C*-Algebras (1986)
102. *M. W. Jeter,* Mathematical Programming (1986)
103. *M. Altman,* A Unified Theory of Nonlinear Operator and Evolution Equations with Applications (1986)
104. *A. Verschoren,* Relative Invariants of Sheaves (1987)
105. *R. A. Usmani,* Applied Linear Algebra (1987)
106. *P. Blass and J. Lang,* Zariski Surfaces and Differential Equations in Characteristic $p > 0$ (1987)
107. *J. A. Reneke et al.,* Structured Hereditary Systems (1987)
108. *H. Busemann and B. B. Phadke,* Spaces with Distinguished Geodesics (1987)
109. *R. Harte,* Invertibility and Singularity for Bounded Linear Operators (1988)
110. *G. S. Ladde et al.,* Oscillation Theory of Differential Equations with Deviating Arguments (1987)
111. *L. Dudkin et al.,* Iterative Aggregation Theory (1987)
112. *T. Okubo,* Differential Geometry (1987)

113. *D. L. Stancl and M. L. Stancl,* Real Analysis with Point-Set Topology (1987)
114. *T. C. Gard,* Introduction to Stochastic Differential Equations (1988)
115. *S. S. Abhyankar,* Enumerative Combinatorics of Young Tableaux (1988)
116. *H. Strade and R. Farnsteiner,* Modular Lie Algebras and Their Representations (1988)
117. *J. A. Huckaba,* Commutative Rings with Zero Divisors (1988)
118. *W. D. Wallis,* Combinatorial Designs (1988)
119. *W. Więsław,* Topological Fields (1988)
120. *G. Karpilovsky,* Field Theory (1988)
121. *S. Caenepeel and F. Van Oystaeyen,* Brauer Groups and the Cohomology of Graded Rings (1989)
122. *W. Kozlowski,* Modular Function Spaces (1988)
123. *E. Lowen-Colebunders,* Function Classes of Cauchy Continuous Maps (1989)
124. *M. Pavel,* Fundamentals of Pattern Recognition (1989)
125. *V. Lakshmikantham et al.,* Stability Analysis of Nonlinear Systems (1989)
126. *R. Sivaramakrishnan,* The Classical Theory of Arithmetic Functions (1989)
127. *N. A. Watson,* Parabolic Equations on an Infinite Strip (1989)
128. *K. J. Hastings,* Introduction to the Mathematics of Operations Research (1989)
129. *B. Fine,* Algebraic Theory of the Bianchi Groups (1989)
130. *D. N. Dikranjan et al.,* Topological Groups (1989)
131. *J. C. Morgan II,* Point Set Theory (1990)
132. *P. Biler and A. Witkowski,* Problems in Mathematical Analysis (1990)
133. *H. J. Sussmann,* Nonlinear Controllability and Optimal Control (1990)
134. *J.-P. Florens et al.,* Elements of Bayesian Statistics (1990)
135. *N. Shell,* Topological Fields and Near Valuations (1990)
136. *B. F. Doolin and C. F. Martin,* Introduction to Differential Geometry for Engineers (1990)
137. *S. S. Holland, Jr.,* Applied Analysis by the Hilbert Space Method (1990)
138. *J. Okniński,* Semigroup Algebras (1990)
139. *K. Zhu,* Operator Theory in Function Spaces (1990)
140. *G. B. Price,* An Introduction to Multicomplex Spaces and Functions (1991)
141. *R. B. Darst,* Introduction to Linear Programming (1991)
142. *P. L. Sachdev,* Nonlinear Ordinary Differential Equations and Their Applications (1991)
143. *T. Husain,* Orthogonal Schauder Bases (1991)
144. *J. Foran,* Fundamentals of Real Analysis (1991)
145. *W. C. Brown,* Matrices and Vector Spaces (1991)
146. *M. M. Rao and Z. D. Ren,* Theory of Orlicz Spaces (1991)
147. *J. S. Golan and T. Head,* Modules and the Structures of Rings (1991)
148. *C. Small,* Arithmetic of Finite Fields (1991)
149. *K. Yang,* Complex Algebraic Geometry (1991)
150. *D. G. Hoffman et al.,* Coding Theory (1991)
151. *M. O. González,* Classical Complex Analysis (1992)
152. *M. O. González,* Complex Analysis (1992)
153. *L. W. Baggett,* Functional Analysis (1992)
154. *M. Sniedovich,* Dynamic Programming (1992)
155. *R. P. Agarwal,* Difference Equations and Inequalities (1992)
156. *C. Brezinski,* Biorthogonality and Its Applications to Numerical Analysis (1992)
157. *C. Swartz,* An Introduction to Functional Analysis (1992)
158. *S. B. Nadler, Jr.,* Continuum Theory (1992)
159. *M. A. Al-Gwaiz,* Theory of Distributions (1992)
160. *E. Perry,* Geometry: Axiomatic Developments with Problem Solving (1992)
161. *E. Castillo and M. R. Ruiz-Cobo,* Functional Equations and Modelling in Science and Engineering (1992)
162. *A. J. Jerri,* Integral and Discrete Transforms with Applications and Error Analysis (1992)
163. *A. Charlier et al.,* Tensors and the Clifford Algebra (1992)
164. *P. Biler and T. Nadzieja,* Problems and Examples in Differential Equations (1992)
165. *E. Hansen,* Global Optimization Using Interval Analysis (1992)
166. *S. Guerre-Delabrière,* Classical Sequences in Banach Spaces (1992)
167. *Y. C. Wong,* Introductory Theory of Topological Vector Spaces (1992)
168. *S. H. Kulkarni and B. V. Limaye,* Real Function Algebras (1992)
169. *W. C. Brown,* Matrices Over Commutative Rings (1993)
170. *J. Loustau and M. Dillon,* Linear Geometry with Computer Graphics (1993)
171. *W. V. Petryshyn,* Approximation-Solvability of Nonlinear Functional and Differential Equations (1993)

172. *E. C. Young*, Vector and Tensor Analysis: Second Edition (1993)
173. *T. A. Bick*, Elementary Boundary Value Problems (1993)
174. *M. Pavel*, Fundamentals of Pattern Recognition: Second Edition (1993)
175. *S. A. Albeverio et al.*, Noncommutative Distributions (1993)
176. *W. Fulks*, Complex Variables (1993)
177. *M. M. Rao*, Conditional Measures and Applications (1993)
178. *A. Janicki and A. Weron*, Simulation and Chaotic Behavior of α-Stable Stochastic Processes (1994)
179. *P. Neittaanmäki and D. Tiba*, Optimal Control of Nonlinear Parabolic Systems (1994)
180. *J. Cronin*, Differential Equations: Introduction and Qualitative Theory, Second Edition (1994)
181. *S. Heikkilä and V. Lakshmikantham*, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations (1994)
182. *X. Mao*, Exponential Stability of Stochastic Differential Equations (1994)
183. *B. S. Thomson*, Symmetric Properties of Real Functions (1994)
184. *J. E. Rubio*, Optimization and Nonstandard Analysis (1994)
185. *J. L. Bueso et al.*, Compatibility, Stability, and Sheaves (1995)
186. *A. N. Michel and K. Wang*, Qualitative Theory of Dynamical Systems (1995)
187. *M. R. Damel*, Theory of Lattice-Ordered Groups (1995)
188. *Z. Naniewicz and P. D. Panagiotopoulos*, Mathematical Theory of Hemivariational Inequalities and Applications (1995)
189. *L. J. Corwin and R. H. Szczarba*, Calculus in Vector Spaces: Second Edition (1995)
190. *L. H. Erbe et al.*, Oscillation Theory for Functional Differential Equations (1995)
191. *S. Agaian et al.*, Binary Polynomial Transforms and Nonlinear Digital Filters (1995)
192. *M. I. Gil'*, Norm Estimations for Operation-Valued Functions and Applications (1995)
193. *P. A. Grillet*, Semigroups: An Introduction to the Structure Theory (1995)
194. *S. Kichenassamy*, Nonlinear Wave Equations (1996)
195. *V. F. Krotov*, Global Methods in Optimal Control Theory (1996)
196. *K. I. Beidar et al.*, Rings with Generalized Identities (1996)
197. *V. I. Arnautov et al.*, Introduction to the Theory of Topological Rings and Modules (1996)
198. *G. Sierksma*, Linear and Integer Programming (1996)
199. *R. Lasser*, Introduction to Fourier Series (1996)
200. *V. Sima*, Algorithms for Linear-Quadratic Optimization (1996)
201. *D. Redmond*, Number Theory (1996)
202. *J. K. Beem et al.*, Global Lorentzian Geometry: Second Edition (1996)
203. *M. Fontana et al.*, Prüfer Domains (1997)
204. *H. Tanabe*, Functional Analytic Methods for Partial Differential Equations (1997)
205. *C. Q. Zhang*, Integer Flows and Cycle Covers of Graphs (1997)
206. *E. Spiegel and C. J. O'Donnell*, Incidence Algebras (1997)
207. *B. Jakubczyk and W. Respondek*, Geometry of Feedback and Optimal Control (1998)
208. *T. W. Haynes et al.*, Fundamentals of Domination in Graphs (1998)
209. *T. W. Haynes et al., eds.*, Domination in Graphs: Advanced Topics (1998)
210. *L. A. D'Alotto et al.*, A Unified Signal Algebra Approach to Two-Dimensional Parallel Digital Signal Processing (1998)
211. *F. Halter-Koch*, Ideal Systems (1998)
212. *N. K. Govil et al., eds.*, Approximation Theory (1998)
213. *R. Cross*, Multivalued Linear Operators (1998)
214. *A. A. Martynyuk*, Stability by Liapunov's Matrix Function Method with Applications (1998)
215. *A. Favini and A. Yagi*, Degenerate Differential Equations in Banach Spaces (1999)
216. *A. Illanes and S. Nadler, Jr.*, Hyperspaces: Fundamentals and Recent Advances (1999)
217. *G. Kato and D. Struppa*, Fundamentals of Algebraic Microlocal Analysis (1999)
218. *G. X.-Z. Yuan*, KKM Theory and Applications in Nonlinear Analysis (1999)
219. *D. Motreanu and N. H. Pavel*, Tangency, Flow Invariance for Differential Equations, and Optimization Problems (1999)
220. *K. Hrbacek and T. Jech*, Introduction to Set Theory, Third Edition (1999)
221. *G. E. Kolosov*, Optimal Design of Control Systems (1999)
222. *N. L. Johnson*, Subplane Covered Nets (2000)
223. *B. Fine and G. Rosenberger*, Algebraic Generalizations of Discrete Groups (1999)
224. *M. Väth*, Volterra and Integral Equations of Vector Functions (2000)
225. *S. S. Miller and P. T. Mocanu*, Differential Subordinations (2000)

226. *R. Li et al.*, Generalized Difference Methods for Differential Equations: Numerical Analysis of Finite Volume Methods (2000)
227. *H. Li and F. Van Oystaeyen*, A Primer of Algebraic Geometry (2000)
228. *R. P. Agarwal*, Difference Equations and Inequalities: Theory, Methods, and Applications, Second Edition (2000)
229. *A. B. Kharazishvili*, Strange Functions in Real Analysis (2000)
230. *J. M. Appell et al.*, Partial Integral Operators and Integro-Differential Equations (2000)
231. *A. I. Prilepko et al.*, Methods for Solving Inverse Problems in Mathematical Physics (2000)
232. *F. Van Oystaeyen*, Algebraic Geometry for Associative Algebras (2000)
233. *D. L. Jagerman*, Difference Equations with Applications to Queues (2000)
234. *D. R. Hankerson et al.*, Coding Theory and Cryptography: The Essentials, Second Edition, Revised and Expanded (2000)
235. *S. Dăscălescu et al.*, Hopf Algebras: An Introduction (2001)
236. *R. Hagen et al.*, C*-Algebras and Numerical Analysis (2001)
237. *Y. Talpaert*, Differential Geometry: With Applications to Mechanics and Physics (2001)
238. *R. H. Villarreal*, Monomial Algebras (2001)
239. *A. N. Michel et al.*, Qualitative Theory of Dynamical Systems: Second Edition (2001)
240. *A. A. Samarskii*, The Theory of Difference Schemes (2001)
241. *J. Knopfmacher and W.-B. Zhang*, Number Theory Arising from Finite Fields (2001)
242. *S. Leader*, The Kurzweil-Henstock Integral and Its Differentials (2001)
243. *M. Biliotti et al.*, Foundations of Translation Planes (2001)
244. *A. N. Kochubei*, Pseudo-Differential Equations and Stochastics over Non-Archimedean Fields (2001)
245. *G. Sierksma*, Linear and Integer Programming: Second Edition (2002)
246. *A. A. Martynyuk*, Qualitative Methods in Nonlinear Dynamics: Novel Approaches to Liapunov's Matrix Functions (2002)
247. *B. G. Pachpatte*, Inequalities for Finite Difference Equations (2002)
248. *A. N. Michel and D. Liu*, Qualitative Analysis and Synthesis of Recurrent Neural Networks (2002)
249. *J. R. Weeks*, The Shape of Space: Second Edition (2002)
250. *M. M. Rao and Z. D. Ren*, Applications of Orlicz Spaces (2002)
251. *V. Lakshmikantham and D. Trigiante*, Theory of Difference Equations: Numerical Methods and Applications, Second Edition (2002)
252. *T. Albu*, Cogalois Theory (2003)
253. *A. Bezdek*, Discrete Geometry (2003)
254. *M. J. Corless and A. E. Frazho*, Linear Systems and Control: An Operator Perspective (2003)
255. *I. Graham and G. Kohr*, Geometric Function Theory in One and Higher Dimensions (2003)
256. *G. V. Demidenko and S. V. Uspenskii*, Partial Differential Equations and Systems Not Solvable with Respect to the Highest-Order Derivative (2003)
257. *A. Kelarev*, Graph Algebras and Automata (2003)
258. *A. H. Siddiqi*, Applied Functional Analysis: Numerical Methods, Wavelet Methods, and Image Processing (2004)
259. *F. W. Steutel and K. van Harn*, Infinite Divisibility of Probability Distributions on the Real Line (2004)
260. *G. S. Ladde and M. Sambandham*, Stochastic Versus Deterministic Systems of Differential Equations (2004)
261. *B. J. Gardner and R. Wiegandt*, Radical Theory of Rings (2004)
262. *J. Haluška*, The Mathematical Theory of Tone Systems (2004)

Additional Volumes in Preparation

E. Hansen and G. W. Walster, Global Optimization Using Interval Analysis: Second Edition, Revised and Expanded (2004)

C. Menini and F. Van Oystaeyen, The ERASMUS Guide to Algebra: From Zero to One (2004)

To Vita, Song (Freek), Jung

To Liesbeth, Paul, Arjan, Marlies

Preface

Several years ago we wrote: "Infinite divisibility was a booming field of research in the seventies and eighties. There still is some interest in the subject, but the boom seems to have subsided. So it is time to have a look at the results". Events beyond our control have delayed completion of this book, but the words are still true. We present new developments together with a full account of the classical theory. We only deal with probability distributions on the real line and restrict ourselves to divisibility with respect to convolution or, equivalently, addition of independent random variables. Space and time do not allow a more general setting.

The theory of infinitely divisible distributions plays a fundamental role in several parts of theoretical probability, such as the central limit problem and the theory of processes with stationary independent increments or Lévy processes, which lie at the roots of our subject. The practical interest of infinite divisibility is mainly in modelling. In models that require random variables to be the sum of several independent quantities with the same distribution, a convenient assumption is infinite divisibility of these random variables; this situation occurs in biology and insurance. Other applications concern deconvolution problems in mathematical physics. Lévy processes are important ingredients in financial mathematics models.

Theoretically, the problem of identifying the infinitely divisible distributions is completely solved by the canonical representations of their characteristic functions. In most practical cases, however, these representations are not useful for deciding on the infinite divisibility of a given distribution. One of the aims of this book is to provide means and methods for making such decisions. Emphasis is on criteria in terms of distribution functions and probability densities, but characteristic functions and canonical representations are often needed to obtain such criteria. Though the concept of infinite divisibility is rather simple, the methods involved are sometimes quite sophisticated.

Following the introductory chapter, I, the basic chapters, II, III and IV, offer separate treatments of infinitely divisible distributions on the nonnegative integers, the nonnegative reals, and the real line, respectively. This is done because the needed tools and methods vary, getting more complex in each subsequent chapter. This separate treatment causes a little overlap, but has the advantage that the basic chapters can be read independently. Chapters V, VI and VII are devoted to three special aspects of infinite divisibility: self-decomposability and stability, intimately related to the central limit problem; the fact that for surprisingly large classes of distributions, infinite divisibility is preserved under mixing; and the frequent occurrence of infinite divisibility in, e.g., queueing and renewal processes, and in certain types of Markov processes. In Chapter V, and to a lesser extent in Chapters VI and VII, there is again a separate treatment of distributions on the nonnegative integers, the nonnegative reals, and the real line. This would make it possible to plan a course covering one or two of the three basic chapters together with corresponding parts of the subsequent chapters on special topics.

The important class of compound-exponential distributions is briefly treated in each of the basic chapters; we consider writing a separate monograph on the subject.

The book is practically self-contained; most results are fully proved. One of the exceptions is the Lévy-Khintchine formula in Chapter IV; its proof is only outlined since the full proof is rather lengthy, not terribly informative, and can be found in several textbooks. At times the reader is referred to one of two appendices, A and B, one establishing notation and giving prerequisites, the other containing a selection of much-used distributions with some of their properties.

Each of Chapters II through VII has a separate section on examples, about one hundred altogether. The main text does not give references to the literature; they are deferred to Notes sections at the end of the chapters and appendices. The Bibliography also contains references to articles and books not mentioned in the Notes.

Over the years a large number of people have given advice and encouragement; we wish to express our thanks to all of them. In order not to dilute our appreciation for him we only name Lennart Bondesson, who provided constant stimulation and read large parts of the typescript.

Eindhoven, Amsterdam — Fred W. Steutel, Klaas van Harn
July 2003

Contents

VI. INFINITE DIVISIBILITY AND MIXTURES

VII. INFINITE DIVISIBILITY IN STOCHASTIC PROCESSES

Appendix A. Prerequisites from probability and analysis

Chapter I

INTRODUCTION AND OVERVIEW

1. Introduction

The concept of infinite divisibility was introduced and developed in a very short period and by a handful of people. It started in 1929 with the introduction by de Finetti of infinite divisibility in the context of processes with stationary independent increments (sii-processes); its first development ended in 1934 with the most general form of the canonical representation of infinitely divisible characteristic functions by Lévy in a paper having the term 'integrals with independent increments' in its (French) title. Even more than sii-processes, questions related to the central limit theorem led to the introduction and study of infinitely divisible and, more specifically, self-decomposable and stable distributions. Little, and mostly 'academic', attention was paid to the subject during the forties and fifties. Since about 1960 there was a renewed interest in infinitely divisible distributions, stimulated by the occurrence of such distributions in applications, most notably in waiting-time theory and in modelling problems. This led to the development of criteria for infinite divisibility in terms of distribution functions and densities rather than in terms of characteristic functions. This development culminated in the appearance of Bondesson's book on generalized gamma convolutions in 1992. In the last ten years much attention went to infinite divisibility on abstract spaces; these developments are outside the scope of this book.

The present chapter serves as an introduction to the subsequent chapters, and also as a binding element. It reviews a number of general concepts and results in, or directly related to, infinite divisibility. This chapter is also meant to put the concept of infinite divisibility in a somewhat wider perspective than will be done in the later chapters. In Section 2 we show how a formal definition emerges from practical problems in modelling; here we also give some elementary properties and a few simple examples. The connection with sii-processes is considered in Section 3; we give some special cases, and show that composition of sii-processes leads to important subclasses of infinitely divisible distributions. Section 4 contains information on canonical representations and their relation to the behaviour of sii-processes. The central limit problem is discussed in Section 5; normed sums of independent random variables give rise to stable, self-decomposable and, most generally, infinitely divisible distributions. In Section 6 some information is collected on types of (infinite) divisibility that will not be discussed further in this book. Section 7 gives bibliographic information.

2. Elementary considerations and examples

Loosely speaking, *divisibility* of a random variable X is the property that X can be divided into independent parts having the same distribution. More precisely, for $n \in \mathbb{N}$ with $n \geq 2$ we shall say that X is *n-divisible* if independent, identically distributed random variables $Y_1, \ldots, Y_n$ exist such that

$$(2.1) \qquad X \stackrel{\mathrm{d}}{=} Y_1 + \cdots + Y_n.$$

If (2.1) holds with Y's that are independent but not necessarily identically distributed, then X is said to be *n-decomposable*. We note that in the older literature the word 'decomposable' is sometimes used for what we call divisible. Clearly, divisibility is a stronger requirement than decomposability. Also, whereas $(n+1)$-decomposability implies n-decomposability, $(n+1)$-divisibility does not imply n-divisibility, as the following example shows for $n = 2$. A random variable X that has a binomial $(3, p)$ distribution, is 3-divisible; it can be obtained as the sum of three independent Bernoulli (p) variables with values 0 and 1. On the other hand, X is easily seen not to be 2-divisible; if we would have $X \stackrel{\mathrm{d}}{=} Y_1 + Y_2$ with Y_1 and Y_2 independent and $Y_i \stackrel{\mathrm{d}}{=} Y$ for $i = 1, 2$, then necessarily $Y \in [0, \frac{3}{2}]$ a.s. with $\mathbb{P}(Y = 0) > 0$ and $\mathbb{P}(Y = \frac{3}{2}) > 0$, which would yield the value $\frac{3}{2}$ for X. We return to a second illustrative example in a moment, but we first make a case for *infinite* divisibility in practical problems.

In modelling it is sometimes required that a random variable is n-divisible for given values of n. Consider, for example, the amount of damage claimed from an insurance company during a year. Then one might want to model this amount as a random variable X that is 12-divisible: X satisfies (2.1) with $n = 12$, where Y_j denotes the amount claimed during month j. Similarly, one might want to consider weekly claims and require X to be 52-divisible, or even daily claims and require 365-divisibility. Since, as we have seen, $(n+1)$-divisibility does not imply n-divisibility, it is much more practical to require that X is *n-divisible for every n*. Clearly, such a property is also desirable for random variables that model quantities, such as rainfall, which are measured in continuous time.

Thus we are led to the following concept, which is the main object of study in this book: A random variable X is said to be *infinitely divisible* if for every $n \in \mathbb{N}$ it can be written (in distribution) as

$$X \stackrel{\mathrm{d}}{=} X_{n,1} + \cdots + X_{n,n}, \tag{2.2}$$

where $X_{n,1}, \ldots, X_{n,n}$ are independent with $X_{n,j} \stackrel{\mathrm{d}}{=} X_n$ for all j and some X_n, the *n-th order factor* of X. Infinite divisibility of X is, in fact, a property of the *distribution* of X. Therefore, the distribution, the distribution function (and density in case of absolute continuity) and transform of an infinitely divisible X will be called *infinitely divisible* as well. It follows that a distribution function F is infinitely divisible iff for every $n \in \mathbb{N}$ it is the n-fold convolution of a distribution function F_n with itself, and that a characteristic function ϕ is infinitely divisible iff for every $n \in \mathbb{N}$ it is the n-th power of a characteristic function ϕ_n:

$$F = F_n^{\star n} \text{ for } n \in \mathbb{N}, \quad \phi(u) = \big\{\phi_n(u)\big\}^n \text{ for } n \in \mathbb{N}. \tag{2.3}$$

Here F_n and ϕ_n are called the *n-th order factor* of F and of ϕ, respectively. From the definition it immediately follows that if X is infinitely divisible, then so is aX for every $a \in \mathbb{R}$, and that $X + Y$ is infinitely divisible if X and Y are independent and both infinitely divisible. The latter property will be used frequently and can be reformulated as follows.

Proposition 2.1. *Infinite divisibility of distributions is preserved under convolution.*

We mention a second elementary result which is somewhat unexpected, but nevertheless simple to prove. Recall (cf. Section A.2) that a distribution function F can be decomposed as $F = \alpha\, F_{\rm d} + (1-\alpha)\, F_{\rm c}$, where $0 \le \alpha \le 1$ and $F_{\rm d}$ is a discrete distribution function and $F_{\rm c}$ a continuous one. Consider the non-trivial case where $0 < \alpha < 1$; then the decomposition is unique.

Proposition 2.2. *The discrete component F_d of an infinitely divisible distribution function F on $\mathbb{R}$ is infinitely divisible.*

PROOF. Let $n \in \mathbb{N}$, and let F_n be the n-th order factor of F. Decompose F_n as F above: $F_n = \alpha_n F_{n,{\rm d}} + (1-\alpha_n) F_{n,{\rm c}}$, and take characteristic functions. Then, by the binomial formula, equation (2.3) leads to

$$F = \alpha_n^n\, F_{n,{\rm d}}^{\star n} + \sum_{k=0}^{n-1} \binom{n}{k} \alpha_n^k (1-\alpha_n)^{n-k}\, F_{n,{\rm d}}^{\star k} \star F_{n,{\rm c}}^{\star(n-k)}.$$

Since all terms under the summation sign are continuous, the uniqueness of the decomposition of F implies that $F_{\rm d} = F_{n,{\rm d}}^{\star n}$. Hence $F_{\rm d}$ is infinitely divisible. ◻

The set of discontinuity points of an infinitely divisible distribution function F is empty or a singleton or unbounded. This immediately follows by combining Proposition 2.2 with the following elementary property, which is of a 'negative' type and shows, for instance, that the *uniform* and *binomial* distributions are *not* infinitely divisible.

Proposition 2.3. *A non-degenerate bounded random variable is not infinitely divisible.*

PROOF. Let X be a random variable with $|X| \le a$ for some $a \in \mathbb{R}$, and suppose that X is infinitely divisible. Then for $n \in \mathbb{N}$, by (2.2), the n-th order factor X_n of X satisfies $|X_n| \le a/n$, and hence the variance $\mathbb{V}{\rm ar}\, X$ of X can be estimated as follows:

$$\mathbb{V}{\rm ar}\, X = n\, \mathbb{V}{\rm ar}\, X_n \le n\, \mathbb{E} X_n^2 \le a^2/n.$$

Letting $n \to \infty$ we conclude that $\mathbb{V}{\rm ar}\, X = 0$, so X is degenerate. ◻

We return to the claim that divisibility is a more restrictive property than decomposability, and show that the *uniform* distribution is n-decomposable for every n, but not n-divisible for any n.

Example 2.4. Let X have a *uniform* distribution on $(-1,1)$. Then in a way very similar to that of the proof of Proposition 2.3 one can show that X is *not* n-divisible for any $n > 2$. For the case $n = 2$ we suppose that $X \stackrel{d}{=} Y_1 + Y_2$ with Y_1 and Y_2 independent and distributed as Y. Then $Y \in (-\frac{1}{2}, \frac{1}{2})$ a.s. and

$$\begin{aligned} \{\mathbb{P}(Y > 0)\}^2 &= \mathbb{P}(Y_1 > 0;\, Y_2 > 0) \geq \\ &\geq \mathbb{P}(Y_1 + Y_2 > \tfrac{1}{2}) = \mathbb{P}(X > \tfrac{1}{2}) = \tfrac{1}{4}, \\ \{\mathbb{P}(Y \leq 0)\}^2 &= \mathbb{P}(Y_1 \leq 0;\, Y_2 \leq 0) \geq \\ &\geq \mathbb{P}(Y_1 + Y_2 < -\tfrac{1}{2}) = \mathbb{P}(X < -\tfrac{1}{2}) = \tfrac{1}{4}. \end{aligned}$$

Taking square-roots and adding shows that the inequalities here are, in fact, equalities, so necessarily $\mathbb{P}(0 < Y \leq \frac{1}{4}) = 0$ and $\mathbb{P}(-\frac{1}{4} \leq Y \leq 0) = 0$, i.e., $\mathbb{P}\big(|Y| \leq \frac{1}{4}\big) = 0$. But this would imply that $\mathbb{P}\big(\frac{1}{4} \leq |X| \leq \frac{1}{2}\big) = 0$. We conclude that X is *not* 2-divisible.
Turning to decomposability we note that the characteristic function ϕ of X can be written as

$$\phi(u) = \frac{\sin u}{u} = \prod_{j=1}^{\infty} \cos\,(u/2^j),$$

so by the continuity theorem X can be obtained (in distribution) as

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} Y_j, \quad \text{or} \quad \sum_{j=1}^{n} Y_j \stackrel{d}{\longrightarrow} X \ \text{ as } n \to \infty, \tag{2.4}$$

where $Y_1, Y_2, \ldots$ are independent and, for $j \in \mathbb{N}$, Y_j is a Bernoulli variable taking the values $-(\frac{1}{2})^j$ and $(\frac{1}{2})^j$ each with probability $\frac{1}{2}$. It follows that X is n-decomposable for every n. ◫

As one can imagine, infinite divisibility entails much more structure than n-divisibility for a large value of n, which can be obtained by just adding n independent, identically distributed random variables. In this light it is surprising that so many of the much used distributions are infinitely divisible. We give three well-known and important examples.

Example 2.5. Let X have a *normal* (μ, σ^2) distribution. Then X is infinitely divisible; for $n \in \mathbb{N}$ its n-th order factor X_n has a normal $(\mu/n, \sigma^2/n)$ distribution. ◫

Example 2.6. Let X have a *Poisson* (λ) distribution. Then X is infinitely divisible; for $n \in \mathbb{N}$ its n-th order factor X_n is Poisson (λ/n). ◻◻

Example 2.7. Let X have a *gamma* (r, λ) distribution. Then X is infinitely divisible; for $n \in \mathbb{N}$ its n-th order factor X_n has a gamma $(r/n, \lambda)$ distribution. Taking $r = 1$ we see that the *exponential* (λ) distribution is infinitely divisible. ◻◻

The distributions in the first two examples will turn out to be the building blocks of general infinitely divisible distributions; those in the third example will be used to construct important subclasses of infinitely divisible distributions. The corresponding characteristic functions are well known to have no zeroes on $\mathbb{R}$ and to have the property that every positive power, as defined in Section A.2, is again a characteristic function. Every infinitely divisible characteristic function has these properties. We formulate this fact as a proposition to be proved later.

Proposition 2.8. *If ϕ is an infinitely divisible characteristic function, then $\phi(u) \neq 0$ for all $u \in \mathbb{R}$, and ϕ^t is a characteristic function for all $t > 0$. Specifically, for $n \in \mathbb{N}$ $\phi^{1/n}$ is the n-th order factor of ϕ.*

This elementary result is mentioned here because it is needed for the construction of continuous-time sii-processes in the next section.

3. Processes with stationary independent increments

Processes with stationary independent increments in continuous time are at the heart of infinite divisibility. They can be regarded as continuous-time analogues of *random walks* started at zero, i.e., of partial-sum processes $(X_n)_{n \in \mathbb{Z}_+}$ with

$$X_n = \sum_{k=1}^{n} Y_k \qquad [\, n \in \mathbb{Z}_+;\; X_0 :\equiv 0\,], \tag{3.1}$$

where $Y_1, Y_2, \ldots$ are independent, identically distributed random variables, the *step-sizes* of the walk. Clearly, such a process (X_n) has the following property:

$$\begin{cases} X_{n_0}, X_{n_1} - X_{n_0}, \ldots \text{ independent for } 0 < n_0 < n_1 < \cdots, \\ X_{m+n} - X_m \overset{\mathrm{d}}{=} X_n \text{ for } m, n \in \mathbb{N}. \end{cases} \tag{3.2}$$

In this case we say that (X_n) is a (discrete-time) *process with stationary independent increments* or, for short, an *sii-process*. Note that the distribution of an sii-process (X_n) is completely determined by that of X_1. Whereas, for $n \geq 2$, X_n is *n-divisible*, the distribution of X_1 can be chosen arbitrarily; in fact, any random variable Y generates uniquely (in distribution) an sii-process (X_n) with $X_1 \stackrel{\mathrm{d}}{=} Y$, and (X_n) can be viewed as a random walk (3.1) with step-sizes distributed as Y.

In the late 1920s the question arose whether there is a continuous-time analogue of (3.1); it was tentatively written as

$$X(t) = \int_{[0,t]} \mathrm{d}Y(s) \qquad [t \geq 0], \tag{3.3}$$

and termed 'integral, the elements of which are independent random variables'. It is, however, more convenient to look for an analogue of the equivalent property (3.2). In doing so we only consider continuous-time processes $X(\cdot) = \big(X(t)\big)_{t\geq 0}$ starting at zero that are *continuous in probability*, i.e., satisfy $X(s+t) \to X(s)$ in probability as $t \to 0$ for all $s \geq 0$. Such a process is said to be a (continuous-time) *process with stationary independent increments* (an *sii-process*) if

$$\begin{cases} X(t_0), X(t_1) - X(t_0), \dots \text{ independent for } 0 < t_0 < t_1 < \cdots, \\ X(s+t) - X(s) \stackrel{\mathrm{d}}{=} X(t) \text{ for } s, t > 0. \end{cases} \tag{3.4}$$

Let us look at the finite-dimensional distributions of an sii-process $X(\cdot)$. First, consider $X(t)$ with $t > 0$ fixed; for any $n \in \mathbb{N}$ with $n \geq 2$ we can write

$$X(t) = \sum_{j=1}^{n} \Big\{ X(j\,t/n) - X\big((j-1)\,t/n\big) \Big\},$$

where by (3.4) the summands are independent and identically distributed, so by definition $X(t)$ is *infinitely divisible*. Next, let ϕ_t be the characteristic function of $X(t)$. Since $X(s+t) = X(s) + \big\{X(s+t) - X(s)\big\}$, (3.4) yields the following relations:

$$\phi_{s+t}(u) = \phi_s(u)\,\phi_t(u) \qquad [s, t > 0].$$

Now, for every $u \in \mathbb{R}$ the function $t \mapsto \phi_t(u)$ is continuous on $\mathbb{R}_+$; this is implied by (and is, in fact, equivalent to) the continuity in probability of $X(\cdot)$. It follows that the multiplicative semigroup $(\phi_t)_{t\geq 0}$ satisfies

$$\phi_t(u) = \big\{\phi_1(u)\big\}^t \qquad [t \geq 0]; \tag{3.5}$$

cf. Proposition 2.8 and see Section A.2 for the definition of ϕ_1^t. Finally, take $0 = t_0 < t_1 < \cdots < t_n$ with $n \geq 2$. Then using (3.4) as above one can show that

$$\big(X(t_1), X(t_2), \ldots, X(t_n)\big) = (Y_1, Y_1 + Y_2, \ldots, Y_1 + \cdots + Y_n), \tag{3.6}$$

where $Y_1, Y_2, \ldots, Y_n$ are independent with $Y_i \stackrel{\mathrm{d}}{=} X(t_i - t_{i-1})$ for $i = 1, \ldots, n$. Combining (3.5) and (3.6) leads to the following result.

Proposition 3.1. *The distribution of an sii-process $X(\cdot)$ is completely determined by the distribution of $X(1)$, which is infinitely divisible.*

So, contrary to the discrete-time case, the distribution of $X(1)$ here may *not* be chosen arbitrarily; it has to be infinitely divisible. There is a converse; any infinitely divisible distribution gives rise to an sii-process.

Proposition 3.2. *Let Y be an infinitely divisible random variable. Then there exists an sii-process $X(\cdot)$ with $X(1) \stackrel{\mathrm{d}}{=} Y$.*

PROOF. Denote the characteristic function of Y by ϕ; then by Proposition 2.8 ϕ^t is a well-defined characteristic function for every $t > 0$. Let $0 = t_0 < t_1 < \cdots < t_n$ with $n \in \mathbb{N}$, and take independent random variables $Y_1, Y_2, \ldots, Y_n$ where Y_i has characteristic function $\phi^{t_i - t_{i-1}}$ for $i = 1, \ldots, n$. Now, let $F_{t_1,\ldots,t_n}$ be the distribution function of the right-hand side of (3.6). Then varying $t_1, \ldots, t_n$ in $(0, \infty)$ and $n \in \mathbb{N}$ we get a consistent collection of distribution functions; Kolmogorov's extension theorem then guarantees the existence of an sii-process $X(\cdot)$ satisfying (3.6) in distribution. ◻

The sii-process, in this proposition constructed from Y, will be called the sii-process *generated by* Y (or by its distribution or its transform). The sii-processes generated by the infinitely divisible distributions from Examples 2.5, 2.6 and 2.7 are of special importance; the corresponding semigroups $(\phi_t)_{t\geq 0}$ of characteristic functions are given below.

Example 3.3. The sii-process generated by the normal (μ, σ^2) distribution is called *Brownian motion* with parameters μ, the *drift*, and σ^2. The semigroup (ϕ_t) is given by

$$\phi_t(u) = \exp\big[\,iu\mu t + \tfrac{1}{2}\sigma^2 u^2 t\,\big].$$

When $\mu = 0$ and $\sigma^2 = 1$, we speak of *standard* Brownian motion. ◻

Example 3.4. The sii-process generated by the Poisson (λ) distribution is called the *Poisson process* with *intensity* λ or of *rate* λ. The semigroup (ϕ_t) is given by

$$\phi_t(u) = \exp\big[\lambda t\,\{e^{iu} - 1\}\big]. \qquad \square$$

Example 3.5. The sii-process generated by the gamma (r, λ) distribution is called the *gamma process* with parameters r and λ. The semigroup (ϕ_t) is given by

$$\phi_t(u) = \Big(\frac{\lambda}{\lambda - iu}\Big)^{rt}.$$

When $r = 1$ and $\lambda = 1$, we speak of the *standard* gamma process. $\square$

One can build a new sii-process from two given ones by *composition* as follows. Let $S(\cdot)$ and $T(\cdot)$ be independent, continuous-time sii-processes; then the generating random variables $S(1)$ and $T := T(1)$ are infinitely divisible. Suppose that $T(\cdot)$ is *nonnegative*; then we can define

$$(3.7) \qquad X(\cdot) := S\big(T(\cdot)\big),$$

and it is easily seen that $X(\cdot)$ is again an sii-process, so its generating random variable $X := S(T)$ is infinitely divisible. In this context the process $T(\cdot)$ is sometimes called a *subordinator*. If one observes an sii-process $T(\cdot)$ only at discrete times, then there is no need for T to be infinitely divisible, and (3.7) leads to a discrete-time sii-process $(X_n)_{n\in\mathbb{Z}_+}$ for which X is not necessarily infinitely divisible. Similar considerations hold when one observes $S(\cdot)$ only at discrete times; now $T(\cdot)$ must be $\mathbb{Z}_+$-valued; to stress this we replace it by $N(\cdot)$ generated by $N := N(1)$. Thus we are led to the following four types of *compound sii-processes* $X(\cdot)$ or $(X_n)_{n\in\mathbb{Z}_+}$:

(i) Sii-processes $S(\cdot)$ and $T(\cdot)$ yield $X(\cdot)$ with $X(t) := S\big(T(t)\big)$ for $t \geq 0$; it is generated by $X := S(T)$, which is infinitely divisible.

(ii) Sii-processes $S(\cdot)$ and (T_n) yield (X_n) with $X_n := S(T_n)$ for $n \in \mathbb{Z}_+$; it is generated by $X := S(T)$, which is not necessarily infinitely divisible.

(iii) Sii-processes (S_n) and $N(\cdot)$ yield $X(\cdot)$ with $X(t) := S_{N(t)}$ for $t \geq 0$; it is generated by $X := S_N$, which is infinitely divisible.

(iv) Sii-processes (S_n) and (N_n) yield (X_n) with $X_n := S_{N_n}$ for $n \in \mathbb{Z}_+$; it is generated by $X := S_N$, which is not necessarily infinitely divisible.

The distribution of the generating random variable $X = S(T)$ in the cases (i) and (ii) is called a *compound-T* distribution. Its characteristic function ϕ_X can be expressed in terms of the pLSt π_T of T and the characteristic function ϕ_Y of $Y := S(1)$ as follows:

$$\begin{aligned}\phi_X(u) &= \int_{\mathbb{R}_+} \mathbb{E}\, \mathrm{e}^{iuS(t)}\, \mathrm{d}F_T(t) = \\ &= \int_{\mathbb{R}_+} \big\{\phi_Y(u)\big\}^t\, \mathrm{d}F_T(t) = \pi_T\big(-\log\, \phi_Y(u)\big).\end{aligned} \tag{3.8}$$

An important special case is obtained by taking T exponentially distributed; cf. Examples 2.7 and 3.5. The resulting class of *compound-exponential* distributions, with characteristic functions of the form

$$\phi_X(u) = \frac{\lambda}{\lambda - \log\, \phi_Y(u)} \tag{3.9}$$

with $\lambda > 0$ (and Y infinitely divisible), turns out to have many properties analogous to the class of all infinitely divisible distributions.

The distribution of the generating random variable $X = S_N$ in the cases (iii) and (iv) above is called a *compound-N* distribution. Its characteristic function ϕ_X can be expressed in terms of the pgf P_N of N and the characteristic function ϕ_Y of $Y := S_1$ as follows:

$$\begin{aligned}\phi_X(u) &= \sum_{n=0}^{\infty} \mathbb{P}(N=n)\, \mathbb{E}\, \mathrm{e}^{iuS_n} = \\ &= \sum_{n=0}^{\infty} \mathbb{P}(N=n)\, \big\{\phi_Y(u)\big\}^n = P_N\big(\phi_Y(u)\big).\end{aligned} \tag{3.10}$$

An important special case is obtained by taking N Poisson distributed; cf. Examples 2.6 and 3.4. The resulting *compound-Poisson* distributions, with characteristic functions of the form

$$\phi_X(u) = \exp\Big[\lambda\,\big\{\phi_Y(u) - 1\big\}\Big] \tag{3.11}$$

with $\lambda > 0$ (and Y arbitrary), are basic in the sense that they are easy to handle and possess most characteristics of general infinitely divisible characteristic functions. In fact, as we shall see in the next section, every infinitely divisible distribution is the weak limit of a sequence of compound-Poisson distributions.

In Chapters II, III and IV separate sections will be devoted to compound distributions on $\mathbb{Z}_+$, on $\mathbb{R}_+$ and on $\mathbb{R}$, respectively. Though there is difference in accent, compound distributions will also be considered in Chapter VI, on *mixtures*; this is due to the fact that compound distributions can also be viewed as *power mixtures*, as will be clear from the computations in (3.8) and (3.10).

Finally we note that processes from more applied branches of probability theory connected to, or giving rise to infinite divisibility, are considered in Chapter VII; we will pay attention to Markov chains, queueing processes, branching processes, renewal processes and shot noise.

4. Canonical representations

By canonical representations we understand here characterizing formulas for the transforms of infinitely divisible distributions. In order to find such formulas, we let ϕ be an infinitely divisible characteristic function with n-th order factor ϕ_n, say, and write

$$\phi(u) = \{\phi_n(u)\}^n = \exp\Big[n \log\big(1 - \{1 - \phi_n(u)\}\big)\Big].$$

Since by Proposition 2.8 $\phi_n(u) \to 1$ as $n \to \infty$ and $-\log(1-z) \sim z$ as $z \to 0$, we conclude that ϕ can be obtained as

$$\phi(u) = \lim_{n\to\infty} \exp\Big[n\,\{\phi_n(u) - 1\}\Big]. \tag{4.1}$$

Comparing with (3.11) shows that we have proved now the claim of the preceding section which can be formulated as follows.

Proposition 4.1. *Every infinitely divisible distribution is the weak limit of a sequence of compound-Poisson distributions.*

To get a canonical representation we would like to actually take the limit in (4.1), which can be written as

$$\phi(u) = \exp\Big[\lim_{n\to\infty} n \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1)\,\mathrm{d}F_n(x)\Big], \tag{4.2}$$

with F_n the distribution function with $\widetilde{F}_n = \phi_n$. A problem here is that nF_n does not converge to a bounded function. If, however, ϕ corresponds to a distribution that has positive mass p_0, say, at zero and has no further atoms, then one can solve this problem in the following way. The distribution function F_n then has a jump $p_0^{1/n}$ at zero, so $\psi_n := (\phi_n - p_0^{1/n})/(1 - p_0^{1/n})$

is a characteristic function. Therefore, using the fact that $n\,(1-p_0^{1/n}) \to -\log p_0 =: \lambda > 0$, by (4.1) and the continuity theorem we can write

$$\phi(u) = \exp\Big[\lambda\,\big\{\lim_{n\to\infty} \psi_n(u) - 1\big\}\Big] = \exp\Big[\lambda\,\big\{\psi(u) - 1\big\}\Big],$$

where ψ is the characteristic function of a distribution without mass at zero. So ϕ is *compound-Poisson.* Now, one can show (this will be proved in Chapter IV) that any infinitely divisible distribution with at least one atom is, up to a shift, compound-Poisson. Since the converse of this trivially holds, we are led to a first canonical representation.

Theorem 4.2. *A $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of an infinitely divisible distribution that is not continuous, iff ϕ is shifted compound-Poisson, i.e., ϕ has the form*

$$\phi(u) = \exp\Big[iu\gamma + \lambda \int_{\mathbb{R}} (e^{iux} - 1)\, dG(x)\Big], \tag{4.3}$$

where $\gamma \in \mathbb{R}$, $\lambda > 0$ and G is a distribution function that is continuous at zero. The canonical triple (γ, λ, G) is unique.

For general ϕ one has to adapt the right-hand side of (4.1) or of (4.2) differently in order to be able to actually take the limit. We do not go into this problem here, and give, without proofs, the results for the infinitely divisible distributions *with finite variances* (so including the normal and gamma distributions) and *all* infinitely divisible distributions.

Theorem 4.3 (Kolmogorov canonical representation). *A $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of an infinitely divisible distribution with finite non-zero variance iff ϕ has the form*

$$\phi(u) = \exp\Big[iu\mu + \kappa \int_{\mathbb{R}} (e^{iux} - 1 - iux)\, \frac{1}{x^2}\, dH(x)\Big], \tag{4.4}$$

where $\mu \in \mathbb{R}$, $\kappa > 0$ and H is a distribution function; for $x = 0$ the integrand is defined by continuity: $-\frac{1}{2}u^2$. The canonical triple (μ, κ, H) is unique.

Theorem 4.4 (Lévy canonical representation). *A $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of an infinitely divisible distribution iff ϕ has the form*

$$\phi(u) = \exp\Big[iua - \tfrac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(e^{iux} - 1 - \frac{iux}{1+x^2}\Big)\, dM(x)\Big], \tag{4.5}$$

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and M is a right-continuous function that is nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$ with $M(x) \to 0$ as $x \to -\infty$ or $x \to \infty$ and satisfying $\int_{(-1,1)\setminus\{0\}} x^2 \, dM(x) < \infty$. The canonical triple (a, σ^2, M) is unique.

In Chapter IV we will give more details of the proof of the Lévy representation; it is then used to give precise proofs of Theorems 4.2 and 4.3, and of other representations like the one characterizing the infinitely divisible distributions that are symmetric. In Chapters II and III canonical representations are derived independently for infinitely divisible distributions on $\mathbb{Z}_+$ and on $\mathbb{R}_+$, respectively.

The canonical quantities in the theorems above not only relate to specific properties of the corresponding infinitely divisible distribution, but they also determine the behaviour of the sii-process generated by that distribution; cf. Proposition 3.2. This is quite easily seen for ϕ in (4.3) with canonical triple (γ, λ, G). The constant γ represents a drift or trend giving rise to a component γt in the process, and what remains is a compound-Poisson process having independent jumps with distribution function G at the jump times of a Poisson process with intensity λ, independent of these jumps; between jumps the process is linear. In the general case where ϕ is given by (4.5) with canonical triple (a, σ^2, M), the interpretation is more complicated. The constants a and σ^2 give rise to a component $at + \sigma\, B(t)$ with $B(\cdot)$ standard Brownian motion. The function M is not only responsible for the frequency and size of possible jumps; it can, for instance, also give rise to a trend that may even offset the trend at; because there can be very many very small jumps, the interpretation of M is in general quite intricate. It can be shown that there is always a version of $X(\cdot)$ for which the paths $t \mapsto X(t)$ are right-continuous with left-hand limits ('càdlàg') with probability one. Brownian motion is the only non-degenerate sii-process with continuous paths.

5. The central limit problem

From the results of the preceding section it will be clear that, as already said in Section 2, the normal and Poisson distributions can be considered as the building blocks of general infinitely divisible distributions. We now show that they are also special solutions to problems that are collected under the term 'central limit problem'. The *normal* distribution provides an answer to the following explicit question (all solutions will be given later in this section): If $Y_1, Y_2, \ldots$ are independent, identically distributed random variables with the property that for some $a_n \in \mathbb{R}$ and $b_n > 0$

$$(5.1) \qquad \frac{1}{b_n}\,(Y_1 + \cdots + Y_n - a_n) \xrightarrow{\mathrm{d}} X \qquad [\,n \to \infty\,],$$

what are the possible distributions of the limit X? A simple and long-known example is obtained by taking the Y_j to be Bernoulli (p) variables with values 0 and 1; then $Y_1 + \cdots + Y_n$ has a binomial (n, p) distribution with mean np and variance $np\,(1-p)$. If we now take $a_n := np$ and $b_n := \sqrt{np\,(1-p)}$, then we have (5.1) with X standard normally distributed. This can be shown by elementary methods and is a special case of the following celebrated result.

Theorem 5.1 (Central limit theorem). *If $Y_1, Y_2, \ldots$ are independent, identically distributed random variables with mean μ and (finite) variance σ^2, then*

$$(5.2) \qquad \frac{1}{\sigma\sqrt{n}}\,(Y_1 + \cdots + Y_n - n\mu) \xrightarrow{\mathrm{d}} X \qquad [\,n \to \infty\,],$$

where X has a standard normal distribution.

The *Poisson* distribution can also be obtained from binomial (n, p) distributions as $n \to \infty$, if one lets the parameter p depend on n in the following way. For $n \in \mathbb{N}$, let $X_{n,1}, \ldots, X_{n,n}$ be independent Bernoulli (p_n) variables with values 0 and 1, so $X_{n,1} + \cdots + X_{n,n}$ has a binomial (n, p_n) distribution with mean np_n. Suppose that $np_n \to \lambda > 0$ as $n \to \infty$; then it is well known and easily verified that

$$(5.3) \qquad X_{n,1} + \cdots + X_{n,n} \xrightarrow{\mathrm{d}} X \qquad [\,n \to \infty\,],$$

where X has a Poisson (λ) distribution.

Now, observe that the limiting relation (5.1) can be put in the form (5.3) by taking

$$(5.4) \qquad X_{n,j} := \frac{1}{b_n}\,(Y_j - a_n/n) \qquad [\,n \in \mathbb{N};\ j = 1, \ldots, n\,],$$

and note that if $Y_1, Y_2, \ldots$ are independent, identically distributed random variables, then so are $X_{n,1}, \ldots, X_{n,n}$ for every fixed n. This leads to considering more general *triangular arrays* $(X_{n,j})_{n\in\mathbb{N},\, j=1,..,k_n}$, where $k_n \uparrow \infty$ as $n \to \infty$ and, for every $n \in \mathbb{N}$, $X_{n,1}, \ldots, X_{n,k_n}$ are independent, identically distributed. We again ask for the possible distributions of limits X as in (5.3). Of course, by definition any infinitely divisible distribution provides an answer to this question. On the other hand, it is not hard to show that there are no other solutions. So we have the following characterization of infinite divisibility.

Theorem 5.2. *A random variable X is infinitely divisible iff it can be obtained as*

$$(5.5) \qquad X_{n,1} + \cdots + X_{n,k_n} \xrightarrow{\text{d}} X \qquad [\, n \to \infty \,],$$

where $k_n \uparrow \infty$ and, for every $n \in \mathbb{N}$, $X_{n,1}, \ldots, X_{n,k_n}$ are independent, identically distributed.

We next wonder whether the limits X in (5.5) are still infinitely divisible if one drops the condition that the random variables in every row of the triangular array are identically distributed. As an example, consider the special case (5.1) with $Y_j \stackrel{\text{d}}{=} Z_j/j$, where $Z_1, Z_2, \ldots$ are independent and standard exponential. Then it can be shown that the choice $b_n = 1$ and $a_n = \log n$ leads to a limit X having a *Gumbel* distribution, which will turn out to be infinitely divisible. On the other hand, from Example 2.4 it is seen that not all limits X of the form (5.1), with $Y_1, Y_2, \ldots$ independent, are infinitely divisible, or even 2-divisible. We return to this specific problem in a moment, and first give, without proof, the following central limit result, which is a solution for the general situation.

Theorem 5.3. *A random variable X is infinitely divisible iff it can be obtained as*

$$(5.6) \qquad X_{n,1} + \cdots + X_{n,k_n} \xrightarrow{\text{d}} X \qquad [\, n \to \infty \,],$$

where $k_n \uparrow \infty$, $X_{n,1}, \ldots, X_{n,k_n}$ are independent for every $n \in \mathbb{N}$, and

$$(5.7) \qquad \lim_{n\to\infty} \max_{1\le j\le k_n} \mathbb{P}\big(|X_{n,j}| \ge \varepsilon\big) = 0 \qquad [\, \varepsilon > 0 \,].$$

A triangular array $(X_{n,j})$ satisfying condition (5.7) is said to be *infinitesimal*, and its elements are called *uniformly almost negligible.* In such an array the role of a single summand becomes vanishingly small as $n \to \infty$; clearly, this is what we need, because otherwise any distribution can appear as the distribution of X in (5.6). It is not hard to show that an array of random variables that are independent, identically distributed in every row, is infinitesimal if (5.6) holds; therefore, Theorem 5.3 generalizes Theorem 5.2.

We return to the question from the beginning of this section, and first consider a sequence $(Y_j)_{j\in\mathbb{N}}$ of independent, identically distributed random variables with $\mathbb{E}\, Y_1 = 0$ and $\mathbb{V}\mathrm{ar}\, Y_1 = 1$. Writing $S_n := Y_1 + \cdots + Y_n$ we have $\mathbb{E}\, S_n = 0$ and $\mathbb{V}\mathrm{ar}\, S_n = n$, so we should look at $S_n/\sqrt{n}$. Now, suppose that $S_n/\sqrt{n} \xrightarrow{\mathrm{d}} X$ as $n \to \infty$. Since for every $n, k \in \mathbb{N}$ we can write

$$\frac{1}{\sqrt{nk}} S_{nk} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{\sqrt{k}} \left(S_{jk} - S_{(j-1)k}\right),$$

by (3.2) we see, letting $k \to \infty$, that the limit X satisfies

$$(5.8) \qquad X \overset{\mathrm{d}}{=} \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) \qquad [n \in \mathbb{N}],$$

where $X_1, X_2, \ldots$ are independent and distributed as X. By taking characteristic functions it follows that $\phi := \phi_X$ satisfies the equation

$$(5.9) \qquad \phi(u) = \left\{\phi(u/\sqrt{n})\right\}^n \qquad [n \in \mathbb{N}],$$

which can be solved with some difficulty yielding $\phi(u) = \exp\left[-\frac{1}{2}u^2\right]$. So X has a standard normal distribution, as it should have because of the central limit theorem.

Relation (5.8) expresses a special case of stability. A random variable X (and its distribution and transform) is said to be *weakly stable* if X can be written (in distribution) as

$$(5.10) \qquad X \overset{\mathrm{d}}{=} c_n\,(X_1 + \cdots + X_n) + d_n \qquad [n \in \mathbb{N}],$$

where $c_n > 0$, $d_n \in \mathbb{R}$ and $X_1, X_2, \ldots$ are independent and distributed as X. The random variable X is called *strictly stable* or just *stable* if it is weakly stable such that in (5.10) $d_n = 0$ for all n. It turns out that (5.10) is possible only for constants c_n of the form $c_n = n^{-1/\gamma}$ with $\gamma \in (0, 2]$; γ is called the *exponent* (of stability) of X. Clearly, a weakly stable distribution that is symmetric is stable. Moreover, it is not hard to show that the characteristic functions ϕ of the *symmetric* stable distributions with exponent γ are given by

$$(5.11)\qquad \phi(u) = \exp\left[-\lambda\,|u|^{\gamma}\right],$$

with $\lambda > 0$. So, taking $\gamma = 2$ we get the *normal* distributions with mean zero; the symmetric *Cauchy* distributions are stable with exponent $\gamma = 1$. In Chapter V we will give more details, and derive canonical representations of general (non-symmetric, weakly) stable distributions.

By (5.4) and Theorem 5.2 any limit X in (5.1), with $Y_1, Y_2, \ldots$ independent, identically distributed, is infinitely divisible. By definition, a weakly stable random variable X is *infinitely divisible* and can appear as a limit in (5.1). It can be shown that there are no other solutions. So we have the following characterization of weak stability.

Theorem 5.4. *A random variable X is weakly stable iff it can be obtained as*

$$(5.12)\qquad \frac{1}{b_n}\,(Y_1 + \cdots + Y_n - a_n) \xrightarrow{\text{d}} X \qquad [\,n \to \infty\,],$$

with $Y_1, Y_2, \ldots$ independent, identically distributed, $a_n \in \mathbb{R}$ and $b_n > 0$.

Here the distribution of the Y_j is said to be in the *domain of attraction* of the weakly stable distribution of X. One can show that if X is weakly stable with exponent $\gamma \neq 1$, then $X - d$ is stable for some $d \in \mathbb{R}$. It follows that for $\gamma \neq 1$ the domains of attraction do not change if ‘weakly stable’ is replaced by ‘stable’. According to the central limit theorem any distribution with finite variance is in the domain of attraction of the normal distribution. Domains of attraction will not be investigated further here.

We finally raise the question what happens if in (5.12) the Y_j are independent, but not necessarily identically distributed. As we have seen above, it is then possible to get in the limit both an infinitely divisible distribution and a distribution that is not even 2-divisible. In order to only get infinitely divisible limits one may require, in view of Theorem 5.3, that $(X_{n,j})$ given by (5.4) is infinitesimal. It is not hard to show that one gets the same limits if one requires (Y_j/b_n) to be infinitesimal, and that in presence of (5.12) this requirement implies the following property of the sequence (b_n):

$$(5.13)\qquad \lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1.$$

Now, we ask for the possible distributions of the limits X in (5.12) if the Y_j are independent and (b_n) satisfies (5.13). By definition any weakly stable distribution provides an answer to this question, and one might think that there are no other solutions, as was the case when generalizing Theorem 5.2 to Theorem 5.3. This is not so; at this moment it is not even clear that every limit is infinitely divisible. To be more explicit, we need the following definition: A random variable X is said to be *self-decomposable* if for every $\alpha \in [0,1]$ it can be written (in distribution) as

$$X \stackrel{\mathrm{d}}{=} \alpha X + X_\alpha, \tag{5.14}$$

where in the right-hand side X and X_α are independent. It is easily shown that the characteristic function ϕ of a self-decomposable random variable X has no zeroes; therefore, $\phi(nu)$ can be decomposed as

$$\phi(nu) = \phi(u)\,\frac{\phi(2u)}{\phi(u)}\cdots\frac{\phi(nu)}{\phi\big((n-1)u\big)} \qquad [\,n \in \mathbb{N}\,],$$

so if $Y_1, Y_2, \ldots$ are independent with $Y_j \stackrel{\mathrm{d}}{=} j\,X_{(j-1)/j}$ for $j \in \mathbb{N}$ (with X_α as in (5.14)), then X can be written (in distribution) as

$$X \stackrel{\mathrm{d}}{=} \frac{1}{n}\,(Y_1 + \cdots + Y_n) \qquad [\,n \in \mathbb{N}\,]. \tag{5.15}$$

Hence any self-decomposable distribution provides an answer to the question above. It can be shown that there are no other solutions, so we have the following result.

Theorem 5.5. *A random variable X is self-decomposable iff it can be obtained as*

$$\frac{1}{b_n}\,(Y_1 + \cdots + Y_n - a_n) \xrightarrow{\mathrm{d}} X \qquad [\,n \to \infty\,], \tag{5.16}$$

with $Y_1, Y_2, \ldots$ independent, $a_n \in \mathbb{R}$, and $b_n > 0$ satisfying (5.13).

We already noted that a weakly stable distribution can be obtained as the distribution of X in (5.16); hence weak stability implies self-decomposability. Moreover, from (5.15), where (Y_j/n) is infinitesimal, and Theorem 5.3 it follows that any self-decomposable distribution is *infinitely divisible.* These facts can also be proved just from the definitions; this will be done in Chapter V, where the self-decomposable distributions will be considered in detail, and canonical representations will be derived for them.

Well-known examples of self-decomposable distributions are the *gamma*, and hence the *exponential* distributions. Another example is the *Gumbel* distribution; above we obtained it from exponential distributions in the way of (5.16) with $b_n = 1$ for all n, which is not in accordance with Theorem 5.5: (5.13) does not hold. Still, it is self-decomposable because self-decomposability is preserved under shift, convolution and weak limits. We further note that self-decomposable distributions are continuous; this is easily seen from the definition. We will also consider, however, analogues of stability and self-decomposability for distributions on $\mathbb{Z}_+$; it will turn out, for instance, that the *Poisson* distribution is stable with exponent $\gamma = 1$.

6. Other types of divisibility

As said in Section 2, our central theme is divisibility of random variables with respect to the operation of 'independent' addition; it is equivalent to divisibility of probability measures on $\mathbb{R}$ with respect to convolution and to divisibility of one-dimensional characteristic functions with respect to multiplication. Now, in this section we very briefly consider some forms of divisibility that will not be treated in this book; in some instances the operation is different, in others the objects operated upon are different. The set of examples given below is not intended to be complete. We shall not be concerned with divisibility on abstract spaces, such as groups or Banach spaces; for a few, rather haphazard, references see Notes.

General measures on $\mathbb{R}$. Measures giving infinite mass to $\mathbb{R}$ can also be examined for divisibility with respect to convolution. The only measure of this kind that has been studied in detail, is Lebesgue measure, which turns out to be 2-divisible and even infinitely divisible. The factors are, however, not at all unique. It seems that the technique used for Lebesgue measure would be applicable to a more general class of measures.

Renewal sequences and p-functions. Here the operation of multiplication is used; a renewal sequence $(u_n)_{n\in\mathbb{Z}_+}$ is said to be 2-divisible if a renewal sequence (v_n) exists such that $u_n = v_n^2$ for all n. It turns out that the positive infinitely divisible renewal sequences are exactly the log-convex sequences (u_n) with $u_0 = 1$. Similarly, the continuous-time counterparts, i.e., the positive infinitely divisible p-functions, are log-convex.

Random vectors. Use, as for random variables, the operation of independent addition, component-wise. Then the theory of infinite divisibility is very similar to the theory in the one-dimensional case. One has the same closure properties, and the form of the canonical representations is almost independent of the dimension. Also, the canonical representations of more-dimensional stable and self-decomposable distributions are similar to those in dimension one. Things get more complicated, however, if stability and self-decomposability are not defined component-wise, but vector-wise; then one speaks of operator-stability and operator-self-decomposability; the definition is in terms of matrices that specialize to diagonal matrices in the simple case. In many respects the more-dimensional situation allows less detail; there are fewer examples, apart from the trivial ones concerning vectors of independent random variables, and very little is known, for example, about mixtures. For random functions, that is, for stochastic processes, the theory is still similar, but much more technical.

Maxima. If addition is replaced by taking maxima, then things change. Since the distribution function of the maximum $X \vee Y$ of two independent random variables X and Y is given by the product of the distribution functions: $F_{X \vee Y} = F_X \, F_Y$, the tool to use is not the characteristic function, but the distribution function itself; we are concerned with divisibility of distribution functions with respect to multiplication. Rather curiously, every one-dimensional distribution function F is max-infinitely divisible; every positive power of F is again a distribution function. The central-limit type theory for normed maxima of independent, identically distributed random variables leads to the three well-known types of max-stable limit distributions, of which the *Gumbel* distribution, considered in Section 5, is an example; see also Theorem A.2.2. More-dimensional distribution functions are not all max-infinitely divisible. Now there is a theory of max-infinite divisibility not unlike the classical (additive) theory.

Matrices. There is some theory of infinite divisibility for nonnegative definite matrices with respect to Schur products, i.e., element-wise multiplication. The theory can be regarded as an extension of infinite divisibility of more-dimensional characteristic functions. The transition matrices $P(t) = \big(p_{ij}(t)\big)$ with $t > 0$ in a continuous-time Markov chain form a semi-group with respect to ordinary matrix multiplication; here $p_{ij}(t)$ denotes the probability of a transition from i to j during a period of length t.

The matrix $P(t)$ can be put in the form $P(t) = \{P(1)\}^t = e^{tQ} = e^{\lambda t\,(\widetilde{Q}-I)}$, where Q and $\widetilde{Q}$ are matrices of a very special type; this formula can be read as a canonical representation of the semi-group. Infinite divisibility of a given transition matrix is equivalent to embeddability of this matrix into a semi-group, just as we have seen in infinite divisibility on $\mathbb{R}$. In branching processes a similar situation occurs for semi-groups of pgf's with respect to composition.

We finally mention a form of divisibility that *will* be considered in this book. For an infinitely divisible $\mathbb{Z}_+$-valued random variable X we shall require that, for every $n \in \mathbb{N}$, the n-th order factor X_n of X is $\mathbb{Z}_+$-valued as well. In Chapter II we will see that this extra requirement is equivalent to the condition that $\mathbb{P}(X = 0) > 0$. For an infinitely divisible $\mathbb{R}_+$-valued random variable no such condition is necessary; the n-th order factors are easily seen to be $\mathbb{R}_+$-valued as well.

7. Notes

The concept of infinite divisibility was introduced by de Finetti (1929), and developed by Kolmogorov (1932), Lévy (1934, 1937) and Khintchine (1937a). Other early papers are Feller (1939) and Itô (1942). Good sources for the history of this period are the obituary of Lévy by Loève (1973) and the appendix on the history of probability in the book by Gnedenko (1991), in German. Survey papers were published by Fisz (1962), who gives many references, and by Steutel (1973, 1979). The book by Bondesson (1992) is on a special subject, but it contains a lot of general information.

Processes with stationary independent increments, also called *Lévy processes*, date back to de Finetti (1929) and Lévy (1934). Two recent books on the subject are by Bertoin (1996) and Sato (1999), both of which contain sections on infinite divisibility. Good treatments, discussing the relation between properties of the canonical measure and of the sample paths, can be found in Breiman (1968) and Stroock (1993).

The first canonical representation was given by Kolmogorov (1932) for distributions with finite variance. Lévy (1934) gave the first general formula; other early contributions are by Khintchine (1937b), Lévy (1937) and Feller (1939). A good source for these developments is the book by Gnedenko and Kolmogorov (1968).

The theorems on triangular arrays are due to Khintchine (1937a) and Lévy (1937). The stable and self-decomposable distributions (the latter are also called distributions *of class L*) are due to Lévy (1923, 1937); see also Khintchine and Lévy (1936). Full discussions in relation to the central limit problem can be found in Gnedenko and Kolmogorov (1968), Feller (1971), Loève (1977), Petrov (1975, 1995) and, especially in terms of characteristic functions, in Lukacs (1970, 1983).

The infinite divisibility of Lebesgue measure on $\mathbb{R}$ is treated in O'Brien and Steutel (1981). Infinitely divisible renewal sequences and p-functions can be found in Kendall and Harding (1973). Infinitely divisible random vectors are briefly treated in Feller (1971); a good account is given in Sato (1999); see also Sato (1973) for tail behaviour; special aspects are considered in Horn and Steutel (1978). More-dimensional self-decomposable and stable distributions can be found in Sato (1999). For operator-stability we refer to Sharpe (1969b), for operator-self-decomposability to Urbanik (1972); see also Jurek and Mason (1993). Infinitely divisible processes were first studied in detail by Kerstan et al. (1978). Max-infinite divisibility is studied by Balkema and Resnick (1977), and by Alzaid and Proschan (1994); see also Resnick (1987). Horn (1967, 1969) considers the infinite divisibility of matrices with respect to element-wise multiplication. For semigroups of transition matrices of continuous-time Markov processes, see Doob (1953), Chung (1960), Rosenblatt (1962), or Freedman (1972). Kazakyavichyus (1996) considers infinitely divisible transition matrices in the context of limit theorems and semigroups. Some references for infinite divisibility in abstract spaces are McCrudden and Walker (1999, 2000), and Yasuda (2000). For results on infinite divisibility and (operator-) self-decomposability in Banach spaces we mention Kumar and Schreiber (1975, 1979), Jurek and Urbanik (1978), Urbanik (1978), Jurek (1983), Jurek and Vervaat (1983), and Kruglov and Antonov (1984). Infinite divisibility modulo one is considered in Wilms (1994).

As a few references on applications we mention Carasso (1987), Gerber (1992), Voudouri (1995), and Barndorff-Nielsen and Shephard (2001).

Chapter II

INFINITELY DIVISIBLE DISTRIBUTIONS ON THE NONNEGATIVE INTEGERS

1. Introduction

Rather than specializing down from the most general case: distributions on $\mathbb{R}$, we start with the simplest case: distributions on $\mathbb{Z}_+$. This case contains most of the essential features of infinite divisibility, but avoides some of the technical problems of the most general situation; the recurrence relations we encounter here, will be integral equations in the $\mathbb{R}_+$-case, and the problems concerning zeroes of densities and tail behaviour are more delicate there and on $\mathbb{R}$. The basic tool here is the probability generating function, rather than the Laplace-Stieltjes transform or the characteristic function.

Let X be a $\mathbb{Z}_+$-valued random variable. According to the general definition in Chapter I, *infinite divisibility* of X means the existence for every $n \in \mathbb{N}$ of a random variable X_n, the *n-th order factor* of X, such that

(1.1) $$X \stackrel{\mathrm{d}}{=} X_{n,1} + \cdots + X_{n,n},$$

where $X_{n,1}, \ldots, X_{n,n}$ are independent and distributed as X_n. It is quite natural and, as we shall see in a moment, no essential restriction to consider only random variables X that are *discrete infinitely divisible* in the sense that their factors X_n are $\mathbb{Z}_+$-valued as well. Such random variables X necessarily have the property that

(1.2) $$\mathbb{P}(X = 0) > 0,$$

because if $\mathbb{P}(X = 0)$ would be zero, then by (1.1) we would have $X_n \geq 1$ a.s. for all n, and hence $X \geq n$ a.s. for all n. On the other hand, if X is infinitely divisible and $n \in \mathbb{N}$, then $\mathbb{P}(X=0) = \{\mathbb{P}(X_n = 0)\}^n$ and hence for $x > 0$

$$(1.3) \qquad \begin{aligned} \mathbb{P}(X = x) &\geq n\,\mathbb{P}(X_n = x)\,\{\mathbb{P}(X_n = 0)\}^{n-1} \geq \\ &\geq n\,\mathbb{P}(X_n = x)\,\mathbb{P}(X = 0), \end{aligned}$$

so in the presence of (1.2) any possible value of X_n, i.e., a value taken with positive probability, is also a possible value of X, so necessarily X_n is $\mathbb{Z}_+$-valued. We conclude that *a $\mathbb{Z}_+$-valued infinitely divisible random variable X is discrete infinitely divisible iff it satisfies (1.2)*. Since shifting a random variable to zero does not affect its possible infinite divisibility, it follows that without essential loss of generality we can restrict attention to discrete infinite divisibility. We therefore agree that throughout this chapter for $\mathbb{Z}_+$-valued random variables X:

(1.4) *infinite divisibility* means *discrete infinite divisibility*, i.e., infinite divisibility of X implies $\mathbb{P}(X = 0) > 0$.

Moreover, mostly we tacitly exclude the trivial case where X is degenerate at zero, so we then assume that $\mathbb{P}(X = 0) < 1$.

Recall from Section I.2 that the distribution and transform of an infinitely divisible random variable will be called *infinitely divisible* as well. From convention (1.4) it follows that a distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ is infinitely divisible iff for every $n \in \mathbb{N}$ there is a distribution $(p_k^{(n)})_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$, the *n-th order factor* of (p_k), such that (p_k) is the n-fold convolution of $(p_k^{(n)})$ with itself:

$$(1.5) \qquad (p_k) = (p_k^{(n)})^{*n} \qquad [n \in \mathbb{N}].$$

Similarly, a probability generating function (pgf, for short) P is infinitely divisible iff for every $n \in \mathbb{N}$ there is a pgf P_n, the *n-th order factor* of P, such that

$$(1.6) \qquad P(z) = \{P_n(z)\}^n \qquad [n \in \mathbb{N}].$$

Note that from (1.4), or directly from (1.6), it follows that an infinitely divisible pgf P necessarily satisfies $P(0) > 0$. For conventions and notations concerning distributions on $\mathbb{Z}_+$ and generating functions (gf's) we refer to Section A.4.

In Section 2 we give elementary properties and treat two much-used special cases, the Poisson distribution and the negative-binomial (including the geometric) distribution. Section 3, on compound distributions, is basic, since on $\mathbb{Z}_+$ the infinitely divisible distributions coincide with the compound-Poisson distributions. Here also the compound-geometric or, equivalently, the compound-exponential distributions are introduced. A slight rewriting of the compound-Poisson pgf yields, in Section 4, a canonical representation which, in turn, leads to an explicit criterion for infinite divisibility in terms of very useful recurrence relations for the probabilities themselves; here the use of absolutely monotone functions is prominent. These tools are shown to have analogues for the compound-exponential distributions (Section 5), and are used to easily prove closure properties in Section 6. Relations between moments of an infinitely divisible distribution and those of the corresponding canonical sequence are given in Section 7. The structure of the support of infinitely divisible distributions is determined in Section 8, and their tail behaviour is considered in Section 9; here the role of Poisson distributions is very manifest. In Section 10 log-convex distributions, all of which are infinitely divisible, and infinitely divisible log-concave distributions are considered together with their canonical sequences. An interesting special case of log-convexity is complete monotonicity, related to mixtures of geometric distributions. Section 11 contains examples and counter-examples, and in Section 12 bibliographical and other supplementary remarks are made.

Finally, we note that this chapter only treats the basic properties of infinitely divisible distributions on $\mathbb{Z}_+$. Results for self-decomposable and stable distributions on $\mathbb{Z}_+$ can be found in Sections V.4, V.5 and V.8, for mixtures of Poisson distributions and of negative-binomial distributions in Sections VI.6 and VI.7, and for generalized negative-binomial convolutions in Section VI.8. Also, the chapter on stochastic processes contains some information on infinitely divisible distributions on $\mathbb{Z}_+$, especially Section VII.5.

2. Elementary properties

We start with giving a few simple properties that will be used frequently. Those in the first proposition follow directly from (1.1) or (1.6).

Proposition 2.1.

(i) *If X is an infinitely divisible $\mathbb{Z}_+$-valued random variable, then so is aX for every $a \in \mathbb{Z}_+$. Equivalently, if P is an infinitely divisible pgf, then so is P_a with $P_a(z) := P(z^a)$ for every $a \in \mathbb{Z}_+$.*

(ii) *If X and Y are independent infinitely divisible $\mathbb{Z}_+$-valued random variables, then $X+Y$ is an infinitely divisible random variable. Equivalently, if P and Q are infinitely divisible pgf's, then their pointwise product PQ is an infinitely divisible pgf.*

In Section 6 we will prove a variant of part (i): If X is an infinitely divisible $\mathbb{Z}_+$-valued random variable, then so is $\alpha \odot X$ for every $\alpha \in (0,1)$; here $\odot$ is the 'discrete multiplication' as defined in Section A.4. Part (ii) states that infinite divisibility of distributions is preserved under convolutions. We now show that it is also preserved under weak convergence.

Proposition 2.2. *If a sequence $(X^{(m)})$ of infinitely divisible $\mathbb{Z}_+$-valued random variables converges in distribution to X, then X is infinitely divisible. Equivalently, if a sequence $(P^{(m)})$ of infinitely divisible pgf's converges (pointwise) to a pgf P, then P is infinitely divisible.*

PROOF. Since by (1.6) for every $m \in \mathbb{N}$ there exists a sequence $(P_n^{(m)})_{n\in\mathbb{N}}$ of pgf's such that $P^{(m)} = \{P_n^{(m)}\}^n$, the limit $P = \lim_{m\to\infty} P^{(m)}$ can be written as

$$P(z) = \lim_{m\to\infty} \{P_n^{(m)}(z)\}^n = \{\lim_{m\to\infty} P_n^{(m)}(z)\}^n = \{P_n(z)\}^n,$$

where $P_n := \lim_{m\to\infty} P_n^{(m)}$ is a pgf for every $n \in \mathbb{N}$ by the continuity theorem; see Theorem A.4.1. Hence P is infinitely divisible. $\square$

Since a pgf P is nonnegative on $[0,1]$, for z in this interval relation (1.6) can be rewritten as

$$\{P(z)\}^{1/n} = P_n(z) \qquad [n \in \mathbb{N}]. \tag{2.1}$$

As a pgf is determined by its values on $[0,1]$, it follows that the factors P_n of an infinitely divisible pgf P, and hence the corresponding distributions, are uniquely determined by P. Mostly we shall consider pgf's only for values of the argument in $[0,1]$. We shall return to the possibility of zeroes for complex arguments later in this section. For any pgf P with $P(0) > 0$ and for any $t > 0$ the function $P^t = \exp[\,t \log P\,]$ is well defined on $[\,0,1\,]$. We now come to a first criterion for infinite divisibility on $\mathbb{Z}_+$. Note that the set of pgf's equals the set of *absolutely monotone* functions P on $[\,0,1)$ with $P(1-) = 1$; see Theorem A.4.3.

Proposition 2.3. *A pgf P with $P(0) > 0$ is infinitely divisible iff P^t is a pgf for all $t \in T$, where $T = (0, \infty)$, $T = \{1/n : n \in \mathbb{N}\}$ or $T = \{a^{-k} : k \in \mathbb{N}\}$ for any fixed integer $a \geq 2$. Equivalently, P is infinitely divisible iff P^t is absolutely monotone for all $t \in T$ with T as above.*

Proof. Let P be infinitely divisible. Then by (2.1) $P^{1/n}$ is a pgf for all $n \in \mathbb{N}$, and hence $P^{m/n}$ is a pgf for all $m, n \in \mathbb{N}$. It follows that P^t is a pgf for all positive $t \in \mathbb{Q}$, and hence, by the continuity theorem, for all $t > 0$. Conversely, from (2.1) we know that P is infinitely divisible if $P^{1/n}$ is a pgf for every $n \in \mathbb{N}$. We can even be more restrictive because, for a given integer $a \geq 2$, any $t \in (0,1)$ can be represented as $t = \sum_{k=1}^{\infty} t_k a^{-k}$ with $t_k \in \{0, \ldots, a-1\}$ for all k, and hence for these t

$$\{P(z)\}^t = \lim_{m \to \infty} \prod_{k=1}^{m} \Big(\{P(z)\}^{1/a^k} \Big)^{t_k}.$$

By the continuity theorem it now follows that if P^t is a pgf for all t of the form a^{-k} with $k \in \mathbb{N}$, then so is P^t for all $t \in (0,1)$. □

Corollary 2.4. *If P is an infinitely divisible pgf, then so is P^t for all $t > 0$. In particular, the factors X_n of an infinitely divisible $\mathbb{Z}_+$-valued random variable X are infinitely divisible.*

The continuous multiplicative semigroup $(P^t)_{t \geq 0}$ of pgf's generated by an infinitely divisible pgf P corresponds to the set of one-dimensional marginal distributions of a $\mathbb{Z}_+$-valued *sii-process*, i.e., a process $X(\cdot)$ with stationary independent increments, started at zero and continuous in probability; see Section I.3. If $X(1)$, with pgf P, has distribution $(p_k)_{k \in \mathbb{Z}_+}$, then for $t > 0$ the distribution of $X(t)$, with pgf P^t, will be denoted by $(p_k^{*t})_{k \in \mathbb{Z}_+}$, so:

$$p_k^{*t} = \mathbb{P}\big(X(t) = k\big), \quad \sum_{k=0}^{\infty} p_k^{*t} z^k = \{P(z)\}^t. \tag{2.2}$$

Note that from (1.3) it follows that the n-th order factor $(p_k^{*(1/n)})$ of (p_k) satisfies

$$p_0^{*(1/n)} = p_0^{1/n}, \quad p_k^{*(1/n)} \leq p_k / (n p_0) \ \text{ for } k \in \mathbb{N}. \tag{2.3}$$

We give two examples that will turn out to be important building blocks for the construction of more general infinitely divisible distributions.

Example 2.5. For $\lambda > 0$, let X have the *Poisson* (λ) distribution, so its distribution $(p_k)_{k\in\mathbb{Z}_+}$ and pgf P are given by

$$p_k = \frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda}, \quad P(z) = \exp\bigl[-\lambda\,(1-z)\bigr].$$

Then for $t > 0$ the t-th power P^t of P is recognized as the pgf of the Poisson (λt) distribution. Hence by Proposition 2.3 the Poisson (λ) distribution is *infinitely divisible.* The corresponding sii-process is the *Poisson process* with intensity λ. □

Example 2.6. For $r > 0$ and $p \in (0,1)$, let X be *negative-binomial* (r,p), so its distribution $(p_k)_{k\in\mathbb{Z}_+}$ and pgf P are given by

$$p_k = \binom{r+k-1}{k}\,p^k\,(1-p)^r, \quad P(z) = \Bigl(\frac{1-p}{1-p\,z}\Bigr)^r.$$

Then for $t > 0$ P^t is recognized as the pgf of the negative-binomial (rt,p) distribution. We conclude that the negative-binomial (r,p) distribution is *infinitely divisible.* The corresponding sii-process is a *negative-binomial process.* Taking $r = 1$ one sees that the *geometric* (p) distribution is *infinitely divisible.* □

For a pgf P the condition $P(0) > 0$ makes it possible to define $\log P(z)$ and $P^t(z)$ also for *complex* values of z with $|z|$ sufficiently small. If one wants to define these quantities for larger values of z, it is important to know about the zeroes of P. Knowledge about zeroes is, however, also of interest because it leads to necessary conditions for infinite divisibility. The property that infinitely divisible characteristic functions have no real zeroes (cf. Proposition I.2.8), implies that infinitely divisible pgf's have no zeroes on the unit circle. It turns out that such pgf's have no zeroes inside this circle either. We shall prove a somewhat stronger result.

Proposition 2.7. *If P is an infinitely divisible pgf with $p_0 := P(0)$, then*

(2.4) $$\bigl|P(z)\bigr| \ge p_0^2 \qquad [\,z\in\mathbb{C} \text{ with } |z| \le 1\,].$$

PROOF. If Q is the pgf of a distribution (q_k) on $\mathbb{Z}_+$ with $q_0 > \frac{1}{2}$, then for $z \in \mathbb{C}$ with $|z| \leq 1$ we can write

$$\left|Q(z)\right| = \left|q_0 + \sum_{k=1}^{\infty} q_k z^k\right| \geq q_0 - \sum_{k=1}^{\infty} q_k |z|^k \geq 2q_0 - 1.$$

Using the resulting inequality for the n-th order factor P_n of an infinitely divisible pgf P, we see that $\left|P(z)\right|$ with $|z| \leq 1$ can be estimated as follows:

$$\left|P(z)\right| = \left|P_n(z)\right|^n \geq \left(2p_0^{1/n} - 1\right)^n,$$

for all n sufficiently large. Now one easily verifies that for every $\alpha > 0$ one has the equality $\lim_{n\to\infty}\left(2\alpha^{1/n} - 1\right)^n = \alpha^2$; taking $\alpha = p_0$ finishes the proof. □

This result can be extended as follows. Let P be an infinitely divisible pgf having radius of convergence $\rho > 1$, and take $\alpha \in (1, \rho)$, so $P(\alpha) < \infty$. Note that because of (2.3) the n-th order factor $P^{1/n}$ of P has radius of convergence $\rho_n \geq \rho$. Therefore, Proposition 2.3 can be used to show that the pgf P_α with $P_\alpha(z) := P(\alpha z)/P(\alpha)$ is infinitely divisible. Now, apply Proposition 2.7 to P_α; then one sees that

$$\left|P(z)\right| \geq p_0^2/P(\alpha) \qquad \left[\, z \in \mathbb{C} \text{ with } |z| \leq \alpha \,\right]. \tag{2.5}$$

The inequalities obtained above immediately yield the following useful result.

Theorem 2.8. *Let P be an infinitely divisible pgf. Then:*

(i) *P has no zeroes in the closed disk $|z| \leq 1$.*

(ii) *If P has radius of convergence $\rho > 1$, then P has no zeroes in the open disk $|z| < \rho$.*

(iii) *If P is an entire function, i.e., if $\rho = \infty$, then P has no zeroes in $\mathbb{C}$.*

Inequality (2.4) is sharp; the Poisson pgf P with $P(z) = \exp\left[-\lambda\,(1 - z)\right]$ satisfies $P(-1) = p_0^2$; cf. Example 2.5. The same example, with λ sufficiently large, shows that an infinitely divisible pgf, though non-zero, may be positive as well as negative inside the unit disk. It also illustrates part (iii) of Theorem 2.8. This theorem may be used to show that a given pgf is not infinitely divisible; examples are given in Section 11. Finally, it follows that if P is an infinitely divisible pgf, then $\log P(z)$ can be defined on $|z| \leq 1$ as a continuous function with $\log P(1) = 0$; see also Section A.4.

3. Compound distributions

In Section I.3 we showed how composition of discrete- or continuous-time sii-processes leads to *compound* distributions. We recall some facts specializing them to the present $\mathbb{Z}_+$-case, and first use discrete-time processes. Let $(S_n)_{n\in\mathbb{Z}_+}$ be an sii-process generated by a $\mathbb{Z}_+$-valued random variable Y (so $S_n = Y_1 + \cdots + Y_n$ for all n with $Y_1, Y_2, \ldots$ independent and distributed as Y), let N be $\mathbb{Z}_+$-valued and independent of (S_n), and consider X such that

$$(3.1) \qquad X \stackrel{d}{=} S_N \quad (\text{so } X \stackrel{d}{=} Y_1 + \cdots + Y_N).$$

Then X is said to have a *compound-N* distribution, and from (I.3.10) one sees that its pgf can be expressed in the pgf's of Y and N by

$$(3.2) \qquad P_X(z) = P_N\big(P_Y(z)\big).$$

In particular, it follows that the composition of two pgf's is again a pgf. Hence, taking t-th powers in (3.2) with $t > 0$ and using Proposition 2.3, we are led to the following simple, but useful result.

Proposition 3.1. *A $\mathbb{Z}_+$-valued random variable X that has a compound-N distribution with N $\mathbb{Z}_+$-valued and infinitely divisible, is infinitely divisible. Equivalently, the composition $P \circ Q$ of an infinitely divisible pgf P with an arbitrary pgf Q is an infinitely divisible pgf.*

The case where N has a Poisson distribution, is especially important to us; we will then speak of a *compound-Poisson* distribution. Its pgf has the form

$$(3.3) \qquad P(z) = \exp\big[-\lambda\big\{1 - Q(z)\big\}\big],$$

where $\lambda > 0$ and Q is a pgf. Here the pair (λ, Q) is not uniquely determined by P, but it is, if we choose Q such that $Q(0) = 0$, which can always be done. From Proposition 3.1 and Example 2.5, or directly from (3.3), it is clear that compound-Poisson distributions are infinitely divisible. In fact, on $\mathbb{Z}_+$, the set of compound-Poisson distributions coincides with the set of infinitely divisible distributions. This important result is the content of the following theorem.

Theorem 3.2. *A pgf P is infinitely divisible iff it is compound-Poisson, i.e., iff it has the form (3.3) with $\lambda > 0$ and Q a pgf with $Q(0) = 0$.*

PROOF. There is only one implication left to prove; let P be an infinitely divisible pgf. Then, for $n \in \mathbb{N}$, $P_n := P^{1/n}$ is a pgf, and hence so is the function Q_n defined by

$$Q_n(z) := \frac{P_n(z) - P_n(0)}{1 - P_n(0)} = 1 - \frac{1 - P(z)^{1/n}}{1 - P(0)^{1/n}}.$$

Now, let $n \to \infty$ and use the fact that $\lim_{n\to\infty} n\left(1 - \alpha^{1/n}\right) = -\log \alpha$ for $\alpha > 0$; then from the continuity theorem we conclude that the function Q defined by

$$(3.4) \qquad Q(z) := 1 - \frac{1}{\lambda}\left\{-\log P(z)\right\} \;\text{ with }\; \lambda := -\log P(0),$$

is a pgf with $Q(0) = 0$. It follows that P can be written as in (3.3). ◻

As we shall see, the basic role played by the compound-Poisson distributions is not limited to the infinitely divisible distributions on $\mathbb{Z}_+$. Many properties of infinitely divisible distributions on $\mathbb{R}$ are, in fact, directly inherited from the Poisson distribution itself. Further, note that from Theorem 3.2 it follows that the semigroups $(P^t)_{t\geq 0}$ with P an infinitely divisible pgf consist of the compound-Poisson semigroups, generated by the compound-Poisson processes.

Example 3.3. Let X have a *geometric* (p) distribution. As shown in Example 2.6, X is infinitely divisible and hence it should be possible to rewrite its pgf in the form (3.3). Indeed, since solving (3.3) for (λ, Q) with $Q(0) = 0$ yields (3.4), we find

$$\lambda = -\log(1-p), \quad Q(z) = -\frac{1}{\lambda}\log(1 - pz),$$

where Q is recognized as the pgf of the *logarithmic-series* (p) distribution shifted to $\mathbb{N}$; cf. Section B.4. ◻

The compound-N distributions with N geometrically distributed will be called *compound-geometric*. Their pgf's have the form

$$(3.5) \qquad P(z) = \frac{1 - p}{1 - p\,Q(z)},$$

where $p \in (0, 1)$ and Q is a pgf. The pair (p, Q) is uniquely determined by P if we choose Q such that $Q(0) = 0$, which can always be done. Since the geometric distribution is infinitely divisible (cf. Examples 2.6 or 3.3), the following theorem is an immediate consequence of Proposition 3.1.

Theorem 3.4. *The compound-geometric distributions on $\mathbb{Z}_+$, with pgf's given by (3.5), are infinitely divisible (and hence compound-Poisson).*

The class of compound-geometric distributions is a proper subclass of the class of compound-Poisson distributions. In fact, as in Example 3.3 one shows that a pgf is compound-geometric iff it is of the compound-Poisson form (3.3) with Q compound-N where N has a logarithmic-series distribution on $\mathbb{N}$. Though one class contains the other, there are several analogies between them; this will be apparent from the results in Sections 4 and 5. We now show that the compound-geometric distributions can also be obtained by using a different type of compounding.

To this end we again recall some facts from Section I.3, specialized to our $\mathbb{Z}_+$-case. Let $S(\cdot)$ be a continuous-time sii-process generated by a $\mathbb{Z}_+$-valued random variable Y $\big($so $S(1) \stackrel{\mathrm{d}}{=} Y$ and Y is infinitely divisible$\big)$, let T be $\mathbb{R}_+$-valued and independent of $S(\cdot)$, and consider X such that

$$(3.6) \qquad X \stackrel{\mathrm{d}}{=} S(T).$$

Then X is said to have a *compound-T* distribution, and from (I.3.8) one sees that its pgf can be expressed in the pgf of Y and the pLSt of T by

$$(3.7) \qquad P_X(z) = \pi_T\big(-\log P_Y(z)\big).$$

Now, apply Proposition 2.3 and its counterpart for $\mathbb{R}_+$, to be given in Proposition III.2.3, or just use (I.2.3) adapted for pgf's and pLSt's; then one immediately obtains the following analogue of Proposition 3.1.

Proposition 3.5. *A $\mathbb{Z}_+$-valued random variable X that has a compound-T distribution with T $\mathbb{R}_+$-valued and infinitely divisible, is infinitely divisible. Equivalently, the composition $\pi \circ (-\log P_0)$ where π is an infinitely divisible pLSt and P_0 is an infinitely divisible pgf, is an infinitely divisible pgf.*

The compound-T distributions with T degenerate at one constitute precisely the set of all infinitely divisible, and hence compound-Poisson, distributions. It is more interesting to take T standard exponentially distributed. Since then $\pi_T(s) = 1/(1+s)$, the resulting *compound-exponential* distributions have pgf's of the form

$$(3.8) \qquad P(z) = \frac{1}{1 - \log P_0(z)},$$

where P_0 is an *infinitely divisible* pgf. This pgf, which is sometimes called the *underlying* (infinitely divisible) pgf of P, is uniquely determined by P because

$$(3.9) \qquad P_0(z) = \exp\big[1 - 1/P(z)\big].$$

Note that taking T exponential (λ) with $\lambda \neq 1$ leads to the same class of distributions; just use Corollary 2.4. Since the exponential distribution is infinitely divisible (see Example I.2.7), Proposition 3.5 guarantees that the compound-exponential distributions are infinitely divisible as well; we can say somewhat more.

Theorem 3.6. *The compound-exponential distributions on $\mathbb{Z}_+$, with pgf's given by (3.8), coincide with the compound-geometric distributions on $\mathbb{Z}_+$, and hence are infinitely divisible.*

PROOF. First, let P be a compound-exponential pgf, so it has the form (3.8). Then using Theorem 3.2 for P_0 shows that P can be rewritten as

$$P(z) = \frac{1}{1 + \lambda\{1 - Q(z)\}} = \frac{1 - \lambda/(1+\lambda)}{1 - \{\lambda/(1+\lambda)\}\, Q(z)},$$

which is of the compound-geometric form (3.5). The converse statement is proved similarly: If P has the form (3.5), then computing the right-hand side of (3.9) shows that P can be written as in (3.8) with P_0 of the form (3.3); take $\lambda = p/(1-p)$. □

Finally we return to Theorem 3.2, and use it to give a representation (in distribution) of an infinitely divisible random variable X as an infinite sum of independent random variables having Poisson distributions on different lattices. Let X be infinitely divisible. Then the pgf P_X of X has the compound-Poisson form (3.3) with $\lambda > 0$ and Q the pgf of a distribution $(q_j)_{j\in\mathbb{N}}$ on $\mathbb{N}$. Putting $\lambda_j := \lambda q_j$ for $j \in \mathbb{N}$, we can rewrite P_X as follows:

$$(3.10) \qquad P_X(z) = \exp\Big[-\sum_{j=1}^{\infty} \lambda_j (1 - z^j)\Big] = \lim_{n\to\infty} P_{V_1+2V_2+\cdots+nV_n}(z),$$

where $V_1, V_2, \ldots$ are independent and V_j has a Poisson (λ_j) distribution with $V_j = 0$ a.s. if $\lambda_j = 0$. By the continuity theorem it follows that $\sum_{j=1}^{n} jV_j \xrightarrow{\text{d}} X$ as $n \to \infty$. Hence $\sum_{j=1}^{\infty} jV_j < \infty$ a.s., and we can represent X as in (3.11) below; the converse is proved similarly, or by use of Proposition 2.2. Also, note that $\mathbb{P}(\sum_{j=1}^{\infty} jV_j = 0) = \exp[-\sum_{j=1}^{\infty} \mathbb{E}V_j]$.

Theorem 3.7. *A $\mathbb{Z}_+$-valued random variable X is infinitely divisible iff a sequence $(V_j)_{j\in\mathbb{N}}$ of independent Poisson variables (possibly degenerate at zero) exists such that*

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} jV_j, \tag{3.11}$$

in which case necessarily $\sum_{j=1}^{\infty} \mathbb{E}V_j < \infty$.

This theorem is of importance in Sections 8 and 9 where we study the support and tail behaviour of an infinitely divisible distribution. The representation for P_X in (3.10) is closely related to the canonical representation in the next section.

4. Canonical representation

According to Theorem 3.2 an infinitely divisible pgf P is characterized by a pair (λ, Q), where λ is a positive number and Q is the pgf of a probability distribution $(q_j)_{j\in\mathbb{N}}$ on $\mathbb{N}$:

$$P(z) = \exp\big[-\lambda\{1 - Q(z)\}\big]. \tag{4.1}$$

As we saw at the end of the previous section, the pair $\big(\lambda, (q_j)\big)$ can be replaced by just one sequence of nonnegative numbers: $(\lambda_j) = (\lambda q_j)$. We now make a somewhat different choice for this sequence, which turns out to be more convenient:

$$r_k := \lambda\,(k+1)\,q_{k+1} \qquad [\,k \in \mathbb{Z}_+\,]. \tag{4.2}$$

Using these r_k shows that P in (4.1) can be rewritten as in (4.3) below. Since the converse is proved similarly, we are led to the following representation result, which will be considered as a *canonical representation.* Here the sequence $(r_k)_{k\in\mathbb{Z}_+}$, which is unique because (λ, Q) is such, is called the *canonical sequence* of the pgf P, and of the corresponding distribution $(p_k)_{k\in\mathbb{Z}_+}$ and of a corresponding random variable X.

Theorem 4.1 (Canonical representation). *A function P on $[0,1]$ with $P(0) > 0$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$ iff P has the form*

$$P(z) = \exp\Big[-\sum_{k=0}^{\infty} \frac{r_k}{k+1}(1 - z^{k+1})\Big] \qquad [\,0 \le z \le 1\,], \tag{4.3}$$

where the quantities r_k with $k \in \mathbb{Z}_+$ are all nonnegative. Here the canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ is unique, and necessarily

$$(4.4) \qquad \sum_{k=0}^{\infty} \frac{r_k}{k+1} = -\log P(0), \quad \textit{and hence} \ \sum_{k=0}^{\infty} \frac{r_k}{k+1} < \infty.$$

One easily shows that if $r_k \geq 0$ for all k, then the infinite sum in the exponent of P in (4.3) can be rewritten as $\int_z^1 R(x)\,\mathrm{d}x$, where R is the gf of the sequence (r_k) and hence is *absolutely monotone*. Now, in view of Theorem A.4.3 we can reverse matters and thus obtain the following reformulation of Theorem 4.1.

Theorem 4.2. *A function P on $[0,1)$ with $P(0) > 0$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$ iff P has the form*

$$(4.5) \qquad P(z) = \exp\Big[-\int_z^1 R(x)\,dx\Big] \qquad [0 \leq z < 1]$$

with R an absolutely monotone function on $[0,1)$. Here R is unique; it is the gf of the canonical sequence of P.

This theorem leads to a useful *criterion* for infinite divisibility, as follows. Solving equation (4.5) for R we find

$$(4.6) \qquad R(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log P(z) = \frac{P'(z)}{P(z)} \qquad [0 \leq z < 1].$$

For lack of a better name we shall call this function the *R-function* of P, also for functions P that are not yet known to be pgf's (but are such that (4.6) makes sense). Note that if P is an infinitely divisible pgf with canonical sequence (r_k), then

(4.7) the gf of (r_k) is the R-function of P.

Theorem 4.2 now immediately yields the following characterization of infinitely divisible pgf's; note that we need not start from a pgf.

Theorem 4.3. *Let P be a positive, differentiable function on $[0,1)$ with $P(1-) = 1$. Then P is an infinitely divisible pgf iff its R-function is absolutely monotone.*

In Proposition 2.3 we saw that a necessary and sufficient condition for infinite divisibility of a pgf P with $P(0) > 0$ is that P^t is absolutely monotone for all $t > 0$. Theorem 4.3 states that this condition is equivalent to the absolute monotonicity of only the R-function of P. We note that this equivalence can be proved directly; we will proceed in this way in the $\mathbb{R}_+$-case in Chapter III, and obtain there the canonical representation as a consequence.

The criterion for infinite divisibility in Theorem 4.3 turns out to be very useful indeed; this will be clear from Example 4.8, from proofs in Section 6 and from examples in Section 11. The following consequence is even more useful, because it gives a characterization of infinite divisibility on $\mathbb{Z}_+$ by means of *recurrence relations* for the probability distribution itself.

Theorem 4.4. *A distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ with $p_0 > 0$ is infinitely divisible iff the quantities r_k with $k \in \mathbb{Z}_+$ determined by*

$$(4.8)\qquad (n+1)\,p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \qquad [\,n \in \mathbb{Z}_+\,],$$

are nonnegative. In this case $(r_k)_{k\in\mathbb{Z}_+}$ is the canonical sequence of (p_k).

PROOF. Let (p_k) have pgf P, and apply Theorem 4.3; note that $P(0) > 0$. First suppose that (p_k) is infinitely divisible. Then the R-function R of P is absolutely monotone; by (4.7) it is the gf of the canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ of (p_k). Now observe that

$$(4.9)\qquad P'(z) = P(z)\,R(z) \qquad [\,0 \le z < 1\,];$$

equating the coefficients in the power series expansions of both members of this identity, we see that the recurrence relations in (4.8) hold. Conversely, let (p_k) satisfy these relations with $r_k \ge 0$ for all k. Then taking gf's shows that (4.9) holds with $R(z) := \sum_{k=0}^{\infty} r_k z^k$. It follows that the R-function of P is given by R, which is absolutely monotone. Hence P is infinitely divisible. □

Theorem 4.4 is especially useful for the construction of examples and counter-examples. By Theorem 4.1, *any* sequence $(r_k)_{k\in\mathbb{Z}_+}$ of nonnegative numbers satisfying $\sum_k r_k/(k+1) < \infty$ determines an infinitely divisible distribution (p_k) via (4.8). It is usually not convenient to use (4.8) for calculating r_k explicitly from the p_k for large values of k. Nevertheless, some useful necessary conditions can be obtained, as follows. Consider the first n equations in (4.8):

$$\begin{cases} p_0 r_0 & = \quad p_1 \\ p_1 r_0 + p_0 r_1 & = \ 2p_2 \\ p_2 r_0 + p_1 r_1 + p_0 r_2 & = \ 3p_3 \\ \qquad \vdots & \qquad \vdots \\ p_{n-1} r_0 + p_{n-2} r_1 + \cdots + p_0 r_{n-1} & = \ np_n. \end{cases}$$

Solving these by Cramer's rule (see Section A.5), one gets $r_{n-1} = D_n / p_0^n$, where D_n is the determinant defined in (4.10) below. Since $r_0 = p_1/p_0$ is always nonnegative, Theorem 4.4 yields the following explicit criterion for infinite divisibility.

Corollary 4.5. *A distribution* $(p_k)_{k \in \mathbb{Z}_+}$ *on* $\mathbb{Z}_+$ *with* $p_0 > 0$ *is infinitely divisible iff for every* $n \geq 2$

$$(4.10) \qquad D_n := \begin{vmatrix} p_0 & 0 & 0 & \cdot & \cdot & 0 & p_1 \\ p_1 & p_0 & 0 & \cdot & \cdot & 0 & 2p_2 \\ p_2 & p_1 & p_0 & \cdot & \cdot & 0 & 3p_3 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & p_0 & \cdot \\ p_{n-1} & p_{n-2} & p_{n-3} & \cdot & \cdot & p_1 & np_n \end{vmatrix} \geq 0.$$

Taking $n = 2$ in (4.10) we get the following necessary condition for a distribution (p_k) to be infinitely divisible:

$$(4.11) \qquad 2p_0 p_2 \geq p_1^2.$$

This implies for instance that no distribution (p_k) with $p_0 > 0$, $p_1 > 0$ and $p_2 = 0$ can be infinitely divisible. We shall return to this in Section 8, when discussing the *support* of (p_k).

The canonical sequence $(r_k)_{k \in \mathbb{Z}_+}$ of an infinitely divisible distribution is most easily determined by first computing the R-function $R = P'/P$ of its pgf P and then expanding R as a power series; cf. (4.7). We show this for the two examples from Section 2, and add a third basic example, which emerges from considerations in Chapter V in a natural way.

Example 4.6. The *Poisson* (λ) distribution of Example 2.5 with pgf P given by

$$P(z) = \exp\big[-\lambda\,(1-z)\big],$$

has R-function $R(z) = \lambda$; hence its canonical sequence (r_k) is given by

$$r_k = \begin{cases} \lambda & , \text{ if } k = 0, \\ 0 & , \text{ if } k \geq 1, \end{cases}$$

so $D_n = 0$ for $n \geq 2$. Thus, within the class of infinitely divisible distributions, the Poisson distribution can be viewed as a borderline case. □

Example 4.7. The *negative-binomial* (r, p) distribution of Example 2.6 with pgf P given by

$$P(z) = \Big(\frac{1-p}{1-p\,z}\Big)^r,$$

has R-function $R(z) = rp/(1-pz)$, so for its canonical sequence (r_k) we get

$$r_k = r\,p^{k+1} \qquad [\,k \in \mathbb{Z}_+\,].$$ □

Example 4.8. For $\lambda > 0$, $\gamma > 0$, let P be the function on $[\,0, 1)$ given by

$$P(z) = \exp\big[-\lambda\,(1-z)^\gamma\big] \qquad [\,0 \leq z < 1\,].$$

For $\gamma = 1$ we get Example 4.6. When $\gamma \neq 1$ we compute the R-function of P and find $R(z) = \lambda\gamma\,(1-z)^{\gamma-1}$ for $z \in [\,0, 1)$. It follows that in case $\gamma > 1$ the function P is *not* a pgf; if it were, then $\lim_{z\uparrow 1} R(z) = 0$ would equal the first moment of P. When $\gamma < 1$, however, the function R is absolutely monotone, so from Theorem 4.3 we conclude that P is an *infinitely divisible pgf* with canonical sequence (r_k) given by

$$r_k = \lambda\gamma\,(-1)^k \binom{\gamma-1}{k} = \lambda\gamma \binom{k-\gamma}{k} \qquad [\,k \in \mathbb{Z}_+\,].$$

Note that (r_k) is completely monotone: $r_k = \lambda\gamma\,\mathbb{E}Z^k$ for all k, where Z has a beta $(1-\gamma, \gamma)$ distribution; cf. Section B.3. For $\lambda > 0$ and $\gamma \leq 1$ the pgf P, and the corresponding distribution, will be called *stable* (λ) *with exponent* γ; this terminology is justified by the results in Section V.5 where the stable distributions on $\mathbb{Z}_+$ are studied separately. □

For further examples we refer to Section 11. New examples can be constructed from old ones by using the closure properties to be given in Section 6. But first, in the next section, we briefly consider the subclass of compound-geometric or, equivalently, compound-exponential distributions (cf. Theorem 3.6), and show that analogues can be given of several results from the present section.

5. Compound-exponential distributions

The *compound-geometric* distributions, introduced in Section 3, are particularly interesting. They occur quite frequently in practice, e.g., in queueing situations (cf. Chapter VII), and they contain the completely monotone and log-convex distributions, the infinite divisibility of which will be proved in Section 10. In the present section we focus on another interesting aspect; the class of compound-geometric distributions turns out to have many properties very similar to the class of *all* infinitely divisible distributions.

To show this we recall the general form of the compound-geometric pgf's P:

$$(5.1) \qquad P(z) = \frac{1-p}{1-p\,Q(z)},$$

where $p \in (0,1)$ and Q is the pgf of a distribution $(q_j)_{j\in\mathbb{N}}$ on $\mathbb{N}$, and rewrite P in terms of the gf S of the sequence $(s_k)_{k\in\mathbb{Z}_+}$ of nonnegative numbers where

$$(5.2) \qquad s_k := p\,q_{k+1} \qquad [\,k \in \mathbb{Z}_+\,].$$

Then using Theorem A.4.3 and Proposition A.4.4 (vi) we arrive at the following analogue to Theorem 4.2; note that P in (4.5) can be rewritten as

$$(5.3) \qquad P(z) = P(0)\,\exp\Big[\int_0^z R(x)\,\mathrm{d}x\Big] \qquad [\,0 \le z < 1\,].$$

Theorem 5.1. *A positive function P on $[\,0,1)$ with $P(1-) = 1$ is the pgf of a compound-geometric distribution on $\mathbb{Z}_+$ iff P has the form*

$$(5.4) \qquad P(z) = P(0)\,\frac{1}{1-z\,S(z)} \qquad [\,0 \le z < 1\,]$$

with S an absolutely monotone function on $[\,0,1)$.

Of course, the function S in this theorem is uniquely determined by P; solving equation (5.4) for S we find

$$S(z) = \frac{1}{z}\Big\{1 - \frac{P(0)}{P(z)}\Big\} \qquad [\,0 \le z < 1\,]. \tag{5.5}$$

This function will be called the *S-function* of P, also for (positive) functions P that are not yet known to be pgf's. Theorem 5.1 now immediately yields the following characterization of compound-geometric pgf's; cf. Theorem 4.3.

Theorem 5.2. *Let P be a positive function on $[\,0,1)$ satisfying $P(1-) = 1$. Then P is a compound-geometric pgf iff its S-function is absolutely monotone.*

This criterion leads to a useful characterization result in terms of the probability distribution itself; it is the analogue of Theorem 4.4 where infinite divisibility of (p_k) is characterized by the nonnegativity of the r_k satisfying

$$(n+1)\,p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \qquad [\,n \in \mathbb{Z}_+\,]. \tag{5.6}$$

Theorem 5.3. *A distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ with $p_0 > 0$ is compound-geometric iff the quantities s_k with $k \in \mathbb{Z}_+$ determined by*

$$p_{n+1} = \sum_{k=0}^{n} p_k\, s_{n-k} \qquad [\,n \in \mathbb{Z}_+\,], \tag{5.7}$$

are nonnegative. In this case the gf of $(s_k)_{k\in\mathbb{Z}_+}$ equals the S-function of the pgf P of (p_k), and necessarily

$$\sum_{k=0}^{\infty} s_k = 1 - p_0, \quad \text{and hence} \quad \sum_{k=0}^{\infty} s_k < 1. \tag{5.8}$$

Proof. Similar to the proof of Theorem 4.4; use Theorems 5.2 and A.4.3, and observe that (5.5) can be rewritten as

$$\frac{P(z) - P(0)}{z} = P(z)\, S(z) \qquad [\,0 \le z < 1\,], \tag{5.9}$$

which clearly is equivalent to the recurrence relations (5.7) if S is the gf of $(s_k)_{k\in\mathbb{Z}_+}$. Condition (5.8) is obtained by letting $z \uparrow 1$ in (5.9). ◻

By Theorem 5.1, *any* sequence $(s_k)_{k\in\mathbb{Z}_+}$ of nonnegative numbers satisfying $\sum_k s_k < 1$ determines a compound-geometric distribution (p_k) via (5.7). A useful inequality for compound-geometric distributions (p_k) is the following:

$$p_0\, p_2 \geq p_1^2; \tag{5.10}$$

it is the condition $E_2 \geq 0$ in the following consequence (by Cramer's rule) of Theorem 5.3.

Corollary 5.4. *A distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ with $p_0 > 0$ is compound-geometric iff for every $n \geq 2$*

$$E_n := \begin{vmatrix} p_0 & 0 & 0 & \cdot & \cdot & 0 & p_1 \\ p_1 & p_0 & 0 & \cdot & \cdot & 0 & p_2 \\ p_2 & p_1 & p_0 & \cdot & \cdot & 0 & p_3 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & p_0 & \cdot \\ p_{n-1} & p_{n-2} & p_{n-3} & \cdot & \cdot & p_1 & p_n \end{vmatrix} \geq 0. \tag{5.11}$$

The Poisson distribution is *not* compound-geometric; it does not satisfy inequality (5.10). Let us consider another basic example.

Example 5.5. The *negative-binomial* (r, p) distribution with pgf P given by

$$P(z) = \Big(\frac{1-p}{1-p\,z}\Big)^r,$$

is *compound-geometric* iff $r \leq 1$. This follows from Theorem 5.2; the S-function of P satisfies $z\,S(z) = 1 - (1 - p\,z)^r$, so S is absolutely monotone iff $r \leq 1$. In this case the sequence $(s_k)_{k\in\mathbb{Z}_+}$ with gf S is given by

$$s_k = (-1)^k \binom{r}{k+1} p^{k+1} \qquad [\,k \in \mathbb{Z}_+\,],$$

so in the geometric case where $r = 1$, we have $s_0 = p$ and $s_k = 0$ for $k \geq 1$; within the class of compound-geometric distributions, the geometric distribution can be viewed as a borderline case. □

By starting from the *compound-exponential* distributions, also introduced in Section 3, and using Theorem 3.6 we can give alternatives for

Theorems 5.2 and 5.3 that are sometimes more convenient. To show this we recall that a compound-exponential pgf P is determined by (and determines) an infinitely divisible pgf P_0, the *underlying* pgf of P, as follows:

$$(5.12) \qquad P(z) = \frac{1}{1 - \log P_0(z)}, \quad \text{so} \quad P_0(z) = \exp\big[1 - 1/P(z)\big].$$

Now observe that the R-function of P_0, i.e., the function $R_0 = (\log P_0)'$, can be expressed in terms of P as follows:

$$(5.13) \qquad R_0(z) = -\frac{\mathrm{d}}{\mathrm{d}z}\{1/P(z)\} = \frac{P'(z)}{\{P(z)\}^2} \qquad [0 \le z < 1].$$

This function will be called the R_0*-function* of P, also for functions P that are not yet known to be pgf's (but are such that (5.13) makes sense). Now applying Theorem 4.3 to P_0 in (5.12) immediately yields the following counterpart to Theorem 5.2.

Theorem 5.6. *Let P be a positive, differentiable function on $[0,1)$ with $P(1-) = 1$. Then P is a compound-exponential pgf iff its R_0-function is absolutely monotone.*

From (5.13) it is seen that the R- and R_0-functions of a pgf P are related by

$$(5.14) \qquad R(z) = P(z)\, R_0(z) \qquad [0 \le z < 1].$$

Now suppose that P is compound-exponential (and hence infinitely divisible). Then both R and R_0 are absolutely monotone; by (4.7) R is the gf of the canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ of P. Since by Proposition I.2.3 the distribution $(p_k)_{k\in\mathbb{Z}_+}$ with pgf P has unbounded support, i.e., $p_k > 0$ for infinitely many k, from (5.14) it follows that the same holds for (r_k). So we have the following result.

Proposition 5.7. *The canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ of a compound-exponential distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ has unbounded support: $r_k > 0$ for infinitely many k.*

Finally, we make another use of (5.13); we rewrite it as

$$(5.15) \qquad P'(z) = \{P(z)\}^2\, R_0(z) \qquad [0 \le z < 1].$$

Then, in a similar way as in the proofs of Theorems 4.4 and 5.3, we can use Theorem 5.6 to obtain the following counterpart to Theorem 5.3; the final statement follows from (4.4) applied to the underlying pgf P_0.

Theorem 5.8. *A distribution* $(p_k)_{k\in\mathbb{Z}_+}$ *on* $\mathbb{Z}_+$ *with* $p_0 > 0$ *is compound-exponential iff the quantities* $r_{0,k}$ *with* $k \in \mathbb{Z}_+$ *determined by*

$$(5.16) \qquad (n+1)\, p_{n+1} = \sum_{k=0}^{n} p_k^{*2}\, r_{0,n-k} \qquad [\, n \in \mathbb{Z}_+ \,],$$

are nonnegative. In this case the gf of $(r_{0,k})_{k\in\mathbb{Z}_+}$ *equals the* R_0*-function of the pgf* P *of* (p_k)*, and necessarily*

$$(5.17) \qquad \sum_{k=0}^{\infty} \frac{r_{0,k}}{k+1} = \frac{1-p_0}{p_0}, \quad \text{and hence} \ \sum_{k=0}^{\infty} \frac{r_{0,k}}{k+1} < \infty.$$

The recurrence relations occurring in this theorem will be used and interpreted in Section VII.5 on renewal processes.

6. Closure properties

The criterion in Theorem 4.3 enables us to derive some more *closure properties*. Propositions 2.1 (ii) and 2.2 state that the class of infinitely divisible pgf's is closed under pointwise *products* and *limits*, provided these limits are pgf's. In particular, if P is an infinitely divisible pgf, then so is P^n for all $n \in \mathbb{N}$. Hence the additional value of the closure property of Corollary 2.4 only concerns powers P^α with $0 < \alpha < 1$. In the following proposition this property is restated together with similar properties related to other operations which preserve infinite divisibility.

Proposition 6.1. *Let* P *be an infinitely divisible pgf and let* $0 < \alpha < 1$. *Then any of the following functions* P_α *is an infinitely divisible pgf:*

(i) $P_\alpha(z) := \{P(z)\}^\alpha$;
(ii) $P_\alpha(z) := P(1-\alpha+\alpha z)$;
(iii) $P_\alpha(z) := P(\alpha z)/P(\alpha)$;
(iv) $P_\alpha(z) := P(\alpha)P(z)/P(\alpha z)$.

Proof. Apply Theorem 4.3 twice in each of the four cases; note that we do not know in advance that the functions P_α in (i) and (iv) are pgf's. The R-function R_α of P_α can be expressed in that of P by

$$R_\alpha(z) = \alpha R(z),\ \ \alpha R(1-\alpha+\alpha z),\ \ \alpha R(\alpha z),\ \ R(z) - \alpha R(\alpha z),$$

respectively. Now, use some elementary properties of absolutely monotone functions as listed in Proposition A.4.4. ◻

The pgf's in (i), (ii) and (iii) can be interpreted in terms of random variables. Let P be the (infinitely divisible) pgf of a random variable X. Then P_α in (i) is the pgf of $X(\alpha)$, where $X(\cdot)$ is the continuous-time sii-process generated by X. The pgf P_α in (ii) is recognized as the pgf of the *α-fraction* $\alpha \odot X$ of X, as defined in Section A.4; hence we have, as a variant of Proposition 2.1 (i):

(6.1) $\qquad X$ infinitely divisible $\implies$ $\alpha \odot X$ infinitely divisible.

Since P_α can be viewed as the composition of P and Q_α with $Q_\alpha(z) := 1-\alpha+\alpha z$, the infinite divisibility of P_α also immediately follows from Proposition 3.1. Reversing the roles of P and Q_α, one is led to considering the pgf $1-\alpha+\alpha P$. Such a pgf, however, can have 'forbidden' zeroes, in which case it is not infinitely divisible by Theorem 2.8; let P correspond to a Poisson distribution, for instance. Hence we have for $\alpha \in (0,1)$:

(6.2) $\qquad P$ infinitely divisible $\not\Longrightarrow$ $1-\alpha+\alpha P$ infinitely divisible.

In particular, it follows that the class of infinitely divisible distributions is not closed under mixing; see, however, Proposition 10.6 and Chapter VI. Turning to the pgf P_α in (iii), we take independent sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of independent random variables such that $X_n \stackrel{\mathrm{d}}{=} X$ and $Y_n \stackrel{\mathrm{d}}{=} Y$ for all n, where Y has a geometric (α) distribution shifted to $\mathbb{N}$, and define $N := \inf\{n \in \mathbb{N} : X_n < Y_n\}$, which is finite a.s. Then, also letting X and Y be independent, we have $X_N \stackrel{\mathrm{d}}{=} (X \mid X < Y)$, and because

$$\mathbb{E}\, z^X 1_{\{X<Y\}} = \sum_{k=0}^{\infty} z^k\, \mathbb{P}(X=k)\, \mathbb{P}(Y>k) = P(\alpha z),$$

and hence $\mathbb{P}(X<Y) = P(\alpha)$, we see that X_N has pgf P_α as given by (iii). Note that in terms of distributions the closure property (iii) reads as follows: If $(p_k)_{k\in\mathbb{Z}_+}$ is an infinitely divisible distribution on $\mathbb{Z}_+$, then so is $(p_k^{(\alpha)})$ for every $\alpha \in (0,1)$, where

(6.3) $$p_k^{(\alpha)} = \frac{1}{P(\alpha)}\, \alpha^k\, p_k \qquad [\,k \in \mathbb{Z}_+\,].$$

There does not seem to exist an obvious interpretation of the pgf P_α in part (iv) of Proposition 6.1. We note that P_α is not necessarily a pgf if P is not infinitely divisible. From the relation between the R-functions of P and P_α, given in the proof of the proposition, it will be clear that the infinite divisibility of P_α is not only necessary for the infinite divisibility of P but also sufficient. For ease of reference we state this result explicitly.

Proposition 6.2. *Let P be a pgf, let $0 < \alpha < 1$, and define the function P_α by*

$$P_\alpha(z) = \frac{P(\alpha)\, P(z)}{P(\alpha z)}.$$

Then P is infinitely divisible iff P_α is an infinitely divisible pgf.

Here the condition that P_α is an infinitely divisible pgf for a fixed $\alpha \in (0,1)$, may be replaced by the condition that P_α is just a pgf for *all* $\alpha \in (0,1)$ or, also sufficient, for all $\alpha \in (1-\varepsilon, 1)$ for some $\varepsilon > 0$. We formulate this result as a 'self-decomposability' characterization of infinite divisibility; cf. Chapter V.

Theorem 6.3. *A pgf P with $P(0) > 0$ is infinitely divisible iff for all $\alpha \in (0,1)$ there exists a pgf P_α such that*

$$P(z) = \frac{P(\alpha z)}{P(\alpha)}\, P_\alpha(z). \tag{6.4}$$

Proof. Suppose that (6.4) holds for all $\alpha \in (0,1)$ with P_α a pgf. Then for all $\alpha \in (0,1)$ the function $z \mapsto P(z)/P(\alpha z)$ is absolutely monotone, and hence so is R_α defined by

$$R_\alpha(z) := \frac{1}{1-\alpha}\, \frac{1}{z} \Big\{ \frac{P(z)}{P(\alpha z)} - 1 \Big\}.$$

Now, let $\alpha \uparrow 1$; since then $R_\alpha(z) \to P'(z)/P(z)$, it follows that the R-function of P is absolutely monotone. Hence P is infinitely divisible by Theorem 4.3. The converse statement immediately follows from Proposition 6.1 (iv). □

The result of Proposition 3.1 can also be viewed as a closure property: If P is a pgf, then for any pgf Q

$$P \text{ infinitely divisible} \implies P \circ Q \text{ infinitely divisible.} \tag{6.5}$$

The closure property of Proposition 6.1 (iii) holds for general $\alpha > 0$ when adapted as follows: If P is a pgf with $P(\alpha) < \infty$, then

$$P \text{ infinitely divisible} \implies P(\alpha\,\cdot)\big/P(\alpha) \text{ infinitely divisible;} \tag{6.6}$$

the same proof can be used, but one can also use Proposition 2.3, as was done for deriving (2.5), or Theorem 3.2. Now, combining the closure properties (6.5) and (6.6) yields the following generalization.

Proposition 6.4. *Let P be a pgf and let $\alpha > 0$ be such that $P(\alpha) < \infty$. Then for any pgf Q the following implication holds:*

$$\text{(6.7)} \qquad P \text{ infinitely divisible} \implies P \circ (\alpha Q)\big/P(\alpha) \text{ infinitely divisible.}$$

The special case with $\alpha = \mathrm{e}$ and $Q(z) = \mathrm{e}^{z-1}$ emerges in a natural way from properties of *moments* given in the next section.

Finally we turn to the counterpart of (6.5) given by Proposition 3.5: If π is an infinitely divisible pLSt, then for pgf's P_0 and P we have

$$\text{(6.8)} \qquad P_0 \text{ infinitely divisible} \implies P := \pi \circ (-\log P_0) \text{ infinitely divisible.}$$

For instance, taking here π exponential shows that if P_0 is an infinitely divisible pgf, then so is P with $P(z) = 1\big/\big(1 - \log\, P_0(z)\big)$; P is compound-exponential. On the other hand, taking for P_0 a stable pgf as considered in Example 4.8, we obtain the following useful special case.

Proposition 6.5. *If π is an infinitely divisible pLSt, then $z \mapsto \pi\big((1-z)^\gamma\big)$ is an infinitely divisible pgf for every $\gamma \in (0, 1\,]$.*

7. Moments

Let X be an infinitely divisible $\mathbb{Z}_+$-valued random variable with distribution $(p_k)_{k \in \mathbb{Z}_+}$. We are interested in the *moments* of X of any nonnegative order; let $\mu_0 := 1$, and for $r > 0$ let

$$\mu_r := \mathbb{E}X^r = \sum_{k=0}^{\infty} k^r p_k = \sum_{n=0}^{\infty} (n+1)^r\, p_{n+1} \qquad [\le \infty\,].$$

We will relate the μ_r to moments of the canonical sequence $(r_k)_{k \in \mathbb{Z}_+}$ of X. To this end we use the recurrence relations of Theorem 4.4:

$$\text{(7.1)} \qquad (n+1)\, p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \qquad [\,n \in \mathbb{Z}_+\,],$$

and recall that $r_k \geq 0$ for all k with $\sum_{k=0}^{\infty} r_k/(k+1) < \infty$. A first result is obtained by summing both members of (7.1) over $n \in \mathbb{Z}_+$; then one sees that the expectation of X itself is given by

$$(7.2) \qquad \mathbb{E}X = \sum_{k=0}^{\infty} r_k.$$

In a similar way one easily shows that for $\alpha > -1$ the moment of order $r = \alpha + 1$ can be obtained as

$$(7.3) \qquad \mu_{\alpha+1} = \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} (k+\ell)^{\alpha}\, p_k\, \overline{r}_\ell,$$

where $(\overline{r}_k)_{k\in\mathbb{N}}$ is the *shifted* canonical sequence with $\overline{r}_k := r_{k-1}$. Here, and in Sections 8 and 9, it is convenient to consider $(\overline{r}_k)$ rather than (r_k). The moment of $(\overline{r}_k)$ of order $\alpha > -1$ is denoted by ν_α, so

$$\nu_\alpha := \sum_{k=1}^{\infty} k^{\alpha}\, \overline{r}_k \qquad [\leq \infty].$$

We first take $\alpha = n \in \mathbb{Z}_+$ in (7.3). Then using the binomial formula for $(k+\ell)^n$ one sees that the moment sequences $(\mu_n)_{n\in\mathbb{Z}_+}$ and $(\nu_n)_{n\in\mathbb{Z}_+}$ are related by

$$(7.4) \qquad \mu_{n+1} = \sum_{j=0}^{n} \binom{n}{j} \mu_j\, \nu_{n-j} \qquad [n \in \mathbb{Z}_+].$$

Since these relations have exactly the same structure as those in (A.2.20) for $(\mu_n)_{n\in\mathbb{Z}_+}$ and the sequence $(\kappa_n)_{n\in\mathbb{N}}$ of *cumulants* of (p_k), one is led to the following result.

Theorem 7.1. *Let $(p_k)_{k\in\mathbb{Z}_+}$ be an infinitely divisible distribution on $\mathbb{Z}_+$ with shifted canonical sequence $(\overline{r}_k)_{k\in\mathbb{N}}$ as defined above, and let $n \in \mathbb{Z}_+$. Then the $(n+1)$-st order moment μ_{n+1} of (p_k) is finite iff the n-th order moment ν_n of $(\overline{r}_k)$ is finite:*

$$(7.5) \qquad \mu_{n+1} < \infty \iff \nu_n < \infty.$$

In this case the $(n+1)$-st order cumulant κ_{n+1} of (p_k) equals ν_n:

$$(7.6) \qquad \kappa_{n+1} = \nu_n.$$

Corollary 7.2. *The cumulants of an infinitely divisible distribution on $\mathbb{Z}_+$ (as far as they exist) are nonnegative.*

Relation (7.4) also yields a curious *closure property* in the following way. Suppose that $\mu_k < \infty$ for all $k \in \mathbb{Z}_+$. Then from (7.4) one easily verifies that for every $c > 0$ the sequence $(\widetilde{p}_k)_{k\in\mathbb{Z}_+}$ with $\widetilde{p}_k := c\,\mu_k/k!$ satisfies the recurrence relations in (7.1) with r_k replaced by $\nu_k/k!$ for $k \in \mathbb{Z}_+$. Further observe that

$$(7.7) \qquad \sum_{k=0}^{\infty} \frac{\mu_k}{k!}\, z^k = \sum_{k=0}^{\infty} \frac{1}{k!}\, \mathbb{E}X^k\, z^k = \mathbb{E}\,\mathrm{e}^{zX} \qquad [\, z \ge 0\,].$$

Now applying Theorem 4.4 proves the following result.

Proposition 7.3. *Let $(p_k)_{k\in\mathbb{Z}_+}$ be an infinitely divisible distribution on $\mathbb{Z}_+$ with pgf P such that $P(\mathrm{e}) < \infty$. Then (p_k) has all its moments μ_k with $k \in \mathbb{Z}_+$ finite, and the sequence $(\widetilde{p}_k)_{k\in\mathbb{Z}_+}$ with*

$$(7.8) \qquad \widetilde{p}_k := \frac{1}{P(\mathrm{e})}\, \frac{\mu_k}{k!} \qquad [\, k \in \mathbb{Z}_+\,]$$

is an infinitely divisible probability distribution on $\mathbb{Z}_+$ with pgf $\widetilde{P}$ given by

$$(7.9) \qquad \widetilde{P}(z) = P(\mathrm{e}^z)/P(\mathrm{e}).$$

So, our considerations on moments lead to Proposition 6.4 for the special case with $Q(z) = \mathrm{e}^{z-1}$. One may also start from an infinitely divisible $\mathbb{R}_+$-valued random variable X with pLSt π such that $\pi(-1) < \infty$. Then the moments μ_k of X are all finite, and in Chapter III we will show that, as in the $\mathbb{Z}_+$-case, the cumulants of X are all nonnegative. Since (7.7) still holds, one is led to the special case $a = 1$ of the following result; it is easily proved by using Proposition 2.3 and its counterpart for $\mathbb{R}_+$, to be given in Proposition III.2.3, or by just using (I.2.3) adapted for pgf's and pLSt's.

Proposition 7.4. *Let $a > 0$, and let π be an infinitely divisible pLSt such that $\pi(-a) < \infty$. Then the function P_a defined below is an infinitely divisible pgf:*

$$(7.10) \qquad P_a(z) := \pi(-az)\,/\pi(-a).$$

For instance, if π is the pLSt of the *gamma* (r, λ) distribution, then for P_a in (7.10) we get the pgf of the *negative-binomial* (r, p) distribution if we choose $a = p\lambda$, which can be done. In a similar way the *degenerate* distributions on $\mathbb{R}_+$ yield the *Poisson* distributions.

Next we return to the basic formula (7.3) for general $\alpha > -1$ and use it to show that the equivalence in (7.5) can be generalized as follows; for a generalization of (7.6) see Notes.

Theorem 7.5. *Let $(p_k)_{k\in\mathbb{Z}_+}$ be an infinitely divisible distribution on $\mathbb{Z}_+$ with shifted canonical sequence $(\overline{r}_k)_{k\in\mathbb{N}}$ as defined above, and let $\alpha > -1$. Then the $(\alpha+1)$-st order moment $\mu_{\alpha+1}$ of (p_k) is finite iff the α-th order moment ν_α of $(\overline{r}_k)$ is finite:*

$$(7.11)\qquad \mu_{\alpha+1} < \infty \iff \nu_\alpha < \infty.$$

Proof. Apply (7.3). Then one readily sees that

$$(7.12)\qquad \mu_{\alpha+1} \geq p_0\,\nu_\alpha \quad (\text{and } \mu_{\alpha+1} \geq \nu_\alpha \text{ when } \alpha \geq 0).$$

Since $p_0 > 0$, the implication to the right in (7.11) now immediately follows. The converse is easily proved when $\alpha \leq 0$, since then $(k+\ell)^\alpha \leq \ell^\alpha$ for all k and ℓ, and hence

$$(7.13)\qquad \mu_{\alpha+1} \leq \nu_\alpha \qquad [-1 < \alpha \leq 0].$$

Note that $\mu_1 = \nu_0$. Finally we let $\alpha > 0$. Then we have the elementary inequality $(k+\ell)^\alpha \leq 2^\alpha\,(k^\alpha + \ell^\alpha)$ for all k and ℓ, so from (7.3) we conclude that

$$(7.14)\qquad \mu_{\alpha+1} \leq 2^\alpha\,(\mu_\alpha\,\nu_0 + \nu_\alpha) \qquad [\alpha > 0].$$

Now, suppose that $\nu_\alpha < \infty$, and let $\lfloor \alpha \rfloor = n \;(\in \mathbb{Z}_+)$. Then $\nu_n < \infty$ and hence, because of (7.5), $\mu_{n+1} < \infty$. But then we also have $\mu_\alpha < \infty$, and so by (7.14) $\mu_{\alpha+1} < \infty$. ◻◻

As is well known, the $(\alpha+1)$-st order moment $\mu_{\alpha+1}$ of a distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ is finite iff the α-th order moment of $(\overline{P}_k)_{k\in\mathbb{Z}_+}$ is finite, where $\overline{P}_k$ is the *tail probability* defined by $\overline{P}_k := \sum_{j=k+1}^{\infty} p_j$. Hence for an infinitely divisible (p_k) Theorem 7.5 gives information on the asymptotic behaviour of $\overline{P}_k$ as $k \to \infty$. In Section 9 we will return to this and give more detailed results. There we also consider the asymptotic behaviour of the probability p_k itself as $k \to \infty$; since p_k can be zero, we first pay attention to the *support* of (p_k).

8. Support

Let $p = (p_k)_{k \in \mathbb{Z}_+}$ be an infinitely divisible probability distribution on $\mathbb{Z}_+$. The *support* $S(p)$ of p, as defined in Section A.4, is given by

$$S(p) = \{k \in \mathbb{Z}_+ : p_k > 0\},$$

so $S(p)$ and the set of zeroes of p are complementary sets (with respect to $\mathbb{Z}_+$). We first derive some properties of $S(p)$ starting from the definition of infinite divisibility. A random variable X with distribution p can be written as

$$X \stackrel{\rm d}{=} X_{n,1} + \cdots + X_{n,n} \qquad [n \in \mathbb{N}], \tag{8.1}$$

where $X_{n,1}, \ldots, X_{n,n}$ are independent with $X_{n,j} \stackrel{\rm d}{=} X_n$, the n-th order factor of X. As usual we exclude the trivial case where X is degenerate at zero. Then the factors X_n are non-degenerate at zero as well, and since by convention (1.4) they are $\mathbb{Z}_+$-valued, there is a sequence $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$ such that $\mathbb{P}(X_n = k_n) > 0$ for all n. From (8.1) it follows that $\mathbb{P}(X = nk_n) > 0$ for all n, so $\{nk_n : n \in \mathbb{N}\} \subset S(p)$. Thus we have given a simpler proof of Proposition I.2.3 for distributions on $\mathbb{Z}_+$.

Proposition 8.1. *The support of a non-degenerate infinitely divisible distribution on $\mathbb{Z}_+$ is unbounded.*

Making better use of (8.1) we can say more. Suppose that $p_k > 0$ and $p_\ell > 0$ for some $k, \ell \in \mathbb{N}$, and apply the equality in (8.1) with $n := k + \ell$. Since the value k for X is obtained by taking k or fewer of the $X_{n,j}$ positive and the value ℓ by taking ℓ or fewer of the remaining $X_{n,j}$ positive, the value $k + \ell$ for X can also be obtained with positive probability. More precisely, since

$$p_k \le \binom{n}{k} \mathbb{P}(X_{n,1} + \cdots + X_{n,k} = k)\, \mathbb{P}(X_{n,k+1} = \cdots = X_{n,n} = 0),$$

we have $\mathbb{P}(X_{n,1} + \cdots + X_{n,k} = k) > 0$; as $\mathbb{P}(X_{n,k+1} + \cdots + X_{n,n} = \ell) > 0$ is proved similarly, it follows that

$$p_{k+\ell} \ge \mathbb{P}(X_{n,1} + \cdots + X_{n,k} = k)\, \mathbb{P}(X_{n,k+1} + \cdots + X_{n,n} = \ell) > 0.$$

Thus we have obtained the following result.

Theorem 8.2. *If $(p_k)_{k\in\mathbb{Z}_+}$ is an infinitely divisible distribution on $\mathbb{Z}_+$, then*

$$p_k > 0,\ p_\ell > 0 \implies p_{k+\ell} > 0 \qquad [\,k,\ell\in\mathbb{Z}_+\,],$$

i.e., the support of an infinitely divisible distribution on $\mathbb{Z}_+$ is closed under addition.

Corollary 8.3. *If $(p_k)_{k\in\mathbb{Z}_+}$ is an infinitely divisible distribution on $\mathbb{Z}_+$, then*

$$p_1 > 0 \implies p_k > 0 \ \textit{ for all }\ k\in\mathbb{Z}_+.$$

Theorem 8.2 implies, for instance, that for any $\mathbb{Z}_+$-valued random variable X and any $n\in\mathbb{N}$ with $n\geq 2$ the random variable X^n cannot be infinitely divisible; in fact, the set $\{k^n : k\in\mathbb{Z}_+\}$ has no subset S with $S\neq\emptyset$ and $\neq\{0\}$ that is closed under addition.

Theorem 8.2 and its corollary can also be obtained by using the recurrence relations of Section 4; according to Theorem 4.4 the infinite divisibility of a distribution $p = (p_k)_{k\in\mathbb{Z}_+}$ with $p_0 > 0$ is characterized by the nonnegativity of the quantities r_k determined by

$$(8.2) \qquad (n+1)\,p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \qquad [\,n\in\mathbb{Z}_+\,].$$

Since $r_0 = p_1/p_0$, it follows that an infinitely divisible distribution p satisfies

$$(8.3) \qquad (k+1)\,p_{k+1}\,p_0 \geq p_k\,p_1 \qquad [\,k\in\mathbb{Z}_+\,],$$

which for $k = 1$ reduces to the well-known inequality (4.11). Clearly, (8.3) immediately yields the result of the corollary above. Now (8.3) can be generalized such that also Theorem 8.2 is immediate.

Proposition 8.4. *If $(p_k)_{k\in\mathbb{Z}_+}$ is an infinitely divisible distribution on $\mathbb{Z}_+$, then*

$$(8.4) \qquad \binom{k+\ell}{\ell}\,p_{k+\ell}\,p_0 \geq p_k\,p_\ell \qquad [\,k,\ell\in\mathbb{Z}_+\,].$$

Proof. We use induction with respect to k. The case $k = 0$ is trivial. Suppose that for some $n\in\mathbb{Z}_+$ the inequality in (8.4) holds for $k = 0,\ldots,n$

and $\ell \in \mathbb{Z}_+$. Then applying (8.2) twice, we can estimate as follows:

$$\begin{aligned}
&\binom{n+1+\ell}{\ell} p_{n+1+\ell}\, p_0 = \\
&\quad = \frac{1}{n+1}\binom{n+\ell}{\ell}(n+\ell+1)\, p_{n+\ell+1}\, p_0 = \\
&\quad = \frac{1}{n+1}\binom{n+\ell}{\ell}\sum_{k=0}^{n+\ell} p_k\, r_{n+\ell-k}\, p_0 \geq \\
&\quad \geq \frac{1}{n+1}\sum_{k=0}^{n}\binom{k+\ell}{\ell} p_{k+\ell}\, p_0\, r_{n-k} \geq \\
&\quad \geq \frac{1}{n+1}\sum_{k=0}^{n} p_k\, p_\ell\, r_{n-k} = p_{n+1} p_\ell,
\end{aligned}$$

i.e., the inequality in (8.4) also holds for $k = n+1$ and $\ell \in \mathbb{Z}_+$. □

Note that (8.4) is equivalent to saying that the sequence $(a_k)_{k\in\mathbb{Z}_+}$ with $a_k := k!\, p_k/p_0$ is super-multiplicative in the sense that $a_{k+\ell} \geq a_k\, a_\ell$ for all k and ℓ. It follows that the Poisson (λ) distribution (p_k) satisfies (8.4) with equality: $a_k = \lambda^k$ for all k.

Let X be an infinitely divisible random variable with distribution $p = (p_k)_{k\in\mathbb{Z}_+}$. Then $S(p)\setminus\{0\}$ has a minimal element m, the smallest value in $\mathbb{N}$ that X can take with positive probability. One might think that $S(p)$ only contains multiples of m, so by Theorem 8.2 that $S(p) = m\,\mathbb{Z}_+$, but this is not so. In fact, if X is of the form

$$X \stackrel{d}{=} 2Y + 3Z, \tag{8.5}$$

where Y and Z are independent, infinitely divisible random variables with $\mathbb{P}(Y=1) > 0$ and $\mathbb{P}(Z=1) > 0$, then X is infinitely divisible with $p_1 = 0$, but $p_k > 0$ for all $k \geq 2$. Rather than the minimal element of $S(p)\setminus\{0\}$ one should consider its greatest common divisor, which is called the *period* $d = d(p)$ of the sequence p:

$$d(p) := \mathrm{GCD}\big(S(p)\setminus\{0\}\big) = \mathrm{GCD}\big(\{k \in \mathbb{N} : p_k > 0\}\big);$$

if $d = 1$, then p is said to be *aperiodic*. Of course, one has $p_k = 0$ for k not a multiple of d, so $S(p) \subset d\,\mathbb{Z}_+$; as the example in (8.5) shows, we need not have equality here. Theorem 8.2 yields, however, the following result.

Proposition 8.5. *If $(p_k)_{k\in\mathbb{Z}_+}$ is an infinitely divisible distribution on $\mathbb{Z}_+$ with period d, then*

$$\exists\, n_0 \in \mathbb{N}\ \forall\, n \geq n_0 : p_{nd} > 0,$$

so the support of an infinitely divisible distribution on $\mathbb{Z}_+$ contains all sufficiently large multiples of its period.

PROOF. The support $S(p)$ of p is closed under addition. Now apply the following well-known result in number theory: A non-empty subset of $\mathbb{N}$ that is closed under addition contains all sufficiently large multiples of its GCD. □

From this property it follows that an aperiodic infinitely divisible distribution has the nice property that its probabilities are eventually non-zero. For many purposes we can confine ourselves to such distributions; if X is an infinitely divisible random variable of period $d \geq 2$, then the random variable X/d is aperiodic and still infinitely divisible.

The support and period of an infinitely divisible distribution p can be found from the canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ of p, or rather the shifted version of it as used in the preceding section, i.e., $\overline{r} = (\overline{r}_k)_{k\in\mathbb{N}}$ with $\overline{r}_k := r_{k-1}$. To show this it is most convenient to use Theorem 3.7: An infinitely divisible $\mathbb{Z}_+$-valued random variable X can be represented as

$$X \stackrel{\mathrm{d}}{=} \sum_{j=1}^{\infty} j V_j, \tag{8.6}$$

where $V_1, V_2, \ldots$ are independent Poisson variables, possibly degenerate at zero, with parameters $\lambda_1, \lambda_2, \ldots$ for which by (4.2)

$$\lambda_j = \overline{r}_j / j \geq 0 \ \text{ for } \ j \in \mathbb{N}, \quad \sum_{j=1}^{\infty} \lambda_j < \infty. \tag{8.7}$$

Of course, the support $S(p)$ of the distribution p of X strongly depends on what terms in the right-hand side of (8.6) are non-zero, i.e., for what $j \in \mathbb{N}$ the parameter λ_j is positive. In view of (8.7) we therefore consider the *support* $S(\overline{r})$ of $\overline{r}$:

$$S(\overline{r}) = \{k \in \mathbb{N} : \overline{r}_k > 0\};$$

by $\operatorname{sg}\big\{S(\overline{r})\big\}$ we denote the additive semigroup generated by $S(\overline{r})$.

Theorem 8.6. *Let $p = (p_k)_{k \in \mathbb{Z}_+}$ be an infinitely divisible distribution on $\mathbb{Z}_+$ with shifted canonical sequence $\overline{r} = (\overline{r}_k)_{k \in \mathbb{N}}$. Then:*

(i) $S(p) = \{0\} \cup \mathrm{sg}\,\{S(\overline{r})\}$.

(ii) *The period of p is given by $GCD\big(S(\overline{r})\big)$.*

PROOF. Part (ii) is an immediate consequence of part (i). To prove (i) we use representation (8.6) for X with distribution p, and first take a non-zero element k in $S(p)$. Then $p_k > 0$, where by (8.7)

$$p_k = \mathbb{P}\Big(\sum_{j=1}^{\infty} jV_j = k\Big) = \mathbb{P}\Big(\sum_{j \in S(\overline{r})} jV_j = k\Big).$$

Hence there exist nonnegative integers k_j with $j \in S(\overline{r})$ such that $k = \sum_{j \in S(\overline{r})} jk_j$, so $k \in \mathrm{sg}\,\{S(\overline{r})\}$. Thus we have proved $S(p) \backslash \{0\} \subset \mathrm{sg}\,\{S(\overline{r})\}$. On the other hand, if $k \in S(\overline{r})$, then by (8.7) the random variable V_k in (8.6) is non-degenerate Poisson, from which it follows that p_k is positive:

$$p_k = \mathbb{P}\Big(\sum_{j=1}^{\infty} jV_j = k\Big) \geq$$
$$\geq \mathbb{P}(V_k = 1)\,\mathbb{P}(V_j = 0 \text{ for } j \neq k) \geq \mathbb{P}(V_k = 1)\,p_0.$$

Thus $S(\overline{r}) \subset S(p)$, and hence by Theorem 8.2 also $\mathrm{sg}\,\{S(\overline{r})\} \subset S(p)$. $\square$

As noted in Section 4, any sequence $(\overline{r}_k)_{k \in \mathbb{N}}$ satisfying (8.7) determines an infinitely divisible distribution (p_k) via the recurrence relations (8.2). From Theorem 8.6 it follows that the possible supports of infinitely divisible distributions on $\mathbb{Z}_+$ are precisely the additive semigroups of positive integers supplemented with 0.

9. Tail behaviour

The necessary conditions for infinite divisibility developed in this section will provide simple tools for deciding that a given distribution on $\mathbb{Z}_+$ is *not* infinitely divisible. Let $p = (p_k)_{k \in \mathbb{Z}_+}$ be an infinitely divisible distribution on $\mathbb{Z}_+$. In Section 8 we essentially determined for what $k \in \mathbb{Z}_+$ the probability p_k is non-zero; we now want to decide how small such a p_k can be for large k. We do so in a rather crude way, and restrict ourselves to the asymptotic behaviour of $-\log p_k$ for $k \to \infty$; here (p_k) is supposed to be *aperiodic* so that the p_k are eventually non-zero (see Proposition 8.5). Of course, we need no such assumption when considering $-\log \overline{P}_k$ for $k \to \infty$, where $\overline{P}_k$ is the (global) *tail probability* defined by $\overline{P}_k := \sum_{j=k+1}^{\infty} p_j$ for $k \in \mathbb{Z}_+$.

In Proposition 8.1 we saw that the support $S(p)$ of an infinitely divisible distribution p is unbounded; together with Proposition 2.2 this seems to suggest that p cannot have an arbitrarily thin tail. Moreover, the remark following Proposition 8.4 suggests that the boundary case is given by the Poisson distribution. We therefore start with looking at this distribution.

Lemma 9.1. *The Poisson distribution* (p_k) *has the following properties:*

(i) $p_k \le 1/e$ *for all* $k \in \mathbb{N}$ *(and all values of the parameter).*

(ii) $-\log p_k \sim k \log k$ *as* $k \to \infty$.

(iii) $-\log \overline{P}_k \sim k \log k$ *as* $k \to \infty$.

PROOF. One easily verifies that for all values of the parameter λ:

$$p_k \le \frac{k^k}{k!}\, \mathrm{e}^{-k} =: \pi_k \qquad [\, k \in \mathbb{N}\,].$$

Since $(1 + 1/k)^k \uparrow \mathrm{e}$ as $k \to \infty$, the sequence $(\pi_k)_{k \in \mathbb{N}}$ is nonincreasing. It follows that $p_k \le \pi_1 = 1/\mathrm{e}$ for all $k \in \mathbb{N}$. The assertion in (ii) is immediately obtained from the fact that by Stirling's formula $\log k! \sim k \log k$ as $k \to \infty$. Because of the inequalities

$$p_{k+1} \le \overline{P}_k \le \mathrm{e}^{\lambda}\, p_{k+1} \qquad [\, k \in \mathbb{Z}_+\,],$$

which are easily verified, part (iii) is a consequence of (ii). ◻

The inequality in part (i) of the lemma turns out to hold for *all* infinitely divisible distributions (p_k) on $\mathbb{Z}_+$. To see this we use Theorem 3.2; an infinitely divisible random variable X is compound-Poisson, and can hence be written as $X \overset{\mathrm{d}}{=} S_N$, where $(S_n)_{n \in \mathbb{Z}_+}$ is an sii-process generated by an $\mathbb{N}$-valued random variable Y, say (so $S_1 \overset{\mathrm{d}}{=} Y$), and N is Poisson distributed and independent of (S_n). It follows that the distribution (p_k) of X satisfies

$$(9.1) \qquad p_0 = \mathbb{P}(N = 0), \;\; p_k = \sum_{n=1}^{k} \mathbb{P}(N = n)\, \mathbb{P}(S_n = k) \;\text{ for } k \in \mathbb{N},$$

and hence by Lemma 9.1 (i) for $k \in \mathbb{N}$

$$p_k \le (1/\mathrm{e}) \sum_{n=1}^{k} \mathbb{P}(S_n = k) = (1/\mathrm{e})\, \mathbb{P}\Big(\bigcup_{n=1}^{k} \{S_n = k\} \Big) \le 1/\mathrm{e}.$$

Proposition 9.2. *If $(p_k)_{k \in \mathbb{Z}_+}$ is an infinitely divisible distribution on $\mathbb{Z}_+$, then $p_k \le 1/e$ for all $k \in \mathbb{N}$.*

From (9.1) one can also get some information on the tails of an infinitely divisible distribution (p_k). On the one hand we have

$$p_k \ge \mathbb{P}(N = 1)\, \mathbb{P}(Y = k) \qquad [\, k \in \mathbb{N}\,], \tag{9.2}$$

from which it follows that an infinitely divisible distribution can have an *arbitrarily thick tail.* On the other hand one sees that

$$p_k \ge \mathbb{P}(N = k)\big\{\mathbb{P}(Y = 1)\big\}^k \ \text{ or } \ p_k \ge p_0 \frac{(p_1/p_0)^k}{k!} \qquad [\, k \in \mathbb{N}\,], \tag{9.3}$$

which also follows by iterating the inequality of (8.3). We conclude that the tails of an infinitely divisible distribution (p_k) satisfying $p_1 > 0$ (and hence $p_k > 0$ for all k) *cannot be thinner than Poissonian tails*, and so by Lemma 9.1 (ii)

$$\limsup_{k \to \infty} \frac{-\log p_k}{k \log k} \le 1. \tag{9.4}$$

We can say more. To this end we again use the representation of an infinitely divisible $\mathbb{Z}_+$-valued random variable X as given by Theorem 3.7 or by (8.6):

$$X \overset{\mathrm{d}}{=} \sum_{j=1}^{\infty} j V_j, \tag{9.5}$$

where $V_1, V_2, \ldots$ are independent Poisson variables, possibly degenerate at zero. As noted just after (8.7), the set of those $j \in \mathbb{N}$ for which V_j is non-degenerate, is equal to the support $S(\overline{r})$ of the shifted canonical sequence $\overline{r} = (\overline{r}_k)_{k \in \mathbb{N}}$ of X. First we observe that the distribution $p = (p_k)$ of X satisfies

$$p_k \ge \mathbb{P}(V_1 = k)\, \mathbb{P}(V_j = 0 \text{ for } j \ge 2) \ge p_0 \frac{\overline{r}_1^{\,k}}{k!} \mathrm{e}^{-\overline{r}_1} \qquad [\, k \in \mathbb{N}\,], \tag{9.6}$$

where we used (8.7). Since $\overline{r}_1 = r_0 = p_1/p_0$, we have obtained an inequality that is slightly weaker than (9.3) but nevertheless yields (9.4) in case $1 \in S(\overline{r})$. Now $S(\overline{r})$ contains no elements but 1 iff X has a Poisson distribution, in which case (9.4) holds with equality and with 'lim sup' replaced by 'lim'. For non-Poissonian distributions with $p_1 > 0$ we have $S(\overline{r}) \supset \{1, m\}$ for some $m \ne 1$, so V_1 and V_m are non-degenerate. Of course, any $k \in \mathbb{N}$ can be written as $k = n_k m + \ell_k$ with $n_k = \lfloor k/m \rfloor$, the integer part of k/m, and $0 \le \ell_k < m$. Therefore, similarly to (9.6) p_k can be estimated as follows:

$$p_k \ge p_0\, \mathbb{P}(V_1 + mV_m = k) \ge p_0\, \mathbb{P}(V_1 = \ell_k)\, \mathbb{P}(V_m = n_k). \tag{9.7}$$

Since by Lemma 9.1 (ii) $-\log \mathbb{P}(V_m = n_k) \sim (k \log k)/m$ and the numbers $\mathbb{P}(V_1 = \ell_k)$ are bounded away from zero, we conclude that any infinitely divisible distribution $p = (p_k)_{k \in \mathbb{Z}_+}$ with $p_1 > 0$ satisfies

$$\limsup_{k\to\infty} \frac{-\log p_k}{k \log k} \le \frac{1}{m} \tag{9.8}$$

for all $m \in S(\overline{r})$, and hence for $m = m_{\overline{r}} := \sup S(\overline{r})$. Here, of course, we agree that $1/m_{\overline{r}} := 0$ if $m_{\overline{r}} = \infty$. When $m_{\overline{r}}$ is finite, p_k can also be estimated as follows:

$$p_k = \mathbb{P}\Big(\sum_{j=1}^{m_{\overline{r}}} jV_j = k\Big) \le \mathbb{P}\Big(\sum_{j=1}^{m_{\overline{r}}} V_j \ge \lfloor k/m_{\overline{r}} \rfloor\Big). \tag{9.9}$$

As $\sum_{j=1}^{m_{\overline{r}}} V_j$ is a Poisson variable, we can apply Lemma 9.1 (iii) to conclude that

$$\liminf_{k\to\infty} \frac{-\log p_k}{k \log k} \ge \frac{1}{m_{\overline{r}}}. \tag{9.10}$$

Combining (9.8) and (9.10) shows that we have proved the following precise result for the special case $p_1 > 0$.

Theorem 9.3. *Let $p = (p_k)_{k \in \mathbb{Z}_+}$ be an aperiodic, infinitely divisible distribution on $\mathbb{Z}_+$ with shifted canonical sequence $\overline{r} = (\overline{r}_k)_{k \in \mathbb{N}}$. Then*

$$\lim_{k\to\infty} \frac{-\log p_k}{k \log k} = \frac{1}{m_{\overline{r}}}, \tag{9.11}$$

where $m_{\overline{r}} := \sup \{k \in \mathbb{N} : \overline{r}_k > 0\}$, possibly ∞, in which case $1/m_{\overline{r}} := 0$.

PROOF. Let p be aperiodic with $p_1 = 0$. Inspecting the proof above for the case $p_1 > 0$, one sees that it is sufficient to prove (9.8) for all $m \in S(\overline{r})$. To do so, we again use representation (9.5) for a random variable X with distribution p. Let $m \in S(\overline{r})$; then $m \ne 1$ because $\overline{r}_1 = r_0 = p_1/p_0 = 0$. Take $k_0 \in \mathbb{N}$ such that $p_k > 0$ for all $k \ge k_0$; such a k_0 exists on account of Proposition 8.5. In order to get an appropriate lowerbound for p_k with $k \ge k_0$, we consider the integer part $n_k := \lfloor (k - k_0)/m \rfloor$ of $(k - k_0)/m$ (rather than that of k/m) and write $k = n_k m + \ell_k$, where the remainder term ℓ_k now satisfies $k_0 \le \ell_k < k_0 + m$. Since $\ell_k \ge k_0$, the random variable $\sum_{j=1}^{\infty} jV_j$ takes the value ℓ_k with positive probability; hence there is a sequence $(v_j(k))_{j\in\mathbb{N}}$ in $\mathbb{Z}_+$ with $v_j(k) = 0$ for all $j \notin S(\overline{r})$ such that $\sum_{j=1}^{\infty} jv_j(k) = \ell_k$ and $\mathbb{P}\big(V_j = v_j(k) \text{ for all } j\big) > 0$. Note that $S(\overline{r})$, apart from m, contains another element $\neq 1$; cf. Theorem 8.6 (ii). It follows that the values $k \ge k_0$ can be written as

$$k = \{n_k + v_m(k)\}m + \sum_{j\neq m} jv_j(k),$$

and hence for $k \ge k_0$

$$(9.12) \qquad p_k \ge \mathbb{P}\big(V_m = n_k + v_m(k)\big)\, \mathbb{P}\big(V_j = v_j(k) \text{ for all } j \neq m\big).$$

Now, since $\ell_k < k_0 + m$, the sequence $(v_j(k))_{j\in\mathbb{N}}$ has finitely many non-zero terms and hence is bounded uniformly in k. Therefore, the first factor $p_k^{(1)}$ in the right-hand side of (9.12) satisfies $-\log p_k^{(1)} \sim (k \log k)/m$, whereas for the second factor $p_k^{(2)}$ we have $-\log p_k^{(2)} = o(k \log k)$ as $k \to \infty$. This concludes the proof of (9.8) and hence of the theorem. ◻

We next consider the global tail probability $\overline{P}_k = \sum_{j=k+1}^{\infty} p_j$ of an infinitely divisible distribution $p = (p_k)_{k\in\mathbb{Z}_+}$ for $k \to \infty$; as noted above, it is now not necessary to require aperiodicity of p. The asymptotic behaviour of $-\log \overline{P}_k$ can be found in the same way as that of $-\log p_k$ in case $p_1 > 0$. In stead of (9.7) we have for any $m \in S(\overline{r})$ and with $n_k = \lfloor k/m \rfloor$

$$(9.13) \qquad \overline{P}_k \ge \mathbb{P}(mV_m > k) \ge \mathbb{P}(V_m > n_k + 1) \qquad [k \in \mathbb{N}],$$

so from Lemma 9.1 (iii) we see that (9.8) holds with p_k replaced by $\overline{P}_k$. On the other hand, when $m_{\overline{r}}$ is finite, we can proceed as for (9.9) and find

$$(9.14) \qquad \overline{P}_k \le \mathbb{P}\Big(\sum_{j=1}^{m_{\overline{r}}} V_j > \lfloor k/m_{\overline{r}} \rfloor\Big),$$

so also (9.10) holds with p_k replaced by $\overline{P}_k$. We summarize.

Theorem 9.4. *Let $p = (p_k)_{k\in\mathbb{Z}_+}$ be an infinitely divisible distribution on $\mathbb{Z}_+$ with shifted canonical sequence $\overline{r} = (\overline{r}_k)_{k\in\mathbb{N}}$. Then the tail probability $\overline{P}_k := \sum_{j=k+1}^{\infty} p_j$ satisfies*

$$(9.15) \qquad \lim_{k\to\infty} \frac{-\log \overline{P}_k}{k \log k} = \frac{1}{m_{\overline{r}}},$$

where $m_{\overline{r}} := \sup\{k \in \mathbb{N} : \overline{r}_k > 0\}$, possibly ∞, in which case $1/m_{\overline{r}} := 0$.

When the quantity $m_{\overline{r}}$ corresponding to an infinitely divisible distribution p is finite, Theorems 9.3 and 9.4 give the exact rate with which the quantities $-\log p_k$ and $-\log \overline{P}_k$ tend to infinity. The special case where $m_{\overline{r}} = 1$, yields, together with Lemma 9.1, the following *characterization of the Poisson distribution.*

Corollary 9.5. *An infinitely divisible distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ is Poisson iff it satisfies one, and then both, of the following conditions:*

$$(9.16)\qquad \limsup_{k\to\infty} \frac{-\log p_k}{k\log k} \geq 1, \quad \limsup_{k\to\infty} \frac{-\log \overline{P}_k}{k\log k} \geq 1,$$

where for the first condition (p_k) needs to be aperiodic.

When $m_{\overline{r}} = \infty$, i.e., when $r_k > 0$ infinitely often, we only know that

$$(9.17)\qquad \lim_{k\to\infty} \frac{-\log p_k}{k\log k} = 0, \quad \lim_{k\to\infty} \frac{-\log \overline{P}_k}{k\log k} = 0.$$

In general not more than this can be said; the convergence in (9.17) can be arbitrarily slow. For the *compound-exponential* distributions, which by Proposition 5.7 satisfy (9.17), it can be shown, however, that $(-\log p_k)/k$ and $(-\log \overline{P}_k)/k$ have finite limits as $k \to \infty$; we shall not do this.

10. Log-convexity

It is important to have easily verifiable sufficient conditions for infinite divisibility in terms of probability distributions (rather than in terms of pgf's). Such conditions will now be derived from Theorem 4.4, which states that infinite divisibility of a distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ with $p_0 > 0$ is characterized by the nonnegativity of the (canonical) quantities r_k determined by

$$(10.1)\qquad (n+1)\, p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \qquad [\, n\in\mathbb{Z}_+\,].$$

Here $r_0 = p_1/p_0$ is always nonnegative. As we did in (4.11), the condition $r_1 \geq 0$ can be written in the form $p_1^2 \leq 2p_0p_2$; this is the first condition in

$$(10.2)\qquad p_k^2 \leq 2\, p_{k-1}\, p_{k+1} \qquad [\, k\in\mathbb{N}\,],$$

and one might wonder whether distributions (p_k) satisfying (10.2) are infinitely divisible. Unfortunately, this is not the case; for a counter-example we refer to Section 11. When we leave out, however, the factor 2 in (10.2), then we do get a sufficient condition for infinite divisibility; see Theorem 10.1 below. The distribution (p_k) then satisfies

$$p_k^2 \le p_{k-1}\, p_{k+1} \qquad [\, k \in \mathbb{N}\,], \tag{10.3}$$

i.e., (p_k) is *log-convex*; see Section A.4. Note that a non-degenerate distribution (p_k) is log-convex iff $p_k > 0$ for all k and $(p_k/p_{k-1})_{k\in\mathbb{N}}$ is a non-decreasing sequence; the last property can be rephrased as

$$p_k\, p_n \le p_{k-1}\, p_{n+1} \qquad [\, k, n \in \mathbb{N} \text{ with } k \le n\,]. \tag{10.4}$$

Theorem 10.1. *If $(p_k)_{k\in\mathbb{Z}_+}$ is a log-convex distribution on $\mathbb{Z}_+$, then in (10.1) one has $r_k \ge 0$ for all k, and hence (p_k) is infinitely divisible.*

PROOF. Let (p_k) be log-convex. Applying (10.1) twice, for $n \in \mathbb{N}$ we can write the quantity $(n+1)\, p_n p_{n+1}$ in two different ways:

$$(n+1)\, p_n p_{n+1} = \begin{cases} \displaystyle\sum_{k=0}^{n} p_n p_k r_{n-k} = p_n p_0 r_n + \sum_{k=1}^{n} p_n p_k r_{n-k}, \\ \displaystyle p_n p_{n+1} + \sum_{k=1}^{n} p_{n+1} p_{k-1} r_{n-k}, \end{cases}$$

and hence

$$p_0 p_n r_n = p_n p_{n+1} + \sum_{k=1}^{n} (p_{k-1} p_{n+1} - p_k p_n)\, r_{n-k} \qquad [\, n \in \mathbb{N}\,]. \tag{10.5}$$

From this and (10.4) it follows by induction that $r_n \ge 0$ for all n. ◻

Next, we give a sufficient condition in terms of the canonical sequence (r_k) for an infinitely divisible distribution (p_k) to be log-convex. The proof of the following theorem is similar to that of Theorem 10.1; it uses induction and a relation like (10.5). Though easily verified, this relation is hard to discover among the many possible relations that can be derived from (10.1). Since $p_0 r_0 = p_1$ and $p_0^2 r_1 = 2p_0 p_2 - p_1^2$, for the first inequality $p_1^2 \le p_0 p_2$ in the log-convexity condition for (p_k) we have

$$p_1^2 \le p_0\, p_2 \iff r_1 \ge r_0^2. \tag{10.6}$$

We conclude that $r_1 \ge r_0^2$ is a necessary condition for (p_k) to be log-convex.

Theorem 10.2. *Let (p_k) be infinitely divisible, i.e., satisfy (10.1) with $r_k \ge 0$ for all k. If (r_k) is log-convex, then (p_k) is log-convex iff $r_1 \ge r_0^2$.*

Proof. Let (r_k) be log-convex with $r_1 \geq r_0^2$. If $r_0 = 0$, then $r_k = 0$ for all k and (p_k) is degenerate. So let $r_0 > 0$; then $r_1 > 0$, and hence $r_k > 0$ for all k. From (10.6) it follows that $p_1^2 \leq p_0 p_2$. To prove the other inequalities in the log-convexity condition for (p_k) we use the following relation which holds for $n \in \mathbb{N}$:

$$(10.7)\qquad \begin{aligned}(n+1)\, r_n (p_n p_{n+2} - p_{n+1}^2) &= p_n (p_{n+1} r_{n+1} - p_{n+2} r_n) + \\ &+ \sum_{k=1}^{n} (p_{n+1} p_{k-1} - p_n p_k)(r_{n+1} r_{n-k} - r_n r_{n-k+1});\end{aligned}$$

this can be verified by splitting the sum over k in the right-hand side into four sums and then using (10.1) for each of them. Now, since $r_0 r_{n+1} \geq r_1 r_n$ and $r_1 \geq r_0^2$, we have $r_{n+1} \geq r_0 r_n$. It follows that

$$\begin{aligned}(n+2)\, p_{n+2} r_n = p_{n+1} r_0 r_n + \sum_{k=0}^{n} p_k r_{n+1-k} r_n &\leq \\ \leq p_{n+1} r_{n+1} + \sum_{k=0}^{n} p_k r_{n-k} r_{n+1} &= (n+2)\, p_{n+1} r_{n+1},\end{aligned}$$

and hence

$$(10.8)\qquad p_{n+1} r_{n+1} \geq p_{n+2} r_n \qquad [\, n \in \mathbb{N}\,].$$

Induction now completes the proof. □□

We illustrate this result by returning to Examples 4.6 and 4.7. The Poisson distribution has a log-convex canonical sequence, but is not log-convex. The negative-binomial distribution also has a log-convex canonical sequence, and is log-convex iff its shape parameter r satisfies $r \leq 1$; cf. Example 5.5.

One easily verifies that the Poisson distribution and the negative-binomial distribution with shape parameter $r \geq 1$ are log-concave; here a distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ is said to be *log-concave* if

$$(10.9)\qquad p_k^2 \geq p_{k-1}\, p_{k+1} \qquad [\, k \in \mathbb{N}\,].$$

Though many infinitely divisible distributions are log-concave, *not* every log-concave distribution is infinitely divisible. For instance, any distribution with $p_1 > 0$ and $p_k = 0$ for $k \geq 2$ is log-concave. For a more interesting counter-example we refer to Section 11. Nevertheless, there is an analogue of Theorem 10.2.

Theorem 10.3. *Let (p_k) be infinitely divisible, i.e., satisfy (10.1) with $r_k \geq 0$ for all k. If (r_k) is log-concave, then (p_k) is log-concave iff $r_1 \leq r_0^2$.*

PROOF. Use induction and the following relation, in which $p_{-1} := 0$ and $n \in \mathbb{N}$:

$$\begin{aligned} 2n(n+2)(p_n p_{n+2} - p_{n+1}^2) &= 2p_{n+1}(p_{n+1} - p_n r_0) + \\ &- \sum_{k=0}^{n} \sum_{\ell=0}^{n} (p_{k-1} p_\ell - p_k p_{\ell-1})(r_{n-k+1} r_{n-\ell} - r_{n-k} r_{n-\ell+1}); \end{aligned} \tag{10.10}$$

this can be verified in a similar way as (10.7). Note that, under the induction hypothesis, the two factors in the summand of the double sum are both nonnegative if $\ell \leq k$ and both nonpositive if $\ell \geq k$. ◻◻

An important subclass of the class of log-convex distributions is given by the completely monotone distributions. Recall that a distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ is said to be *completely monotone* if

$$(-1)^n \Delta^n p_k \geq 0 \qquad [\, n \in \mathbb{Z}_+;\ k \in \mathbb{Z}_+ \,], \tag{10.11}$$

where $\Delta^0 p_k := p_k$, $\Delta p_k := p_{k+1} - p_k$ and $\Delta^n := \Delta \circ \Delta^{n-1}$ for $n \in \mathbb{N}$. By Hausdorff's theorem (Theorem A.4.5) (p_k) is completely monotone iff there exists a (finite) measure ν on $[\,0,1\,]$ such that

$$p_k = \int_{[0,1]} x^k \, \nu(\mathrm{d}x) \qquad [\, k \in \mathbb{Z}_+ \,]. \tag{10.12}$$

Combining the Hausdorff representation and Schwarz's inequality shows that a completely monotone sequence is log-convex; see Proposition A.4.7. Together with Theorem 10.1 this leads to the following result.

Theorem 10.4. *If $(p_k)_{k \in \mathbb{Z}_+}$ is a completely monotone distribution on $\mathbb{Z}_+$, then it is log-convex, and hence infinitely divisible.*

The completely monotone distributions can be identified within the class of all infinitely divisible distributions in the following way.

Theorem 10.5. *A distribution (p_k) on $\mathbb{Z}_+$ is completely monotone iff it is infinitely divisible having a canonical sequence (r_k) such that $\big(r_k/(k+1)\big)$ is completely monotone with Hausdorff representation of the form*

$$\frac{1}{k+1}\, r_k = \int_0^1 x^k \, w(x)\, dx \qquad [\, k \in \mathbb{Z}_+ \,], \tag{10.13}$$

where w is a measurable function on $(0,1)$ satisfying $0 \le w \le 1$ and necessarily

$$\int_0^1 \frac{1}{1-x}\, w(x)\, dx < \infty. \tag{10.14}$$

In this theorem, which we do not prove here, the extra condition $0 \le w \le 1$ on the Hausdorff density w looks rather curious; we shall return to it in Chapter VI in the context of mixtures of geometric distributions, where it emerges rather naturally. The theorem can be used, for instance, to easily show that the (log-convex) *negative-binomial* distribution with shape parameter $r \le 1$ is completely monotone; cf. Examples 4.7 and 11.7.

We return to log-convexity, and treat some *closure properties*. Clearly, log-convexity is preserved under *pointwise multiplication*: If (p_k) and (q_k) are log-convex distributions, then so is $(c\, p_k q_k)$, where $c > 0$ is a norming constant. The following consequence of Proposition A.4.9 is less trivial: If (p_k) and (q_k) are log-convex distributions, then so is $\big(\alpha\, p_k + (1-\alpha)\, q_k\big)$ for every $\alpha \in [\,0,1\,]$. This property extends to general *mixtures*, because log-convexity is defined in terms of weak inequalities and hence is preserved under taking limits.

Proposition 10.6. *The class of log-convex distributions on $\mathbb{Z}_+$ is closed under mixing: If, for every θ in the support Θ of a distribution function G, $\big(p_k(\theta)\big)$ is a log-convex distribution on $\mathbb{Z}_+$, then so is (p_k) given by*

$$p_k = \int_\Theta p_k(\theta)\, dG(\theta) \qquad [\,k \in \mathbb{Z}_+\,]. \tag{10.15}$$

We use this result to prove the infinite divisibility of a large class of distributions that occur in renewal theory.

Proposition 10.7. *Let $(q_k)_{k\in\mathbb{Z}_+}$ be a log-convex distribution on $\mathbb{Z}_+$ having a finite first moment μ. Then the sequence $(p_k)_{k\in\mathbb{Z}_+}$ defined by*

$$p_k = \frac{1}{\mu} \sum_{j=k+1}^{\infty} q_j \qquad [\,k \in \mathbb{Z}_+\,],$$

is a probability distribution on $\mathbb{Z}_+$ that is log-convex and hence infinitely divisible.

PROOF. From (A.4.2) it follows that (p_k) is a probability distribution on $\mathbb{Z}_+$. Since it can be written as

$$p_k = \frac{1}{\mu} \sum_{\ell=1}^{\infty} q_{\ell+k} \qquad [\, k \in \mathbb{Z}_+ \,],$$

it is seen that (p_k) is a mixture of log-convex distributions on $\mathbb{Z}_+$, and hence is log-convex, by Proposition 10.6. ◻

Proposition 10.6 can also be used to show that certain log-convex distributions on $\mathbb{Z}_+$ can be obtained from probability densities f on $(0, \infty)$ that are *log-convex*, i.e. (cf. Section A.3), satisfy the inequalities

$$\big\{ f\big(\tfrac{1}{2}(x+y)\big) \big\}^2 \le f(x)\, f(y) \qquad [\, x > 0,\, y > 0 \,].$$

Proposition 10.8. *Let Y be an $\mathbb{R}_+$-valued random variable having a density f that is log-convex on $(0, \infty)$, and consider X such that*

$$X \stackrel{\mathrm{d}}{=} \lfloor Y \rfloor,$$

the integer part of Y. Then X is $\mathbb{Z}_+$-valued having a distribution $(p_k)_{k \in \mathbb{Z}_+}$ that is log-convex and hence infinitely divisible.

PROOF. The distribution (p_k) of X can be written as

$$p_k = \mathbb{P}\big(Y \in [\, k, k+1)\big) = \int_k^{k+1} f(x)\, \mathrm{d}x = \int_0^1 f(k+\theta)\, \mathrm{d}\theta.$$

Now one easily verifies that for every $\theta \in (0,1)$ the sequence $\big(f(k+\theta)\big)$ is log-convex, and hence can be normalized to a log-convex distribution. It follows that (p_k) is a mixture of log-convex distributions, and hence is log-convex, by Proposition 10.6. ◻

Of course, there is a simpler way to obtain a log-convex distribution on $\mathbb{Z}_+$ from a log-convex density on $(0, \infty)$.

Proposition 10.9. *Let f be a log-convex probability density on $(0, \infty)$. Then there exists $c > 0$ such that $(p_k)_{k \in \mathbb{Z}_+}$ defined by*

$$p_k = c\, f(k+1) \qquad [\, k \in \mathbb{Z}_+ \,],$$

is a probability distribution on $\mathbb{Z}_+$, and (p_k) is log-convex and hence infinitely divisible.

The last four propositions also hold with 'log-convexity' replaced by 'complete monotonicity'; this is easily verified by using Proposition A.4.6 and Bernstein's theorem as given by Theorem A.3.6.

11. Examples

In the preceding sections we have considered only two classes of examples, viz. the *negative-binomial* distributions (including *geometric*) and the *stable* distributions (including *Poisson*). These much used distributions are infinitely divisible, and the negative-binomial distribution with shape parameter $r \le 1$ is compound-exponential, log-convex, and even completely monotone. We now present some other distributions, and use the results obtained so far to determine whether they are infinitely divisible or not. In doing so we will frequently deal with the recurrence relations of Theorem 4.4:

$$(11.1) \qquad (n+1)\, p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \qquad [\, n \in \mathbb{Z}_+ \,].$$

Recall that infinite divisibility of a distribution (p_k) on $\mathbb{Z}_+$ is characterized by the nonnegativity of the (canonical) quantities r_k determined by (11.1).

We start with three examples, each of which is obtained from Poisson distributions in a simple way, but nevertheless is *not* infinitely divisible.

Example 11.1. Let Y be Poisson(λ) distributed, and consider X such that

$$X \stackrel{\rm d}{=} (Y - 1 \,|\, Y \ge 1).$$

Then the distribution $(p_k)_{k \in \mathbb{Z}_+}$ and the pgf P of X are given by

$$p_k = c_\lambda\, \frac{\lambda^{k+1}}{(k+1)!}, \quad P(z) = c_\lambda\, \frac{{\rm e}^{\lambda z} - 1}{z} \;\text{ for } z \in \mathbb{C},$$

where $c_\lambda := 1/({\rm e}^\lambda - 1)$. Since $P(z_0) = 0$ for $z_0 = 2\pi i/\lambda$, it follows from Theorem 2.8 that X is *not* infinitely divisible. Alternatively, one might use the results on tail behaviour of Section 9; by Lemma 9.1 (ii) one has $-\log p_k \sim k \log k$ as $k \to \infty$, so if X would be infinitely divisible, then by Corollary 9.5 X would be Poisson distributed, which it is not. □

Example 11.2. Let Y and Z be Poisson distributed with different parameters λ and μ, respectively, and consider X such that

$$X \stackrel{d}{=} (1-A)\, Y + A\, Z,$$

where A has a Bernoulli$\,(\alpha)$ distribution on $\{0,1\}$ and is independent of (Y,Z). The pgf P of X is then given by the following mixture of Poisson pgf's:

$$P(z) = (1-\alpha)\, \mathrm{e}^{-\lambda(1-z)} + \alpha\, \mathrm{e}^{-\mu(1-z)} \qquad [\, z \in \mathbb{C}\,].$$

As in Example 11.1, both Theorem 2.8 and the results of Section 9 show that X is *not* infinitely divisible. In fact, letting $z_0 := 1 + (\gamma + \pi i)/(\lambda - \mu)$ with $\gamma := \log\{\alpha/(1-\alpha)\}$, one easily verifies that $P(z_0) = 0$. Also, if $\lambda < \mu$, then for $k \in \mathbb{Z}_+$ we have $\mathbb{P}(Y > k) \le \mathbb{P}(Z > k)$ (cf. Section B.4), and hence

$$\mathbb{P}(X > k) = (1-\alpha)\, \mathbb{P}(Y > k) + \alpha\, \mathbb{P}(Z > k) \begin{cases} \ge \mathbb{P}(Y > k), \\ \le \mathbb{P}(Z > k); \end{cases}$$

now apply Lemma 9.1 (iii) and Corollary 9.5. Similarly one shows that *mixtures of finitely many* (different) *Poisson distributions are not infinitely divisible.* Mixtures of infinitely many Poisson distributions may be infinitely divisible; this will be clear from Proposition 3.5. □

Example 11.3. Let Y and Z be independent and Poisson distributed with the same parameter λ, and consider X such that

$$X \stackrel{d}{=} Y Z.$$

Suppose that X is infinitely divisible. Then from the recurrence relations (11.1) it follows that the canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ of the distribution $(p_k)_{k\in\mathbb{Z}_+}$ of X satisfies $r_0 = p_1/p_0 > 0$ and

$$p_0 r_n \le (n+1)\, p_{n+1} - r_0 p_n \qquad [\, n \in \mathbb{N}\,].$$

Now, take $n = 2m$ with $m > 2$ such that $n+1$ is a prime number. Then

$$p_{n+1} = 2\, \mathbb{P}(Y = 1)\, \mathbb{P}(Z = 2m+1) = 2\, \frac{\lambda^{2m+2}}{(2m+1)!}\, \mathrm{e}^{-2\lambda},$$

whereas p_n can be estimated below by

$$p_n \ge 2\, \mathbb{P}(Y = 2)\, \mathbb{P}(Z = m) = \frac{\lambda^{m+2}}{m!}\, \mathrm{e}^{-2\lambda}.$$

It follows that for all n as indicated

$$p_0 r_n \leq \Big(\frac{2\lambda^m}{(2m)(2m-1)\cdots(m+1)} - r_0\Big)\,\frac{\lambda^{m+2}}{m!}\,\mathrm{e}^{-2\lambda},$$

which is negative for large m; we have obtained a contradiction. We conclude that *the product of two independent random variables with the same Poisson distribution is not infinitely divisible.* ◫

We proceed with some examples of a positive character, and recall the fact that completely monotone and, more generally, log-convex distributions are infinitely divisible. Also some further results from Sections 10 and A.4 will be used, like the Hausdorff representation (10.12) for a completely monotone sequence.

Example 11.4. Let the distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ be given by

$$p_k = \frac{1}{(k+1)\,(k+2)}.$$

Since $1/(k{+}1) = \mathbb{E}\,U^k$ for all $k \in \mathbb{Z}_+$, where U has a uniform distribution on $(0,1)$, the sequences $\big(1/(k+1)\big)$ and $\big(1/(k+2)\big)$ are completely monotone. From Proposition A.4.6 we conclude that (p_k) is *completely monotone*, and hence *infinitely divisible.* Also, the Hausdorff representation for (p_k) is easily found:

$$p_k = \mathbb{E}\,U^k - \mathbb{E}\,U^{k+1} = \mathbb{E}\,U^k(1{-}U) = \int_0^1 x^k\,(1-x)\,\mathrm{d}x.$$

Next, consider the following variant $(\widetilde{p}_k)_{k\in\mathbb{Z}_+}$ of (p_k):

$$\widetilde{p}_k = c\,\frac{1}{(k+1)^2} \qquad [\,c := 6/\pi^2\,].$$

Then, as above, it follows that $(\widetilde{p}_k)$ is *completely monotone*, and hence *infinitely divisible.* The Hausdorff representation is now found by taking independent U and V with a uniform distribution on $(0,1)$ and writing

$$\widetilde{p}_k = c\,(\mathbb{E}\,U^k)\,(\mathbb{E}\,V^k) = c\,\mathbb{E}\,(UV)^k = c\int_0^1 x^k\,(-\log x)\,\mathrm{d}x.$$

A second natural variant is considered in the next example; it is more difficult to handle. ◫

Example 11.5. Let the distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ be given by

$$p_k = c\,\frac{1}{k^2+1} \qquad \big[\, c := 2/(1+\pi \coth \pi)\,\big].$$

Then (p_k) is *not* completely monotone; it is *not* even log-convex because $p_1^2 > p_0 p_2$. On the other hand, as one easily verifies, we do have $p_k^2 \le p_{k-1}p_{k+1}$ for all $k \ge 2$, and hence, as in (10.4),

$$p_k\, p_n \le p_{k-1}\, p_{n+1} \qquad [\, k, n \in \mathbb{N} \text{ with } 2 \le k \le n\,].$$

Also, note that this inequality holds for $k = 1$ and $n \ge 2$. Now, we can easily adapt the proof of Theorem 10.1 in the following way. As seen there, the quantities r_k determined by (11.1) satisfy

$$p_0 p_n r_n = p_n p_{n+1} + \sum_{k=1}^{n} (p_{k-1}p_{n+1} - p_k p_n)\, r_{n-k} \qquad [\, n \in \mathbb{N}\,].$$

Using this equality for $n \ge 2$ and the inequalities above, we see by induction that the r_k are nonnegative as soon as $r_1 \ge 0$, i.e., $p_1^2 \le 2p_0p_2$. Since this inequality does indeed hold, we conclude that (p_k) is *infinitely divisible.* Note that (p_k), though not log-convex, is *convex.* □

Example 11.6. Let $(p_k)_{k \in \mathbb{Z}_+}$ be the *discrete Pareto* distribution on $\mathbb{Z}_+$ with parameter $r > 1$, so (p_k) is given by

$$p_k = c_r\,\frac{1}{(k+1)^r} \qquad \big[\, c_r := 1/\zeta(r),\;\; \zeta(r) := \textstyle\sum_{n=1}^{\infty} 1/n^r\,\big].$$

Then, generalizing the 'Hausdorff' derivation in the second part of Example 11.4, one is led to the easily verified fact that

$$p_k = c_r\, \mathbb{E}\, \mathrm{e}^{-k\,Y_r} = \frac{c_r}{\Gamma(r)} \int_0^1 x^k\, (-\log x)^{r-1}\, \mathrm{d}x \qquad [\, k \in \mathbb{Z}_+\,],$$

with Y_r standard gamma (r). We conclude that (p_k) is *completely monotone*, and hence *infinitely divisible.* Of course, one can also use the 'completely monotone variant' of Proposition 10.9; cf. the remark at the end of Section 10. □

Example 11.7. Let $(p_k)_{k \in \mathbb{Z}_+}$ be the *logarithmic-series* distribution on $\mathbb{Z}_+$ with parameter $p \in (0,1)$, so (p_k) and its pgf P are given by

$$p_k = c_p\,\frac{p^{k+1}}{k+1}, \qquad P(z) = c_p\,\frac{-\log(1-pz)}{z},$$

where $c_p := 1/\{-\log(1-p)\}$. Then (p_k) can be represented in the Hausdorff form, as follows:

$$p_k = c_p \int_0^1 x^k \, 1_{(0,p)}(x)\, \mathrm{d}x \qquad [\, k \in \mathbb{Z}_+ \,],$$

so (p_k) is *completely monotone*, and hence *infinitely divisible.* Mixing (p_k) with the degenerate distribution at zero, one obtains a distribution $(\widetilde{p}_k)_{k\in\mathbb{Z}_+}$ of the form

$$\widetilde{p}_k = \frac{a}{1+a}\,\delta_{0,k} + \frac{c_p}{1+a}\,\frac{p^{k+1}}{k+1},$$

with parameters $a>0$ and $p\in(0,1)$ (and c_p as above). By Proposition A.4.6 $(\widetilde{p}_k)$ is *completely monotone*, and hence *infinitely divisible.* In Example 11.9 we will consider a similar perturbation of (p_k) which is, however, not always infinitely divisible. ◫

Example 11.8. Let the distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ be given by

$$p_k = c_p\, \frac{p^{k+1}}{1-p^{k+1}} \qquad [\, c_p > 0 \text{ a norming constant} \,],$$

where $p \in (0,1)$ is a parameter. Then (p_k) can be written in the Hausdorff form:

$$p_k = c_p \sum_{n=1}^{\infty} (p^n)^k p^n = \int_0^1 x^k \, \nu(\mathrm{d}x) \qquad [\, k \in \mathbb{Z}_+ \,],$$

where ν is the discrete measure on the set $\{p^n : n \in \mathbb{N}\}$ with $\nu\big(\{p^n\}\big) = c_p\, p^n$. We conclude that (p_k) is *completely monotone*, and hence *infinitely divisible.* ◫

Example 11.9. Let $(p_k)_{k\in\mathbb{Z}_+}$ be the following *perturbed logarithmic-series* distribution on $\mathbb{Z}_+$ with parameters $a > 0$ and $p \in (0,1)$:

$$p_0 = \frac{a}{1+a}, \quad p_k = \frac{c_p}{1+a}\,\frac{p^k}{k} \quad \text{for } k \in \mathbb{N},$$

where $c_p := 1/\{-\log(1-p)\}$; cf. Example 11.7. Since for every $k \geq 2$ the inequality $p_k^2 \leq p_{k-1}\, p_{k+1}$ is trivially satisfied, (p_k) is *log-convex* iff $p_1^2 \leq p_0\, p_2$ or, equivalently, $a \geq 2c_p$. It follows that (p_k) is *infinitely divisible* if $a \geq 2c_p$.

In order to determine whether (p_k) is infinitely divisible for other values of the parameters, we proceed as in Example 11.5, and consider the quantities r_k determined by (11.1). Of course, $r_0 = p_1/p_0 \geq 0$. Further, we have $r_1 \geq 0$ iff $p_1^2 \leq 2p_0p_2$, so iff $a \geq c_p$. Now, suppose that this inequality is satisfied; then we also have $p_1 \leq p\,p_0$, so $r_0 \leq p$. Applying (11.1) twice, we can write for $n \in \mathbb{N}$

$$\begin{cases} p_0 r_n = (n+1)\,p_{n+1} - \sum_{k=2}^{n} p_k r_{n-k} - p_1 r_{n-1}, \\ p_0 r_{n-1} = n\,p_n - \sum_{k=2}^{n} p_{k-1} r_{n-k}, \end{cases}$$

from which it follows that for $n \in \mathbb{N}$

$$p_0 r_n - (p - r_0)\,p_0 r_{n-1} = \\ = \big\{(n+1)\,p_{n+1} - p\,n\,p_n\big\} + \sum_{k=2}^{n} \{p\,p_{k-1} - p_k\}\,r_{n-k}.$$

Since in our case $(n+1)\,p_{n+1} - p\,n\,p_n = 0$ and $p\,p_{k-1} - p_k \geq 0$ for $k \geq 2$, we can use induction to conclude that

$$r_n \geq (p - r_0)\,r_{n-1} \geq 0 \qquad [n \in \mathbb{N}].$$

Thus (p_k) is *infinitely divisible* iff $a \geq c_p$. Note that in this case (p_k) is *monotone*, but not necessarily convex. ◫

Next we consider some functions P, and determine whether they are pgf's and, in such a case, whether they are infinitely divisible.

Example 11.10. Let P be the quotient of the pgf's of two different geometric distributions, so P is of the form

$$P(z) = \frac{1-p}{1-p\,z} \Big/ \frac{1-\alpha p}{1-\alpha p\,z} \qquad [0 \leq z \leq 1],$$

where $p \in (0,1)$ and $\alpha \in (0, 1/p)$, $\alpha \neq 1$. Observe that if P is a pgf, then $P(0) < 1$ and hence $\alpha < 1$. Now, for such an α we can apply Proposition 6.1 (iv); since P can be written as

$$P(z) = \frac{Q(\alpha)\,Q(z)}{Q(\alpha z)} \quad \text{with} \quad Q(z) = \frac{1-p}{1-p\,z},$$

it follows that P is a pgf iff $\alpha < 1$, and that in this case P is *infinitely divisible*. By computing the S-function of P and applying Theorem 5.2 one sees that P is even *compound-geometric* when $\alpha < 1$. ◫

Example 11.11. Let the function P on $[0,1]$ be given by

$$P(z) = \frac{1-\sqrt{1-z}}{z} \qquad [\,0 < z \le 1;\; P(0) = \tfrac{1}{2}\,].$$

Then it will be clear that P is a pgf and that the corresponding distribution $(p_k)_{k\in\mathbb{Z}_+}$ is given by

$$p_k = (-1)^k \binom{\frac{1}{2}}{k+1} = \tfrac{1}{2}\,\frac{1}{k+1}\binom{k-\frac{1}{2}}{k} = \frac{1}{k+1}\binom{2k}{k}\big(\tfrac{1}{2}\big)^{2k+1}.$$

This distribution can be recognized as the distribution of the 'reduced' first-passage time $\frac{1}{2}\,(T_1-1)$ from 0 to 1 in the symmetric Bernoulli walk; see the beginning of Section VII.2, also for the asymmetric case. Now, let U and Z be independent random variables where U has a uniform distribution on $(0,1)$ and Z an arcsine distribution on $(0,1)$. Then (cf. Section B.3) the middle representation for p_k above can be rewritten as

$$p_k = \tfrac{1}{2}\,(\mathbb{E}\,U^k)\,(\mathbb{E}Z^k) = \tfrac{1}{2}\,\mathbb{E}\,(UZ)^k,$$

where by Proposition A.3.12 the product UZ has density f given by

$$f(x) = \int_x^\infty \frac{1}{z}\,f_Z(z)\,\mathrm{d}z = 2\,(1-x)\,f_Z(x) \qquad [\,0 < x < 1\,];$$

here the last equality is verified by using the explicit expression for f_Z as given in Section B.3. It follows that (p_k) is *completely monotone*, and hence *infinitely divisible*, with Hausdorff representation given by

$$p_k = \int_0^1 x^k\,(1-x)\,f_Z(x)\,\mathrm{d}x \qquad [\,k\in\mathbb{Z}_+\,].$$

To prove just the infinite divisibility of P one might also use Theorem 4.3 and compute the R-function of P:

$$R(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log P(z) = \frac{1}{2z}\Big(\frac{1}{\sqrt{1-z}} - 1\Big),$$

so R is indeed absolutely monotone; it is the gf of the (canonical) sequence $(r_k)_{k\in\mathbb{Z}_+}$ with

$$r_k = \tfrac{1}{2}\,(-1)^{k+1}\binom{-\frac{1}{2}}{k+1} = \tfrac{1}{2}\binom{k+\frac{1}{2}}{k+1} = \binom{2k+2}{k+1}\big(\tfrac{1}{2}\big)^{2k+3}.$$

Note that (r_k) is completely monotone as well: $r_k = \frac{1}{2}\,\mathbb{E}Z^{k+1}$ for all k, with Z as above. Also note that there is a simple relation between (p_k) and its canonical sequence: $r_k = (k+\frac{1}{2})\,p_k$ for all k. $\square$

Example 11.12. Let P be the function on $[0,1)$ given by

$$P(z) = \frac{1}{1+(1-z)^{\gamma}} \qquad [\,0 \le z < 1\,],$$

where $\gamma \in (0,1\,]$. Since $P(z) = \pi\big((1-z)^{\gamma}\big)$ with π the pLSt of the standard exponential distribution, from Proposition 6.5 it follows that P is an *infinitely divisible pgf.* Use of Example 4.8 shows that P is even *compound-exponential.* Note that P for $\gamma = \frac{1}{2}$ reduces to the first-passage-time pgf of Example 11.11. ◫

We now turn to some examples that are obtained by choosing the canonical sequence of an infinitely divisible distribution in a particular way. Recall that *any* sequence $(r_k)_{k\in\mathbb{Z}_+}$ of nonnegative numbers satisfying $\sum_{k=0}^{\infty} r_k/(k+1) < \infty$ determines an infinitely divisible distribution $(p_k)_{k\in\mathbb{Z}_+}$ via the recurrence relations (11.1) or, equivalently, an infinitely divisible pgf P via the (canonical) representations (4.3) or (4.5). The examples will show that infinitely divisible distributions need not be monotone or unimodal, and may have indivisible factors.

Example 11.13. Let $(p_k)_{k\in\mathbb{Z}_+}$ be an *infinitely divisible* distribution for which the canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ satisfies

$$r_0 = r_1 = \cdots = r_{m-1} = 1,$$

for some $m \in \mathbb{N}$. Then from the recurrence relations (11.1) it easily follows that

$$p_0 = p_1 = \cdots = p_m.$$

So *an infinitely divisible distribution* (p_k) *may begin with an arbitrarily long sequence of constants.* ◫

Example 11.14. Let $(p_k)_{k\in\mathbb{Z}_+}$ be an *infinitely divisible* distribution for which the canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ satisfies

$$r_k = \begin{cases} \left(\frac{1}{2}\right)^{k+1} & \text{, if } k \text{ is even and } k \le m-1, \\ 2 - \left(\frac{1}{2}\right)^{k+1} & \text{, if } k \text{ is odd and } k \le m-1, \end{cases}$$

for some $m \in \mathbb{N}$. Then using the recurrence relations again and induction one easily verifies that

$$p_k = \tfrac{1}{4}\left(3 + (-1)^k\right) p_0 \qquad [\,k = 0,1,\ldots,m\,].$$

So *an infinitely divisible distribution* (p_k) *may begin with an arbitrarily long sequence* $p_0, \frac{1}{2}p_0, p_0, \frac{1}{2}p_0, \ldots$. ◫

Example 11.15. Let the sequence $(r_k)_{k \in \mathbb{Z}_+}$ with gf R be given by

$$r_k = \begin{cases} (\frac{1}{2})^k, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise,} \end{cases} \qquad \text{or} \quad R(z) = \frac{4}{4 - z^2} = \frac{1}{2 - z} + \frac{1}{2 + z}.$$

Applying (4.5) we see that the pgf P of the *infinitely divisible* distribution $(p_k)_{k \in \mathbb{Z}_+}$ with canonical sequence (r_k) is given by

$$P(z) = \tfrac{1}{3}\,\frac{2 + z}{2 - z}, \quad \text{so} \ \ p_0 = \tfrac{1}{3}, \ \ p_k = \tfrac{2}{3}\big(\tfrac{1}{2}\big)^k \ \text{for} \ k \in \mathbb{N}.$$

Note that P can be decomposed into two factors as follows:

$$P(z) = \big(\tfrac{2}{3} + \tfrac{1}{3}z\big)\, Q(z),$$

where Q is the pgf of the geometric $(\frac{1}{2})$ distribution. So *an infinitely divisible distribution may have an indivisible factor*. The pgf P can also be written as $1 - \alpha + \alpha\, Q$ with $\alpha = \frac{4}{3}$ and Q as above. More generally, if P is of the form

$$P(z) = 1 - \alpha + \alpha\, \frac{1 - p}{1 - pz}$$

with $\alpha > 0$ and $p \in (0, 1)$, then P is a pgf with $p_0 > 0$ iff $\alpha < 1/p$, in which case P is *infinitely divisible* iff $\alpha \le 2/(1 + p)$. This is easily shown by computing the R-function of P and using Theorem 4.3. Note that the corresponding distribution is *log-convex* iff $\alpha \le 1$. ◫

Example 11.16. We ask whether for $\lambda > 0$ there exists an infinitely divisible distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ with a canonical sequence $(r_k)_{k \in \mathbb{Z}_+}$ that is related to (p_k) by $r_k = \lambda\,(k+1)\,p_k$ for all k; cf. (the end of) Example 11.11. By (4.2), in terms of gf's this means that we look for a pgf P of the compound-Poisson form (3.3) with $Q(z) = z\,P(z)$, so we want to solve the equation

$$P(z) = \exp\big[-\lambda\,\{1 - z\,P(z)\}\big],$$

or in terms of Q and with G_λ the Poisson (λ) pgf:

$$Q(z) = z\,G_\lambda\big(Q(z)\big).$$

This can be done by use of Bürmann-Lagrange expansion (see Section A.5; the conditions necessary for this are easily verified; take $G = G_\lambda$); for $|z| < 1$ there is a unique solution $Q(z)$ with $|Q(z)| \leq 1$ given by

$$Q(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} (n\lambda)^{n-1} \mathrm{e}^{-n\lambda} \qquad \big[\, |z| < 1 \,\big].$$

Moreover, as is easily verified from the equation for Q above, $a := Q(1-)$ is the smallest nonnegative root of the equation $G_\lambda(z) = z$, so $a = 1$ if $\lambda \leq 1$, and $a = a_\lambda < 1$ if $\lambda > 1$. Since $z\,P(z) = Q(z)$, for $\lambda \leq 1$ one is led to the following probability distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$, known as the *Borel* distribution with parameter λ:

$$p_k = \mathrm{e}^{-\lambda} \, (\lambda \, \mathrm{e}^{-\lambda})^k \, \frac{(k+1)^k}{(k+1)!} \qquad [\, k \in \mathbb{Z}_+ \,];$$

by construction, it is *infinitely divisible*. For $\lambda > 1$ the sequence $(\widetilde{p}_k)_{k \in \mathbb{Z}_+}$ with $\widetilde{p}_k := p_k / a_\lambda$ is a probability distribution on $\mathbb{Z}_+$ with pgf $\widetilde{P}$ satisfying the equation for P above with λ replaced by $\widetilde{\lambda} := \lambda a_\lambda$, so $(\widetilde{p}_k)$ is the Borel distribution with parameter $\widetilde{\lambda}$ (< 1); note that $\widetilde{\lambda}$ satisfies $\widetilde{\lambda} \mathrm{e}^{-\widetilde{\lambda}} = \lambda \mathrm{e}^{-\lambda}$. The Borel distribution is even *completely monotone*. To show this we use the fact that for $r > 0$

$$\frac{r^r}{\Gamma(r+1)} = \frac{1}{\pi} \int_0^{\pi} \big\{ g(t) \big\}^r \, \mathrm{d}t, \quad \text{with } g(t) := \frac{\sin t}{t} \, \mathrm{e}^{t \cot t};$$

note that g decreases from e to 0 on $[\, 0, \pi \,]$. By partial integration it follows that p_k can be written as

$$\begin{aligned} p_k &= \frac{1}{\pi} \, \mathrm{e}^{-\lambda} \, (\lambda \, \mathrm{e}^{-\lambda})^k \, \frac{1}{k+1} \int_0^{\pi} \big\{ g(t) \big\}^{k+1} \, \mathrm{d}t = \\ &= \frac{1}{\pi} \, \mathrm{e}^{-\lambda} \int_0^{\pi} \big\{ \lambda \, \mathrm{e}^{-\lambda} \, g(t) \big\}^k \, \big\{ -g'(t) \big\} \, t \, \mathrm{d}t, \end{aligned}$$

from which the Hausdorff representation for (p_k) is obtained by putting $\lambda \, \mathrm{e}^{-\lambda} \, g(t) =: x$; note that $\lambda \, \mathrm{e}^{-\lambda} \leq \mathrm{e}^{-1}$ for all λ. ◻◻

The final example is a counter-example again; it shows that both log-concavity and the condition

$$p_k^2 \leq 2 \, p_{k-1} p_{k+1} \qquad [\, k \in \mathbb{N} \,]$$

are *not* sufficient for infinite divisibility of a distribution $(p_k)_{k \in \mathbb{Z}_+}$. See Section 10 for the appropriate context.

Example 11.17. Take equality in the inequalities above; then we obtain by induction

$$p_k = p_0\,(r_0)^k\left(\tfrac{1}{2}\right)^{k(k-1)/2} = p_0\left(r_0\sqrt{2}\right)^k\left(\tfrac{1}{2}\sqrt{2}\right)^{k^2} \qquad [\,k \in \mathbb{Z}_+\,],$$

which indeed yields a probability distribution for suitable values of p_0 and $r_0 = p_1/p_0 > 0$. Note that (p_k) is log-concave. It is, however, not infinitely divisible:

$$\lim_{k\to\infty} \frac{-\log p_k}{k\log k} = \infty,$$

so by Theorem 9.3 the distribution (p_k) has too thin a tail to be infinitely divisible. One might also use Proposition 8.4; (p_k) does not satisfy the inequality $3p_3p_0 \geq p_2p_1$, which is necessary for (p_k) to be infinitely divisible. ◫

12. Notes

Infinitely divisible random variables with values in $\mathbb{Z}_+$ were first studied in some detail by Feller (1968), where it is shown that, on $\mathbb{Z}_+$, the infinitely divisible distributions coincide with the compound-Poisson distributions; see Theorem 3.2. The result on zeroes of pgf's in Theorem 2.8 can be regarded as a special case of a similar result for characteristic functions in Lukacs (1970). An important step was made by Katti (1967), who first characterized infinite divisibility on $\mathbb{Z}_+$ by means of recurrence relations as in Theorem 4.4. His work was streamlined and extended in Steutel (1970) and van Harn (1978); this has led to recurrence relations for the compound-geometric distributions as given in Section 5, and to the 'self-decomposability' result in Theorem 6.3.

Relations between the moments of infinitely divisible distributions and the moments of their canonical sequences are given in Wolfe (1971b) and in Sato (1973); formula (7.4) occurs in Steutel and Wolfe (1977). Equation (7.6) opens the possibility of defining 'fractional cumulants' $\kappa_\alpha = \kappa_\alpha(X)$, $\alpha > 0$, of an infinitely divisible random variable X by putting $\kappa_\alpha := \nu_{\alpha-1}$; then one indeed has the cumulating property $\kappa_\alpha(X+Y) = \kappa_\alpha(X) + \kappa_\alpha(Y)$ for independent X and Y. Embrechts and Hawkes (1982) compare the tail of an infinitely divisible distribution with the tail of its canonical sequence.

A proof of Theorem 8.2 is given in Steutel (1970) in the context of zeroes of infinitely divisible densities on $\mathbb{R}_+$. The tail behaviour of infinitely divisible distributions on $\mathbb{R}$ has been studied by many authors; here we only name Kruglov (1970) and Sato (1973). As in the case of supports, the representation of Theorem 3.7 makes our proofs on $\mathbb{Z}_+$ in Section 9 essentially simpler.

The log-convexity result of Theorem 10.1 is the analogue of such a result for renewal-type sequences, which was first proved by Kaluza (1928) in a non-probabilistic context. It was obtained independently by Steutel (1970) and by Warde and Katti (1971). Somewhat more general conditions are given in Katti (1979) and in Danial (1988); see, however, Chang (1989). Theorems 10.2 and 10.3 are due to Hansen (1988b), Theorem 10.5 to Hansen and Steutel (1988). The result in Proposition 10.8 on integer parts can be found in Bondesson et al. (1996).

Example 11.3, on products of independent Poisson variables, occurs in Rohatgi et al. (1990). The infinite divisibility of the perturbed logarithmic-series distribution in Example 11.9 was shown by Katti (1967); a generalized logarithmic-series distribution was shown to be log-convex, and hence infinitely divisible, by Hansen and Willekens (1990). Example 11.16 is taken from Hansen and Steutel (1988); the integral representation given there is due to Bouwkamp (1986). This distribution is a special case of the Borel-Tanner distribution, which occurs in queueing; see Johnson et al. (1992). It is already given in Dwass (1967) who represents a corresponding random variable as in Theorem 3.7. Other Lagrange-type distributions are discussed in Dwass (1968). Bondesson and Steutel (2002) study infinitely divisible distributions (p_k) for which the canonical sequence (r_k) satisfies $r_k = (k + c)\, p_k$ for all k, where $c > 0$; they generalize the cases with $c = \frac{1}{2}$ and $c = 1$ as considered in Examples 11.11 and 11.16.

Chapter III

INFINITELY DIVISIBLE DISTRIBUTIONS ON THE NONNEGATIVE REALS

1. Introduction

This chapter parallels the preceding one on the basic properties of infinitely divisible distributions on $\mathbb{Z}_+$; many results are similar. Distributions on $\mathbb{Z}_+$ are, of course, special cases of distributions on $\mathbb{R}_+$. The $\mathbb{Z}_+$-case has been dealt with separately because it is mathematically simpler and sometimes leads to more detailed results. Our aim that the present chapter be accessible independent of the previous one, as much as possible, makes it necessary to repeat some definitions and arguments given earlier. Even so, we will sporadically have to make use of a result in Chapter II.

We recall the definition of infinite divisibility. An $\mathbb{R}_+$-valued random variable X is said to be *infinitely divisible* if for every $n \in \mathbb{N}$ a random variable X_n exists, the *n-th order factor* of X, such that

$$X \overset{\mathrm{d}}{=} X_{n,1} + \cdots + X_{n,n}, \tag{1.1}$$

where $X_{n,1}, \ldots, X_{n,n}$ are independent and distributed as X_n. Here the factors X_n of X are necessarily $\mathbb{R}_+$-valued as well, and no extra condition as for a $\mathbb{Z}_+$-valued X is needed; cf. (II.1.4). In many cases, however, it will be convenient to suppose that X has left extremity $\ell_X = 0$. This is not an essential restriction because shifting X to zero does not affect the possible infinite divisibility of X. It has the additional advantage that in the special case of $\mathbb{Z}_+$-valued random variables infinite divisibility is equivalent to *discrete* infinite divisibility; convention (II.1.4) is then maintained also in the present chapter. Mostly we tacitly exclude the trivial case of a distribution degenerate at zero, so we then assume that $\mathbb{P}(X = 0) < 1$.

As agreed in Section I.2, the distribution and transform of an infinitely divisible random variable X will be called *infinitely divisible* as well. The distribution of X will mostly be represented by its distribution function F. It follows that a distribution function F on $\mathbb{R}_+$ is infinitely divisible iff for every $n \in \mathbb{N}$ there is a distribution function F_n on $\mathbb{R}_+$, the *n-th order factor* of F, such that F is the n-fold convolution of F_n with itself:

$$(1.2) \qquad F(x) = F_n^{\star n}(x) \qquad [n \in \mathbb{N}].$$

A similar reformulation can be given for a density f of F in case of absolute continuity (with respect to Lebesgue measure) of all factors; this case yields, of course, the closest analogue to the results in Chapter II. As a tool we use the probability Laplace-Stieltjes transform (pLSt), where we used the probability generating function in Chapter II. A pLSt π is infinitely divisible iff for every $n \in \mathbb{N}$ there is a pLSt π_n, the *n-th order factor* of π, such that

$$(1.3) \qquad \pi(s) = \big\{\pi_n(s)\big\}^n \qquad [n \in \mathbb{N}].$$

The correspondence between a distribution function F on $\mathbb{R}_+$ and its pLSt π will be expressed by $\pi = \widehat{F}$. Further conventions and notations concerning distributions on $\mathbb{R}_+$ and Laplace-Stieltjes transforms (LSt's) are given in Section A.3.

For easy comparison of the corresponding results, this chapter is divided in sections in the same way as Chapter II. In order not to distract the reader who wishes to start with the present chapter, we will not discuss these correspondences when we meet them; we do sometimes indicate differences between the two cases. In Section 2 we give some elementary results and treat the degenerate and gamma distributions as first examples of infinitely divisible distributions on $\mathbb{R}_+$. The important classes of compound-Poisson, compound-geometric and compound-exponential distributions are introduced in Section 3. A useful characterization of infinite divisibility by means of complete monotonicity is given in Section 4; it leads to a canonical representation for infinitely divisible pLSt's and to a functional equation for infinitely divisible distribution functions. These tools are shown to have analogues for the compound-exponential distributions (Section 5) and are used to easily prove closure properties in Section 6. Relations between moments of an infinitely divisible distribution and those of the corresponding canonical function are given in Section 7.

The structure of the support of infinitely divisible distributions is determined in Section 8, and their tail behaviour is considered in Section 9. In Section 10 log-convex densities, all of which are infinitely divisible, and infinitely divisible log-concave densities are considered together with their canonical functions. An interesting special case of log-convexity is complete monotonicity, related to mixtures of exponential distributions. Section 11 contains examples and counter-examples, and in Section 12 bibliographical and other supplementary remarks are made.

Finally we note that this chapter only treats the basic properties of infinitely divisible distributions on $\mathbb{R}_+$. Results for self-decomposable and stable distributions on $\mathbb{R}_+$ can be found in Sections V.2, V.3 and V.8, for mixtures of gamma distributions in Sections VI.3 and VI.4, and for generalized gamma convolutions in Section VI.5. Also the remaining sections of Chapter VI and some sections of Chapter VII contain information on infinitely divisible distributions on $\mathbb{R}_+$.

2. Elementary properties

We start with showing that infinite divisibility of distributions is preserved under changing scale, under convolutions and under weak convergence. The first two of these properties follow directly from (1.1) or (1.3).

Proposition 2.1.

(i) *If X is an infinitely divisible $\mathbb{R}_+$-valued random variable, then so is aX for every $a \in \mathbb{R}_+$. Equivalently, if π is an infinitely divisible pLSt, then so is π_a with $\pi_a(s) := \pi(as)$ for every $a \in \mathbb{R}_+$.*

(ii) *If X and Y are independent infinitely divisible $\mathbb{R}_+$-valued random variables, then $X+Y$ is an infinitely divisible random variable. Equivalently, if π_1 and π_2 are infinitely divisible pLSt's, then their pointwise product $\pi_1\pi_2$ is an infinitely divisible pLSt.*

Proposition 2.2. *If a sequence $(X^{(m)})$ of infinitely divisible $\mathbb{R}_+$-valued random variables converges in distribution to X, then X is infinitely divisible. Equivalently, if a sequence $(\pi^{(m)})$ of infinitely divisible pLSt's converges (pointwise) to a pLSt π, then π is infinitely divisible.*

Proof. Since by (1.3) for every $m \in \mathbb{N}$ there exists a sequence $(\pi_n^{(m)})_{n\in\mathbb{N}}$ of pLSt's such that $\pi^{(m)} = \{\pi_n^{(m)}\}^n$, the limit $\pi = \lim_{m\to\infty} \pi^{(m)}$ can be written as

$$\pi(s) = \lim_{m\to\infty} \{\pi_n^{(m)}(s)\}^n = \{\lim_{m\to\infty} \pi_n^{(m)}(s)\}^n = \{\pi_n(s)\}^n,$$

where $\pi_n := \lim_{m\to\infty} \pi_n^{(m)}$ is a pLSt for every $n \in \mathbb{N}$ by the continuity theorem; see Theorem A.3.1. Hence π is infinitely divisible. ◻

Since a pLSt π is positive on $\mathbb{R}_+$, for $s \geq 0$ relation (1.3) can be rewritten in the following way:

$$\text{(2.1)} \qquad \{\pi(s)\}^{1/n} = \pi_n(s) \qquad [n \in \mathbb{N}].$$

As a pLSt is determined by its values on $\mathbb{R}_+$, it follows that the factors π_n of an infinitely divisible pLSt π, and hence the corresponding distributions, are uniquely determined by π. For the time being we shall consider pLSt's only for values of the argument in $\mathbb{R}_+$. We shall return to the possibility of zeroes for complex arguments later in this section. For any pLSt π and any $t > 0$ the function $\pi^t = \exp[\,t \log \pi\,]$ is well defined on $\mathbb{R}_+$. We now come to a first criterion for infinite divisibility. Note that by Bernstein's theorem (Theorem A.3.6) the set of pLSt's equals the set of *completely monotone* functions π on $(0,\infty)$ with $\pi(0+) = 1$.

Proposition 2.3. *A pLSt π is infinitely divisible iff π^t is a pLSt for all $t \in T$, where $T = (0,\infty)$, $T = \{1/n : n \in \mathbb{N}\}$ or $T = \{a^{-k} : k \in \mathbb{N}\}$ for any fixed integer $a \geq 2$. Equivalently, π is infinitely divisible iff π^t is completely monotone for all $t \in T$ with T as above.*

PROOF. Let π be infinitely divisible. Then by (2.1) $\pi^{1/n}$ is a pLSt for all $n \in \mathbb{N}$, and hence $\pi^{m/n}$ is a pLSt for all $m, n \in \mathbb{N}$. It follows that π^t is a pLSt for all positive $t \in \mathbb{Q}$, and hence, by the continuity theorem, for all $t > 0$. Conversely, from (2.1) we know that π is infinitely divisible if $\pi^{1/n}$ is a pLSt for every $n \in \mathbb{N}$. We can even be more restrictive because, for a given integer $a \geq 2$, any $t \in (0,1)$ can be represented as $t = \sum_{k=1}^{\infty} t_k a^{-k}$ with $t_k \in \{0,\ldots,a-1\}$ for all k, and hence for these t

$$\{\pi(s)\}^t = \lim_{m\to\infty} \prod_{k=1}^{m} \Big(\{\pi(s)\}^{1/a^k}\Big)^{t_k}.$$

By the continuity theorem it now follows that if π^t is a pLSt for all t of the form a^{-k} with $k \in \mathbb{N}$, then so is π^t for all $t \in (0,1)$. ◻

Corollary 2.4. *If π is an infinitely divisible pLSt, then so is π^t for all $t > 0$. In particular, the factors X_n of an infinitely divisible $\mathbb{R}_+$-valued random variable X are infinitely divisible.*

The continuous multiplicative semigroup $(\pi^t)_{t\geq 0}$ of pLSt's generated by an infinitely divisible pLSt π, corresponds to the set of one-dimensional marginal distributions of an $\mathbb{R}_+$-valued *sii-process*, i.e., a process $X(\cdot)$ with stationary independent increments, started at zero and continuous in probability; see Section I.3. If $X(1)$, with pLSt π, has distribution function F, then for $t > 0$ the distribution function of $X(t)$, with pLSt π^t, will be denoted by $F^{\star t}$, so:

$$(2.2)\qquad F^{\star t}(x) = \mathbb{P}\big(X(t) \leq x\big), \quad \int_{\mathbb{R}_+} \mathrm{e}^{-sx}\, \mathrm{d}F^{\star t}(x) = \big\{\pi(s)\big\}^t.$$

Directly from (1.1), or from (1.2) and the fact that $G^{\star n} \leq G$ for any distribution function G on $\mathbb{R}_+$, it follows that the n-th order factor $F^{\star(1/n)}$ of F satisfies

$$(2.3)\qquad F^{\star(1/n)}(x) \geq F(x) \qquad [x \in \mathbb{R}].$$

Of course, examples of infinitely divisible distributions on $\mathbb{R}_+$, and hence of $\mathbb{R}_+$-valued sii-processes, are provided by the two examples from Section II.2 containing the *Poisson* distributions and the *negative-binomial* distributions. These distributions have the following analogues on $\mathbb{R}_+$; this will be clear from the results in Sections V.5 and VI.6.

Example 2.5. For $\lambda > 0$, let X have the *degenerate* (λ) distribution, so if π is the pLSt of X, then

$$\mathbb{P}(X = \lambda) = 1, \quad \pi(s) = \mathrm{e}^{-\lambda s}.$$

Then for $t > 0$ the t-th power π^t of π is the pLSt of the degenerate (λt) distribution, so X is *infinitely divisible.* The corresponding sii-process is the *deterministic process* $X(\cdot)$ with $X(t) = \lambda t$. ◻◻

Example 2.6. For $r > 0$ and $\lambda > 0$, let X have the *gamma* (r, λ) distribution, so its density f on $(0, \infty)$ and pLSt π are given by

$$f(x) = \frac{\lambda^r}{\Gamma(r)}\, x^{r-1}\, \mathrm{e}^{-\lambda x}, \quad \pi(s) = \Big(\frac{\lambda}{\lambda + s}\Big)^r.$$

Then for $t > 0$ the t-th power π^t of π is the pLSt of the gamma (rt, λ) distribution. From Proposition 2.3 we conclude that the gamma (r, λ) distribution is *infinitely divisible.* The corresponding sii-process is a *gamma process.* Taking $r = 1$ one sees that the *exponential* (λ) distribution is *infinitely divisible.* □

Sometimes we want to consider a pLSt π for *complex* values of its argument; $\pi(z)$ is well defined at least for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq 0$. Of course, π has no zeroes on $\mathbb{R}_+$. The following inequality implies that there are no zeroes in the closed half-plane $\operatorname{Re} z \geq 0$ if π is infinitely divisible with

$$p_0 := \lim_{s \to \infty} \pi(s) > 0;$$

note that p_0 is the mass at zero of the distribution corresponding to π.

Proposition 2.7. *If π is an infinitely divisible pLSt, then with p_0 as above*

$$\text{(2.4)} \qquad \big|\pi(z)\big| \geq p_0^2 \qquad [\,\operatorname{Re} z \geq 0\,].$$

PROOF. If G is a distribution function on $\mathbb{R}_+$ with $G(0) > \frac{1}{2}$, then for $z \in \mathbb{C}$ with $\operatorname{Re} z \geq 0$:

$$\big|\widehat{G}(z)\big| \geq G(0) - \int_{(0,\infty)} |\mathrm{e}^{-zx}|\,\mathrm{d}G(x) \geq 2G(0) - 1.$$

Using the resulting inequality for the n-th order factor π_n of an infinitely divisible pLSt π, we see that $\big|\pi(z)\big|$ with $\operatorname{Re} z \geq 0$ can be estimated as follows:

$$\big|\pi(z)\big| = \big|\pi_n(z)\big|^n \geq \big(2p_0^{1/n} - 1\big)^n,$$

for all n sufficiently large. Now one easily verifies that for every $\alpha > 0$ one has the equality $\lim_{n \to \infty} \big(2\alpha^{1/n} - 1\big)^n = \alpha^2$; taking $\alpha = p_0$ finishes the proof. □

This result can be extended as follows. Any pLSt π has an abscissa of convergence, this is the smallest $s_\pi \in [-\infty, 0\,]$ such that $\pi(z)$ is well defined for all $z \in \mathbb{C}$ with $\operatorname{Re} z > s_\pi$. Moreover, as noted in Section A.3, s_π is determined by the tail of the distribution function F corresponding to π in the following way:

$$s_\pi = -\liminf_{x \to \infty} \frac{-\log\big\{1 - F(x)\big\}}{x}.$$

Now, let π be infinitely divisible with $s_\pi < 0$, and take $a \in (s_\pi, 0)$, so we have $\pi(a) < \infty$. Note that from (2.3) and the formula for s_π above it follows that the n-th order factor $\pi^{1/n}$ of π has abscissa of convergence $\leq s_\pi$. Therefore, Proposition 2.3 can be used to show that the pLSt π_a with $\pi_a(s) := \pi(a+s)/\pi(a)$ is infinitely divisible. Hence we can apply Proposition 2.7 for π_a and find

$$(2.5) \qquad \big|\pi(z)\big| \geq p_0^2/\pi(a) \qquad [\operatorname{Re} z \geq a].$$

We conclude that if $p_0 > 0$ then π has no zeroes in its (open) half-plane of convergence $\operatorname{Re} z > s_\pi$. There seems to be no non-trivial result analogous to (2.5) when $p_0 = 0$. Nevertheless, the non-zero property of π also holds in this case.

Theorem 2.8. *Let π be an infinitely divisible pLSt. Then:*

(i) *π has no zeroes in the closed half-plane $\operatorname{Re} z \geq 0$.*

(ii) *π has no zeroes in the open half-plane $\operatorname{Re} z > s_\pi$, where s_π is the abscissa of convergence of π.*

(iii) *If π is an entire function, i.e., if $s_\pi = -\infty$, then π has no zeroes in $\mathbb{C}$.*

PROOF. We use the well-known fact that π is analytic on $\operatorname{Re} z > s_\pi$; as seen above, the factors $\pi_n := \pi^{1/n}$ of π are well defined on this half-plane, and hence are analytic there as well. Now, suppose that $\pi(z_0) = 0$ for some z_0 satisfying $\operatorname{Re} z_0 > s_\pi$; on some neighbourhood B of z_0, π can then be represented as a power series of the form

$$\pi(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^k \qquad [z \in B].$$

Since also $\pi_n(z_0) = 0$, π_n can be represented similarly. By equating coefficients in the identity $\pi = (\pi_n)^n$ one then sees that necessarily $a_1 = \cdots = a_{n-1} = 0$. Since this holds for all n, π would be zero on B and hence on all of $\operatorname{Re} z > s_\pi$; we have obtained a contradiction. So π has no zeroes in its half-plane of convergence.

The only thing left to prove now is the property that π (with $s_\pi = 0$ and $p_0 = 0$) has no zeroes *on* the line $\operatorname{Re} z = 0$. This is a special case of the fact that an infinitely divisible characteristic function ϕ has no real zeroes; see Propositions I.2.8 and IV.2.4, also for a proof. ◻

This theorem may be used to show that a given pLSt is not infinitely divisible; an example is given in Section 11. It also implies that if π is an infinitely divisible pLSt, then $\log \pi(z)$, and hence $\pi(z)^t = \exp\bigl[t \log \pi(z)\bigr]$ with $t > 0$, can be defined for $\operatorname{Re} z \geq 0$ as a continuous function with $\log \pi(0) = 0$; see also Section A.3.

3. Compound distributions

Random stopping of processes with stationary independent increments can be used for constructing new infinitely divisible distributions from given ones. First we will show this in the *discrete-time* case. To this end we recall some facts from Section I.3 specializing them to the present $\mathbb{R}_+$-case. Let $(S_n)_{n \in \mathbb{Z}_+}$ be an sii-process generated by an $\mathbb{R}_+$-valued random variable Y (so $S_n = Y_1 + \cdots + Y_n$ for all n with $Y_1, Y_2, \ldots$ independent and distributed as Y), let N be $\mathbb{Z}_+$-valued and independent of (S_n), and consider X such that

$$X \overset{\mathrm{d}}{=} S_N \quad (\text{so } X \overset{\mathrm{d}}{=} Y_1 + \cdots + Y_N). \tag{3.1}$$

Then X is said to have a *compound-N* distribution, and from (I.3.10) one sees that its distribution function and pLSt can be expressed in similar characteristics of Y and N as follows:

$$F_X(x) = \sum_{n=0}^{\infty} \mathbb{P}(N = n)\, F_Y^{\star n}(x), \qquad \pi_X(s) = P_N\bigl(\pi_Y(s)\bigr). \tag{3.2}$$

In particular, it follows that the composition of a pgf with a pLSt is a pLSt. Hence, by applying Proposition 2.3 and its discrete counterpart in Chapter II, we immediately obtain the following general result on infinite divisibility.

Proposition 3.1. *An $\mathbb{R}_+$-valued random variable X that has a compound-N distribution with N $\mathbb{Z}_+$-valued and (discrete) infinitely divisible $\bigl($so $\mathbb{P}(N = 0) > 0\bigr)$, is infinitely divisible. Equivalently, the composition $P \circ \widehat{G}$ of an infinitely divisible pgf P with an arbitrary pLSt $\widehat{G}$ is an infinitely divisible pLSt.*

The *compound-Poisson* distributions, obtained by choosing N Poisson distributed, are of special importance. Their pLSt's have the form

$$\pi(s) = \exp\bigl[-\lambda\{1 - \widehat{G}(s)\}\bigr], \tag{3.3}$$

where $\lambda > 0$ and G is a distribution function on $\mathbb{R}_+$. Here the pair (λ, G) is not uniquely determined by π, but it is, if we choose G such that $G(0) = 0$, which can always be done. From Proposition 3.1, or directly from (3.3), it is clear that the compound-Poisson distributions are infinitely divisible. Contrary to the $\mathbb{Z}_+$-case, not all infinitely divisible distributions on $\mathbb{R}_+$ are compound-Poisson, simply because any compound-N distribution with $\mathbb{P}(N = 0) > 0$ is a distribution with positive mass at zero. Interestingly, restriction to such distributions does lead to a converse, as the following result shows.

Theorem 3.2. *An $\mathbb{R}_+$-valued random variable X is infinitely divisible with $p_0 := \mathbb{P}(X = 0) > 0$ iff X has a compound-Poisson distribution.*

PROOF. Let X be infinitely divisible with $p_0 > 0$ and with pLSt π. Then, for $n \in \mathbb{N}$, $\pi_n := \pi^{1/n}$ is a pLSt with $\lim_{s\to\infty} \pi_n(s) = p_0^{1/n}$, and hence there exists a distribution function G_n on $\mathbb{R}_+$ such that

$$\widehat{G}_n(s) = \frac{\pi_n(s) - p_0^{1/n}}{1 - p_0^{1/n}} = 1 - \frac{1 - \pi(s)^{1/n}}{1 - p_0^{1/n}}.$$

Now, let $n \to \infty$ and use the fact that $\lim_{n\to\infty} n\,\big(1 - \alpha^{1/n}\big) = -\log\alpha$ for $\alpha > 0$. Then from the continuity theorem we conclude that there exists a distribution function G on $\mathbb{R}_+$ such that

$$\text{(3.4)} \qquad \widehat{G}(s) = 1 - \frac{1}{\lambda}\,\big\{-\log\,\pi(s)\big\} \;\; \text{with} \;\; \lambda := -\log p_0;$$

note that $G(0) = 0$. It follows that π can be written in the form (3.3). The converse statement has already been proved. ◻

In the proof just given we saw that $n\,(1-\pi^{1/n})$ tends (pointwise) to $-\log\pi$ as n tends to ∞. For π this means that

$$\text{(3.5)} \qquad \pi(s) = \lim_{n\to\infty} \exp\big[-n\,\big\{1 - \pi^{1/n}(s)\big\}\big].$$

In view of Proposition 2.2 we obtain the following result, which emphasizes the important role played by the compound-Poisson distributions, also on $\mathbb{R}_+$.

Theorem 3.3. *A distribution on $\mathbb{R}_+$ is infinitely divisible iff it is the weak limit of compound-Poisson distributions.*

If in (3.1) we take N geometrically distributed, we obtain the *compound-geometric* distributions on $\mathbb{R}_+$; by (3.2) their pLSt's have the form

$$\pi(s) = \frac{1-p}{1-p\,\widehat{G}(s)}, \tag{3.6}$$

where $p \in (0,1)$ and G is a distribution function on $\mathbb{R}_+$. The pair (p, G) is uniquely determined by π if we choose G such that $G(0) = 0$, which can always be done. Since the geometric distribution is infinitely divisible (cf. Example II.2.6), the following theorem is an immediate consequence of Proposition 3.1 (and Theorem 3.2).

Theorem 3.4. *A compound-geometric distribution on $\mathbb{R}_+$, with pLSt of the form (3.6), is infinitely divisible (and hence compound-Poisson).*

We can say more; the compound-geometric distributions turn out to belong to a class of infinitely divisible distributions that is introduced below. For this class the compound-geometric distributions play the same role as do the compound-Poisson distributions for the class of all infinitely divisible distributions. In fact, we will give analogues to Theorems 3.2 and 3.3.

To this end we again recall some facts from Section I.3, specialized to our $\mathbb{R}_+$-case. Let $S(\cdot)$ be a continuous-time sii-process generated by an $\mathbb{R}_+$-valued random variable Y $\big($so $S(1) \overset{\mathrm{d}}{=} Y$ and Y is infinitely divisible$\big)$, let T be $\mathbb{R}_+$-valued and independent of $S(\cdot)$, and consider X such that

$$X \overset{\mathrm{d}}{=} S(T). \tag{3.7}$$

Then X is said to have a *compound-T* distribution, and from (I.3.8) one sees that the distribution function and pLSt of X can be expressed in similar characteristics of Y and T by

$$F_X(x) = \int_{\mathbb{R}_+} F_Y^{\star t}(x)\,\mathrm{d}F_T(t), \quad \pi_X(s) = \pi_T\big(-\log \pi_Y(s)\big). \tag{3.8}$$

Applying Proposition 2.3, both for π_X and for π_T, yields the following general result.

Proposition 3.5. *An $\mathbb{R}_+$-valued random variable X that has a compound-T distribution with T $\mathbb{R}_+$-valued and infinitely divisible, is infinitely divisible. Equivalently, the composition $\pi \circ (-\log \pi_0)$ where π and π_0 are infinitely divisible pLSt's, is an infinitely divisible pLSt.*

The compound-T distributions with T degenerate at one constitute precisely the set of all infinitely divisible distributions. It is more interesting to take T standard exponentially distributed. Since then $\pi_T(s) = 1/(1+s)$, the resulting *compound-exponential* distributions have pLSt's of the form

$$\pi(s) = \frac{1}{1 - \log \pi_0(s)}, \tag{3.9}$$

where π_0 is an *infinitely divisible* pLSt. This pLSt, which is sometimes called the *underlying* (infinitely divisible) pLSt of π, is uniquely determined by π because

$$\pi_0(s) = \exp\big[1 - 1/\pi(s)\big]. \tag{3.10}$$

Note that taking T exponential (λ) with $\lambda \neq 1$ leads to the same class of distributions; just use Corollary 2.4. The following theorem is an immediate consequence of Proposition 3.5 and the infinite divisibility of the exponential distribution; cf. Example 2.6.

Theorem 3.6. *A compound-exponential distribution on $\mathbb{R}_+$, with pLSt of the form (3.9), is infinitely divisible.*

The compound-exponential distributions are related to the compound-geometric ones. As announced above, we prove analogues to Theorems 3.2 and 3.3; we do so by using these theorems.

Theorem 3.7. *An $\mathbb{R}_+$-valued random variable X is compound-exponential with $p_0 := \mathbb{P}(X = 0) > 0$ iff X has a compound-geometric distribution.*

PROOF. First, let X be compound-exponential with $p_0 > 0$. Then the pLSt π of X has the form (3.9), and letting $s \to \infty$ one sees that the infinitely divisible distribution function F_0 with pLSt π_0 satisfies $F_0(0) > 0$; cf. (A.3.6). From Theorem 3.2 it follows that π_0 is of the form (3.3), so π can be written as

$$\pi(s) = \frac{1}{1 + \lambda\{1 - \widehat{G}(s)\}} = \frac{1 - \lambda/(1+\lambda)}{1 - \{\lambda/(1+\lambda)\}\,\widehat{G}(s)},$$

which is of the compound-geometric form (3.6). The converse statement is proved similarly: If π has the form (3.6), then computing the right-hand side of (3.10) shows that π can be written as in (3.9) with π_0 of the form (3.3); take $\lambda = p/(1-p)$. ◻

Because of Theorems 3.2 and 3.3, a distribution function F_0 on $\mathbb{R}_+$ is infinitely divisible iff it is the weak limit of (a sequence of) infinitely divisible distribution functions $F_{0,n}$ on $\mathbb{R}_+$ with $F_{0,n}(0) > 0$. From representation (3.9) and the continuity theorem it follows that a similar characterization holds for the compound-exponential distributions: A distribution function F on $\mathbb{R}_+$ is compound-exponential iff it is the weak limit of compound-exponential distribution functions F_n on $\mathbb{R}_+$ with $F_n(0) > 0$. In view of the preceding theorem we can reformulate this as follows.

Theorem 3.8. *A distribution on $\mathbb{R}_+$ is compound-exponential iff it is the weak limit of compound-geometric distributions.*

Finally, we return to compound-Poisson random variables X, and transfer representation (3.1) for X, with N Poisson, into an integral representation in terms of a Poisson process. For the definition of the generalized inverse of a distribution function see Section A.2.

Theorem 3.9. *Let $\lambda > 0$, let $N_\lambda(\cdot)$ be the Poisson process with intensity λ, and let G be a distribution function with $G(0) = 0$. Then an $\mathbb{R}_+$-valued random variable X is compound-Poisson (λ, G), i.e., its pLSt π is given by (3.3), iff X can be represented (in distribution) as*

$$(3.11) \qquad X \stackrel{\mathrm{d}}{=} \int_{\mathbb{R}_+} y \, dN_\lambda\big(G(y)\big).$$

Equivalent forms, where $(T_k)_{k\in\mathbb{N}}$ is the sequence of jump-points of $N_\lambda(\cdot)$ and H is the generalized inverse of G:

$$(3.12) \qquad X \stackrel{\mathrm{d}}{=} \int_{[0,1]} H(t) \, dN_\lambda(t) = \sum_{k=1}^{N_\lambda(1)} H(T_k).$$

Proof. The compound-Poisson (λ, G) random variables correspond to the random variables X of the form

$$(3.13) \qquad X \stackrel{\mathrm{d}}{=} Y_1 + \cdots + Y_N,$$

where $Y_1, Y_2, \ldots$ are independent with distribution function G and N is independent of (Y_j) with $N \stackrel{\mathrm{d}}{=} N_\lambda(1)$. Letting $U_1, U_2, \ldots$ be independent random variables with a uniform distribution on $(0,1)$ and independent of N, we can rewrite (3.13) as

$$(3.14) \qquad X \stackrel{\mathrm{d}}{=} H(U_1) + \cdots + H(U_N).$$

Now, we use the well-known fact that the order statistics $U_{n,1}, \ldots, U_{n,n}$ of $U_1, \ldots, U_n$ satisfy

$$(U_{n,1}, \ldots, U_{n,n}) \stackrel{\text{d}}{=} \big(T_1, \ldots, T_n \,\big|\, N_\lambda(1) = n\big). \tag{3.15}$$

It follows that for Borel sets B

$$\begin{aligned}
&\mathbb{P}\big(H(U_1) + \cdots + H(U_N) \in B\big) = \\
&= \sum_{n=0}^{\infty} \mathbb{P}\big(H(U_1) + \cdots + H(U_n) \in B\big)\, \mathbb{P}(N = n) = \\
&= \sum_{n=0}^{\infty} \mathbb{P}\big(H(U_{n,1}) + \cdots + H(U_{n,n}) \in B\big)\, \mathbb{P}(N = n) = \\
&= \sum_{n=0}^{\infty} \mathbb{P}\big(H(T_1) + \cdots + H(T_n) \in B \,\big|\, N_\lambda(1) = n\big)\, \mathbb{P}\big(N_\lambda(1) = n\big) = \\
&= \mathbb{P}\big(H(T_1) + \cdots + H(T_{N_\lambda(1)}) \in B\big).
\end{aligned}$$

We conclude that (3.14) is equivalent to (3.12). The alternative form (3.11) follows by substitution. ◫

Combining Theorems 3.2 and 3.9 shows that an $\mathbb{R}_+$-valued random variable X is infinitely divisible with $p_0 := \mathbb{P}(X = 0) > 0$ iff X can be represented as in (3.11) or (3.12) for some λ and G. A similar representation result can be given for infinitely divisible random variables X with $p_0 = 0$. We will not do so; in deriving such a result the canonical representation of the next section is needed.

4. Canonical representation

According to Proposition 2.3 a necessary and sufficient condition for a pLSt π to be infinitely divisible is that all functions π^t with $t > 0$ are completely monotone. We will show that this condition is equivalent to the complete monotonicity of a *single* function. In doing so we use, without further comment, some elementary properties of completely monotone functions as reviewed in Proposition A.3.7. First, note that if π^t is completely monotone for all $t > 0$, then so is the following function of $s > 0$:

$$\lim_{t \downarrow 0} -\frac{1}{t}\,\frac{\text{d}}{\text{d}s}\pi^t(s) = -\frac{\pi'(s)}{\pi(s)} = -\frac{\text{d}}{\text{d}s}\log \pi(s) =: \rho(s). \tag{4.1}$$

This function will be called the *ρ-function* of π. For the converse statement: complete monotonicity of the ρ-function implies complete monotonicity of all functions π^t, we need not start from a pLSt π; we allow π to be any function on $(0, \infty)$ with $\pi(0+) = 1$ such that (4.1) makes sense. The ρ-function of π is then well defined, and if we can integrate in (4.1) (e.g., if ρ is nonnegative, so π nonincreasing), then we can express π in terms of ρ by

$$(4.2) \qquad \pi(s) = \exp\Big[-\int_0^s \rho(u)\,du\Big] \qquad [\,s > 0\,].$$

Now, this relation yields the desired result; it can be formulated as follows.

Theorem 4.1. *Let π be a positive, differentiable function on $(0, \infty)$ with $\pi(0+) = 1$. Then π is an infinitely divisible pLSt iff its ρ-function is completely monotone.*

PROOF. That the ρ-function of an infinitely divisible pLSt π is completely monotone, we have already seen above; use the limiting relation (4.1).
So suppose, conversely, that the ρ-function ρ of π is completely monotone. Then ρ is nonnegative, and hence we can use (4.2) to conclude that, for any $t > 0$, π^t can be written as

$$\pi^t = \rho_t \circ \sigma, \quad \text{with } \rho_t(s) := \mathrm{e}^{-ts} \text{ and } \sigma(s) := \int_0^s \rho(u)\,du.$$

Since ρ_t is completely monotone and σ is a function with $\sigma' = \rho$ completely monotone and $\sigma(0+) \geq 0$, it follows that also π^t is completely monotone. Taking $t = 1$, together with $\pi(0+) = 1$, implies that π is a pLSt. Applying Proposition 2.3 yields the infinite divisibility of π. □

The *criterion* for infinite divisibility given by this theorem turns out to be very useful, both theoretically and practically. We will use it in Example 4.9, in proving the closures properties in Section 6 and in some examples in Section 11. Because of (4.2) it can be reformulated so as to obtain the following *representation theorem* for infinitely divisible pLSt's.

Theorem 4.2. *A function π on $\mathbb{R}_+$ is the pLSt of an infinitely divisible distribution on $\mathbb{R}_+$ iff π has the form*

$$(4.3) \qquad \pi(s) = \exp\Big[-\int_0^s \rho(u)\,du\Big] \qquad [\,s \geq 0\,]$$

with ρ a completely monotone function on $(0, \infty)$.

It is now easy to prove the following representation result, which will be considered as a *canonical representation.* Here, as defined in Section A.3, a function $K : \mathbb{R} \to \mathbb{R}$ is said to be an *LSt-able* function if K is right-continuous and nondecreasing with $K(x) = 0$ for $x < 0$ and such that its LSt $\widehat{K}$ is finite on $(0, \infty)$.

Theorem 4.3 (Canonical representation). *A function π on $\mathbb{R}_+$ is the pLSt of an infinitely divisible distribution on $\mathbb{R}_+$ iff π has the form*

$$\pi(s) = \exp\Big[-\int_{\mathbb{R}_+} (1 - e^{-sx})\, \frac{1}{x}\, dK(x)\Big] \qquad [\, s \geq 0\,] \tag{4.4}$$

with K an LSt-able function; the integrand for $x = 0$ is defined by continuity. Here the function K is unique, and necessarily

$$\int_{(1,\infty)} \frac{1}{x}\, dK(x) < \infty. \tag{4.5}$$

Proof. Let π be an infinitely divisible pLSt. Then by Theorem 4.2 π has the form (4.3) with ρ completely monotone. From Bernstein's theorem, i.e., Theorem A.3.6, it follows that there exists an LSt-able function K such that $\rho = \widehat{K}$. Now, observe that by Fubini's theorem

$$\int_0^s \widehat{K}(u)\, \mathrm{d}u = \int_{\mathbb{R}_+} (1 - \mathrm{e}^{-sx})\, \frac{1}{x}\, \mathrm{d}K(x) \qquad [\, s \geq 0\,], \tag{4.6}$$

so (4.3) takes the form (4.4). Since on $(1, \infty)$ the function $x \mapsto 1 - \mathrm{e}^{-sx}$ is bounded below by a positive constant, we also have condition (4.5) satisfied. The function K is unique because ρ is uniquely determined by π. Suppose, conversely, that π has the form (4.4) with K an LSt-able function. Then by (4.6) π can be written as in (4.3) with $\rho = \widehat{K}$ completely monotone. Applying Theorem 4.2 shows that π is an infinitely divisible pLSt. ◻

There is an alternative (more classical) way to obtain this canonical representation: Start from the fact that an infinitely divisible pLSt π is the limit of compound-Poisson pLSt's as in (3.5), and proceed as indicated around (I.4.2); then one can show that π has the form (4.4) with K given by

$$K(x) = \lim_{n\to\infty} n \int_{[0,x]} y\, \mathrm{d}F^{\star(1/n)}(y) \tag{4.7}$$

for continuity points x of K, where F is the distribution function with $\widehat{F} = \pi$. We further note that Theorem 4.3 guarantees that *any* LSt-able function K satisfying (4.5) gives rise, via (4.4), to an infinitely divisible pLSt π. On the other hand, if a function π has the form (4.4) with K of bounded variation but *not* monotone, then π is *not* an infinitely divisible pLSt, because its ρ-function, which is $\widehat{K}$, is not completely monotone. Here we used the fact that two different functions of bounded variation, both vanishing on $(-\infty, 0)$ and right-continuous, cannot have the same LSt. For a concrete example we refer to Section 11.

The function K in Theorem 4.3 will be called the *canonical function* of π, and of the corresponding distribution function F and of a corresponding random variable X. Often, it is most easily determined by noting that its LSt equals the ρ-function of π:

$$\widehat{K} = \rho. \tag{4.8}$$

Before giving three simple but basic examples, we prove some useful relations between (characteristics of) an infinitely divisible distribution and its canonical function. Note that (4.4) can be rewritten as

$$\pi(s) = \mathrm{e}^{-sK(0)} \exp\Big[-\int_{(0,\infty)} (1 - \mathrm{e}^{-sx}) \frac{1}{x}\, \mathrm{d}K(x)\Big] \qquad [\, s \geq 0\,]. \tag{4.9}$$

Proposition 4.4. *Let X be an infinitely divisible $\mathbb{R}_+$-valued random variable with left extremity ℓ_X and canonical function K. Then $\ell_X = K(0)$, and*

$$\mathbb{P}(X = \ell_X) = \mathrm{e}^{-\lambda}, \quad \text{where } \lambda := \int_{(0,\infty)} \frac{1}{x}\, dK(x) \quad (\leq \infty). \tag{4.10}$$

So $\mathbb{P}(X = \ell_X) > 0$ iff $\lambda < \infty$, in which case $X - \ell_X$ (if non-degenerate) is compound-Poisson; its pLSt has the form (3.3) where G is the distribution function with $G(0) = 0$ and

$$G(x) = \frac{1}{\lambda} \int_{(0,x]} \frac{1}{y}\, dK(y) \qquad [\, x > 0\,]. \tag{4.11}$$

Proof. We use several properties of LSt's of LSt-able functions; they can be found in Section A.3. Let ρ be the ρ-function of the pLSt π of X. Then by Proposition A.3.3 ℓ_X is given by $\lim_{s\to\infty} \rho(s)$. Since in our case $\rho = \widehat{K}$, it follows that $\ell_X = K(0)$. The equality in (4.10) is now proved by using

(4.9), the monotone convergence theorem and the fact that $\mathbb{P}(X = \ell_X)$ can be obtained as $\lim_{s\to\infty} \pi(s)\, \mathrm{e}^{s\ell_X}$. The final assertion follows by rewriting (4.9); see also Theorem 3.2. ◻

Proposition 4.5. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with canonical function K. Then:*

(i) *For $t > 0$ the canonical function K_t of $F^{\star t}$ is given by $K_t = t\,K$.*

(ii) *If F_1 and F_2 are infinitely divisible distribution functions on $\mathbb{R}_+$ with canonical functions K_1 and K_2, respectively, then $F = F_1 \star F_2$ iff $K = K_1 + K_2$.*

Proof. Both statements immediately follow from representation (4.4) and the fact that the canonical function is unique. ◻

Proposition 4.6. *Let $F, F_1, F_2, \ldots$ be infinitely divisible distribution functions on $\mathbb{R}_+$ with canonical functions $K, K_1, K_2, \ldots$, respectively. Then (as $n \to \infty$):*

(i) *If $F_n \to F$ weakly, then $K_n \to K$ at all continuity points of K, and there exists a nonnegative function σ on $(0,\infty)$ such that $\widehat{K}_n \le \sigma$ for every n.*

(ii) *If $K_n \to K$ at all continuity points of K, and there exists a nonnegative function σ on $(0,\infty)$ with $\int_0^s \sigma(u)\,du < \infty$ for $s > 0$ such that $\widehat{K}_n \le \sigma$ for every n, then $F_n \to F$ weakly.*

Proof. We use the continuity theorem and an extended version of it, as stated in Theorems A.3.1 and A.3.5, and put $\widehat{F} =: \pi$, $\widehat{F}_n =: \pi_n$, $\widehat{K} =: \rho$ and $\widehat{K}_n =: \rho_n$. First, let $F_n \to F$ weakly. Then $\pi_n \to \pi$ and $\pi_n' \to \pi'$, so by (4.8) we have

$$\rho_n(s) = \frac{-\pi_n'(s)}{\pi_n(s)} \longrightarrow \frac{-\pi'(s)}{\pi(s)} = \rho(s) \qquad [\, s > 0\,],$$

which implies the conclusion in (i) for K_n and K. Conversely, if we start from such functions K_n and K, where now, in addition, the bound σ is integrable over $(0, s\,]$ for $s > 0$, then $\rho_n \to \rho$ and we can use the dominated convergence theorem, together with (4.3), to conclude that for $s > 0$

$$\pi_n(s) = \exp\Big[-\int_0^s \rho_n(u)\,\mathrm{d}u\Big] \longrightarrow \exp\Big[-\int_0^s \rho(u)\,\mathrm{d}u\Big] = \pi(s),$$

so $F_n \to F$ weakly. ◻

The need for the integrability condition on the bound σ in (ii) is shown e.g. by the special case where $K_n = \mathrm{e}^{\sqrt{n}}\, 1_{[n,\infty)}$.

Example 4.7. The *degenerate* (λ) distribution of Example 2.5 with pLSt π given by

$$\pi(s) = \mathrm{e}^{-\lambda s},$$

has canonical function $K = \lambda\, 1_{\mathbb{R}_+}$; just use (4.9). □

Example 4.8. The *gamma* (r, λ) distribution of Example 2.6 with pLSt π given by

$$\pi(s) = \Big(\frac{\lambda}{\lambda + s}\Big)^r,$$

has a canonical function K that is absolutely continuous with density k given by

$$k(x) = r\,\mathrm{e}^{-\lambda x} \qquad [\,x > 0\,].$$

This follows from (4.8); the ρ-function of π is $\rho(s) = r/(\lambda + s)$. □

Example 4.9. For $\lambda > 0$, $\gamma > 0$, let π be the function on $(0, \infty)$ given by

$$\pi(s) = \exp\,[-\lambda\, s^\gamma\,] \qquad [\,s > 0\,].$$

In case $\gamma > 1$ this function is *not* a pLSt; it is not log-convex. For $\gamma = 1$ we get Example 4.7. So, let $\gamma < 1$. Then we apply Theorem 4.1 and compute the ρ-function of π: $\rho(s) = \lambda\gamma\, s^{\gamma-1}$ for $s > 0$, so ρ is completely monotone. We conclude that π is an *infinitely divisible pLSt* with (completely monotone) canonical density k given by

$$k(x) = \frac{\lambda\gamma}{\Gamma(1-\gamma)}\, x^{-\gamma} \qquad [\,x > 0\,];$$

see (4.8). For $\lambda > 0$ and $\gamma \le 1$ the pLSt π, and the corresponding distribution, will be called *stable* (λ) *with exponent* γ; this terminology is justified by the results in Section V.3 where the stable distributions on $\mathbb{R}_+$ are studied separately. □

We make another use of the criterion of Theorem 4.1; it yields the following useful characterization of infinite divisibility on $\mathbb{R}_+$ by means of a *functional equation* for the distribution function.

Theorem 4.10. *A distribution function F on $\mathbb{R}_+$ is infinitely divisible iff it satisfies*

$$(4.12) \qquad \int_{[0,x]} u \, dF(u) = \int_{[0,x]} F(x-u) \, dK(u) \qquad [\, x \geq 0 \,]$$

for some LSt-able function K. In this case K is the canonical function of F.

PROOF. Apply Theorem 4.1. First, suppose that F is infinitely divisible. Then the ρ-function of $\widehat{F}$ is completely monotone; by (4.8) it is the LSt of the canonical function K of F. It follows that F satisfies

$$(4.13) \qquad -\widehat{F}'(s) = \widehat{F}(s)\, \widehat{K}(s) \qquad [\, s > 0 \,];$$

Laplace inversion now yields the functional equation (4.12). The converse is proved similarly; if F satisfies (4.12), then we have (4.13), and hence the ρ-function of $\widehat{F}$ equals $\widehat{K}$, which is completely monotone. ◫

The functional equation in this theorem can be used to get information on *Lebesgue properties* of an infinitely divisible distribution function F; we will give necessary and/or sufficient conditions in terms of the canonical function K for F or $F - F(0)$ to be *discrete* or *continuous* or *absolutely continuous*. Here we use, without further comment, some properties from Section A.2, and we take $\ell_F = 0$, which is no essential restriction. Then, by Proposition 4.4, $K(0) = 0$, so K is continuous at 0. If K is continuous everywhere, then so is the convolution $F \star K$, and hence, by the functional equation (4.12), $F - F(0)$ is continuous. In case $F(0) > 0$ the converse also holds. This can be seen by writing

$$(4.14) \qquad F \star K = \big(F - F(0)\big) \star K + F(0)\, K;$$

if $F - F(0)$ is continuous, then so are $\big(F - F(0)\big) \star K$ and, by (4.12), $F \star K$, which because of (4.14) implies the continuity of K, as $F(0) \neq 0$. In a similar way one shows that if K is absolutely continuous, then so is $F - F(0)$, and that in case $F(0) > 0$ the converse holds. When $F(0) > 0$, one can also obtain these results by using Proposition 4.4: Then we have $\lambda := \int_{(0,\infty)} (1/x)\, \mathrm{d}K(x) < \infty$ and for Borel sets B

$$(4.15) \qquad m_F(B) = \sum_{n=1}^{\infty} \Big(\frac{\lambda^n}{n!}\, \mathrm{e}^{-\lambda} \Big)\, m_{G^{\star n}}(B) \qquad \big[\, B \subset (0,\infty) \,\big],$$

where G is the distribution function given by (4.11); now apply (4.15) twice, with $B = \{b\}$ and with B having Lebesgue measure zero: $m(B) = 0$. By this approach it also follows that if $F(0) > 0$ and K is not continuous, then the set of discontinuity points of F is unbounded; cf. the remark following Proposition I.2.2. In a similar way part (iii) of the following summarizing proposition is obtained.

Proposition 4.11. *Let F be an infinitely divisible distribution function with $\ell_F = 0$ and with canonical function K, and let $F(0) > 0$ or, equivalently, let K satisfy $\int_{(0,\infty)} (1/x)\, dK(x) < \infty$. Then:*

(i) *$F - F(0)$ is continuous iff K is continuous.*

(ii) *$F - F(0)$ is absolutely continuous iff K is absolutely continuous.*

(iii) *F is discrete iff K is discrete.*

Part (iii) of this proposition leads to the following characterization of the *infinitely divisible distributions on $\mathbb{Z}_+$* which are considered separately in Chapter II; it follows by comparing the canonical representations (II.4.3) and (4.4).

Proposition 4.12. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with canonical function K. Then F corresponds to an infinitely divisible distribution $(p_j)_{j\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ (with $p_0 > 0$) iff K is discrete with discontinuities restricted to $\mathbb{N}$ and $\int_{(0,\infty)} (1/x)\, dK(x) < \infty$. In this case the canonical sequence $(r_j)_{j\in\mathbb{Z}_+}$ of (p_j) and the canonical function K of F are related by*

$$(4.16) \qquad r_j = K(j+1) - K(j), \quad K(x) = \sum_{j=0}^{\infty} r_j\, 1_{[j+1,\infty)}(x).$$

When $F(0) = 0$ or, equivalently, $\int_{(0,\infty)} (1/x)\, \mathrm{d}K(x) = \infty$, the situation is much more complicated. As we saw above, the continuity of K then is sufficient for the continuity of F; similarly for absolute continuity. Applying these results for the continuous and absolutely continuous components of K and using Proposition 4.5 (ii), we get the following sufficient conditions for continuity and absolute continuity of F.

Proposition 4.13. *Let F be an infinitely divisible distribution function with $\ell_F = 0$ and with canonical function K. Then:*

(i) *F is continuous if the continuous component K_c of K has the property that $\int_{(0,\infty)} (1/x)\, dK_c(x) = \infty$.*

(ii) *F is absolutely continuous if the absolutely continuous component K_{ac} of K has the property that $\int_{(0,\infty)} (1/x)\, dK_{ac}(x) = \infty$.*

Part (i) of this proposition can be improved, however, considerably, as follows.

Theorem 4.14. *Let F be an infinitely divisible distribution function with $\ell_F = 0$ and with canonical function K. Then F is continuous iff it is continuous at zero, so iff $\int_{(0,\infty)} (1/x)\, dK(x) = \infty$.*

Clearly, for proving this theorem it is sufficient to show that an infinitely divisible distribution function F with $\ell_F = 0$ satisfies

$$(4.17) \qquad F \text{ not continuous} \implies F(0) > 0, \text{ i.e., } \int_{(0,\infty)} \frac{1}{x}\, dK(x) < \infty.$$

Unfortunately, attempts to apply the functional equation (4.12) for this only led to partial results. For instance, if D_F is the set of discontinuity points of F, then

$$(4.18) \qquad x_0 \in D_F \text{ with } x_0 > 0 \implies \exists\, x_1 \in D_F : x_1 < x_0.$$

In fact, if $x_0 \in D_F$ with $x_0 > 0$, then by (4.12) $\big($note that $K(0) = 0\big)$

$$m_{F\star K}\big(\{x_0\}\big) = \sum_{x \in D_F \cap [0,x_0)} m_F\big(\{x\}\big)\, m_K\big(\{x_0 - x\}\big) > 0.$$

It follows that in proving (4.17) we may restrict ourselves to *discrete* distribution functions F. To see this, let F be infinitely divisible with $\ell_F = 0$ and $D_F \neq \emptyset$, and apply (4.17) to $F_{\rm d}(\,\cdot + x_0)$, where $F_{\rm d}$ is the discrete component of F and $x_0 := \ell_{F_{\rm d}}$; note that $F_{\rm d}$ is infinitely divisible, too, because of Proposition I.2.2. Then $F_{\rm d}(x_0) > 0$, so $x_0 \in D_F$. Now, because of (4.18), assuming $x_0 > 0$ would contradict the definition of x_0; so $x_0 = 0$, and hence $F(0) > 0$. Not able to prove (4.17) (for discrete F) using $\mathbb{R}_+$-methods, we appeal to its $\mathbb{R}$-counterpart to be given in Theorem IV.4.20 (which indeed is not based on Theorem 4.14): The Lévy function M of an infinitely divisible distribution function F that is not continuous, satisfies $M(0+) > -\infty$. Implication (4.17) then follows because the integral there equals $-M(0+)$ if $\ell_F \geq 0$; cf. Corollary IV.4.14. Now Theorem 4.14 is proved. Combining it with Theorem 3.2 yields the following generalization of the latter result.

Theorem 4.15. *Let F be an infinitely divisible distribution function with $\ell_F = 0$. Then F has at least one discontinuity point iff F is compound-Poisson.*

Turning to part (ii) of Proposition 4.13 we consider the special case where K itself is absolutely continuous, in more detail.

Proposition 4.16. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$, and suppose that its canonical function K is absolutely continuous with density k satisfying $\int_0^\infty (1/x)\,k(x)\,dx = \infty$. Then F is absolutely continuous and has a unique density f for which*

$$(4.19) \qquad x\,f(x) = \int_0^x f(x-u)\,k(u)\,du \qquad [\,x > 0\,].$$

PROOF. Note that $\ell_F = 0$ because $K(0) = 0$; the condition on k yields $F(0) = 0$. As K is absolutely continuous with density k, $F \star K$ $(= K \star F)$ is absolutely continuous with density $h := k \star F$, i.e.,

$$h(x) = \int_{[0,x)} k(x-u)\,\mathrm{d}F(u) \qquad [\,x > 0\,];$$

see Section A.2, also for the convolutions $\star$ and $*$. Now, according to Theorem 4.10 we have $F \star K = G$, where $G(x) := \int_{[0,x]} u\,\mathrm{d}F(u)$ for $x > 0$. It follows that G is absolutely continuous with density h, and hence, as $F(0) = 0$, F is absolutely continuous with density f given by $f(x) = h(x)/x$ for $x > 0$. Clearly, h can now be rewritten as $h = k * f = f * k$; this yields the functional equation (4.19) for f. ◻

There seems to be no simple necessary and sufficient condition on K for F to be absolutely continuous. In case of absolute continuity, however, infinite divisibility can be characterized by means of a functional equation for a density similar to (4.19), as follows; 'almost all' is, of course, with respect to Lebesgue measure m.

Theorem 4.17. *Let F be an absolutely continuous distribution function on $\mathbb{R}_+$ with density f. Then F is infinitely divisible iff f satisfies*

$$(4.20) \qquad x\,f(x) = \int_{[0,x)} f(x-u)\,dK(u) \qquad [\,\text{almost all } x > 0\,]$$

for some LSt-able function K. In this case K is the canonical function of F; if f can be chosen to be continuous on $\mathbb{R}$ (so $f(0+) = 0$ also when $\ell_F = 0$), then the equality in (4.20) holds for all $x > 0$.

PROOF. Let F be infinitely divisible with canonical function K, and let G be as in the proof of the preceding proposition. Then G is absolutely continuous; as is well known (m is σ-finite), its densities are equal m-almost-everywhere (m-a.e.). One density is given by g with $g(x) = x\, f(x)$ for $x > 0$. Since by Theorem 4.10 we have $G = F \star K$, another is given by $h := f \star K$, i.e.,

$$h(x) = \int_{[0,x)} f(x-u)\,\mathrm{d}K(u) \qquad [\,x > 0\,].$$

It follows that $g = h$ m-a.e., i.e., (4.20) holds. The converse is trivial; if $g = h$ m-a.e., then $G = F \star K$, so F is infinitely divisible.
Finally, let F be infinitely divisible having a density f that is continuous on $\mathbb{R}$. Then f is bounded on every finite interval, so the dominated convergence theorem can be applied to conclude that h is continuous on $\mathbb{R}$. Since g is also continuous and by (4.20) $g = h$ m-a.e., it follows that $g = h$ everywhere. ◻◻

Before using the criterion of Theorem 4.1 for obtaining *closure properties*, in the next section we briefly consider the subclass of compound-exponential distributions, and show that analogues can be given of several results from the present section.

5. Compound-exponential distributions

The *compound-exponential* distributions, introduced in Section 3, are particularly interesting. They occur quite frequently in practice, e.g., in queueing situations (cf. Chapter VII), and they contain the distributions with completely monotone or log-convex densities, the infinite divisibility of which will be proved in Section 10. In the present section we focus on another interesting aspect; the class of compound-exponential distributions turns out to have many properties very similar to the class of *all* infinitely divisible distributions.

To show this we recall that a compound-exponential pLSt π is determined by (and determines) an infinitely divisible pLSt π_0, the *underlying* pLSt of π, as follows:

(5.1) $$\pi(s) = \frac{1}{1 - \log \pi_0(s)}, \quad \text{so} \quad \pi_0(s) = \exp\bigl[1 - 1/\pi(s)\bigr].$$

Now observe that the ρ-function of π_0, i.e., the function $\rho_0 = -(\log \pi_0)'$, can be expressed in terms of π as follows:

$$(5.2) \qquad \rho_0(s) = \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{\pi(s)} = -\frac{\pi'(s)}{\{\pi(s)\}^2} \qquad [\,s > 0\,].$$

This function will be called the ρ_0-*function* of π, also for functions π that are not yet known to be pLSt's (but are such that (5.2) makes sense). The following analogue to Theorem 4.1 is easily proved by applying this theorem to π_0 in (5.1).

Theorem 5.1. *Let π be a positive, differentiable function on $(0,\infty)$ with $\pi(0+) = 1$. Then π is the pLSt of a compound-exponential distribution on $\mathbb{R}_+$ iff its ρ_0-function is completely monotone.*

Let π be a compound-exponential pLSt with corresponding distribution function F and canonical function K. Let F_0 and K_0 be the analogous quantities for the underlying infinitely divisible pLSt π_0 in (5.1). From (5.2) it is seen that the ρ- and ρ_0-function of π are related by

$$(5.3) \qquad \rho(s) = \pi(s)\,\rho_0(s) \qquad [\,s > 0\,].$$

Now use the fact that the left extremity ℓ_F of F can be obtained as $\ell_F = \lim_{s\to\infty} \rho(s)$, and similarly for ℓ_{F_0}; cf. Proposition A.3.3. Then one sees that $\ell_F = F(0)\,\ell_{F_0}$, so $\ell_F = 0$ also when $F(0) = 0$. Moreover, combining (5.3) with (4.8) shows that

$$(5.4) \qquad K = F \star K_0\,;$$

since by Proposition I.2.3 F has unbounded support, it follows that the same holds for K. We summarize.

Proposition 5.2. *Let F be a compound-exponential distribution function on $\mathbb{R}_+$ with canonical function K. Then F has zero left extremity ℓ_F (and hence $K(0) = 0$), and K has unbounded support.*

An analogue to Theorem 4.10 is obtained, as in the proof of this theorem, by rewriting (5.2) as

$$(5.5) \qquad -\pi'(s) = \{\pi(s)\}^2\,\rho_0(s) \qquad [\,s > 0\,],$$

and using Theorem 5.1 and Bernstein's theorem. The resulting functional equation will be used and interpreted in Section VII.5 on renewal processes.

Theorem 5.3. *A distribution function F on $\mathbb{R}_+$ is compound-exponential iff it satisfies*

$$\int_{[0,x]} u\, dF(u) = \int_{[0,x]} F^{\star 2}(x-u)\, dK_0(u) \qquad [\, x \geq 0\,] \tag{5.6}$$

for some LSt-able function K_0. In this case the LSt of K_0 equals the ρ_0-function of $\widehat{F}$, and necessarily

$$\int_{(1,\infty)} \frac{1}{x}\, dK_0(x) < \infty. \tag{5.7}$$

The degenerate distribution of Example 2.5 is *not* compound-exponential because of Proposition 5.2. The gamma distribution of Example 2.6 turns out not to be compound-exponential for all values of the shape parameter.

Example 5.4. The *gamma* (r, λ) distribution with pLSt π given by

$$\pi(s) = \Big(\frac{\lambda}{\lambda + s}\Big)^r,$$

is *compound-exponential* iff $r \leq 1$. This follows from Theorem 5.1; the ρ_0-function of π is given by $\rho_0(s) = r\lambda^{-r}\,(\lambda + s)^{r-1}$, which is completely monotone iff $r \leq 1$. In case $r < 1$ the canonical function K_0 of the underlying infinitely divisible distribution, which satisfies $\widehat{K}_0 = \rho_0$, is absolutely continuous with density k_0 given by

$$k_0(x) = \frac{r\lambda^{-r}}{\Gamma(1-r)}\, x^{-r}\, \mathrm{e}^{-\lambda x} \qquad [\, x > 0\,].$$

In the exponential case where $r = 1$, we have $K_0 = (1/\lambda)\, 1_{\mathbb{R}_+}$. ◫

Next we consider the subclass of compound-exponential distributions on $\mathbb{R}_+$ with positive mass at zero; according to Theorem 3.7 they correspond to the *compound-geometric* distributions on $\mathbb{R}_+$. Their pLSt's π can be characterized by the complete monotonicity of a function that is somewhat simpler than the ρ_0-function of π; cf. Theorem 5.1. This is due to the fact that now $\pi(\infty) := \lim_{s\to\infty} \pi(s) > 0$; the complete monotonicity of ρ_0 is equivalent to that of the function $s \mapsto 1/\pi(\infty) - 1/\pi(s)$, so to that of the *$\sigma$-function* of π, which is defined by

$$\sigma(s) := 1 - \frac{\pi(\infty)}{\pi(s)} \qquad [\, s > 0\,]. \tag{5.8}$$

We may start here from any function π on $(0,\infty)$ for which (5.8) makes sense. Thus we arrive at the following criterion; it may also be derived directly from representation (3.6) for a compound-geometric pLSt π.

Theorem 5.5. *Let π be a positive function on $(0,\infty)$ with $\pi(0+) = 1$ and such that $\pi(\infty) := \lim_{s\to\infty} \pi(s)$ exists in $(0,\infty)$. Then π is the pLSt of a compound-geometric distribution on $\mathbb{R}_+$ iff its σ-function is completely monotone.*

The relation between a pLSt π and its σ-function can be rewritten as

$$(5.9) \qquad \pi(s) - \pi(\infty) = \pi(s)\,\sigma(s) \qquad [\, s > 0\,].$$

Laplace-inversion, together with Bernstein's theorem and Theorem 5.5, now yields the following characterization of the compound-geometric distributions on $\mathbb{R}_+$ by means of a *functional equation* for the distribution function.

Theorem 5.6. *A distribution function F on $\mathbb{R}_+$ is compound-geometric iff it satisfies $F(0) > 0$ and*

$$(5.10) \qquad F(x) - F(0) = \int_{[0,x]} F(x-u)\, dL(u) \qquad [\, x \geq 0\,]$$

for some LSt-able function L. In this case the LSt of L equals the σ-function of $\widehat{F}$, and necessarily $L(0) = 0$ and

$$(5.11) \qquad \lim_{x\to\infty} L(x) = 1 - F(0), \quad \textit{and hence} \;\; \lim_{x\to\infty} L(x) < 1.$$

Note that the condition $F(0) > 0$ in this theorem is essential; equation (5.10) with $F(0) = 0$ has only trivial solutions, whereas its analogue (4.12) characterizes *all* infinitely divisible distributions on $\mathbb{R}_+$, not only the compound-Poisson distributions.

Finally we state one instance of a Lebesgue property. Because of Proposition 5.2 we can apply Theorem 4.14 to conclude that a compound-exponential distribution function F on $\mathbb{R}_+$ is continuous iff it is continuous at zero. Combining this with Theorem 3.7 yields the following generalization of the latter result; cf. Theorem 4.15.

Theorem 5.7. *Let F be a compound-exponential distribution function on $\mathbb{R}_+$. Then F has at least one discontinuity point iff F is compound-geometric.*

6. Closure properties

In Propositions 2.1 and 2.2 we saw that the class of infinitely divisible pLSt's is closed under pointwise *products* and *limits*, provided these limits are pLSt's. Corollary 2.4 states that if π is an infinitely divisible pLSt, then so is π^a for all $a > 0$. In the following proposition this property is restated together with similar properties.

Proposition 6.1. *Let π be an infinitely divisible pLSt and let $a > 0$. Then π_a is an infinitely divisible pLSt as well, where π_a is any of the following functions:*

(i) $\pi_a(s) = \{\pi(s)\}^a$;

(ii) $\pi_a(s) = \pi(as)$;

(iii) $\pi_a(s) = \pi(a+s)/\pi(a)$;

(iv) $\pi_a(s) = \pi(a)\,\pi(s)/\pi(a+s)$.

Proof. Apply Theorem 4.1 twice in each of the four cases; note that we do not know in advance that the functions π_a in (i) and (iv) are pLSt's. The ρ-function ρ_a of π_a can be expressed in that of π by

$$\rho_a(s) = a\,\rho(s),\ \ a\,\rho(as),\ \ \rho(a+s),\ \ \rho(s) - \rho(a+s),$$

respectively. Now, use some elementary properties of completely monotone functions as listed in Proposition A.3.7. ◻◻

The pLSt's in (i), (ii) and (iii) can be interpreted in terms of random variables. Let X be a random variable with (infinitely divisible) pLSt π. Then π_a in (i) is the pLSt of $X(a)$, where $X(\cdot)$ is the continuous-time sii-process generated by X. The function in (ii) is, of course, the pLSt of the multiple aX of X. We note in passing that the more general multiple AX, where A is independent of X with $\mathbb{P}(A=a) = 1 - \mathbb{P}(A=0) = \alpha$ for some $\alpha \in [0,1]$, need not be infinitely divisible; take X degenerate, for instance. For a more interesting counter-example we refer to Section 11. Thus we have for $\alpha \in (0,1)$:

$$(6.1) \qquad \pi \text{ infinitely divisible } \not\Longrightarrow\ 1-\alpha+\alpha\,\pi \text{ infinitely divisible.}$$

In particular, it follows that the class of infinitely divisible distributions is not closed under mixing; see, however, Proposition 10.9 and Chapter VI.

Turning to π_a in (iii), we take independent sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of independent random variables such that $X_n \stackrel{\mathrm{d}}{=} X$ and $Y_n \stackrel{\mathrm{d}}{=} Y$ for all n, where Y is independent of X and exponentially distributed with parameter $\lambda = a$. Define $N := \inf\{n \in \mathbb{N} : X_n < Y_n\}$; then N is finite a.s. and for X_N we have $X_N \stackrel{\mathrm{d}}{=} (X \mid X < Y)$. Since

$$\mathbb{E}\, \mathrm{e}^{-sX} 1_{\{X<Y\}} = \int_{\mathbb{R}_+} \mathrm{e}^{-sx}\, \mathbb{P}(Y > x)\, \mathrm{d}F_X(x) = \pi(a+s),$$

and hence $\mathbb{P}(X < Y) = \pi(a)$, we conclude that the random variable X_N has pLSt π_a as given by (iii). Note that in terms of distribution functions the closure property (iii) reads as follows: If F is an infinitely divisible distribution function on $\mathbb{R}_+$, then so is F_a for every $a > 0$, where

$$F_a(x) = \frac{1}{\pi(a)} \int_{[0,x]} \mathrm{e}^{-ay}\, \mathrm{d}F(y) \qquad [\,x \geq 0\,]. \tag{6.2}$$

There does not seem to exist an obvious interpretation of the pLSt π_a in part (iv) of Proposition 6.1. We note that π_a is not necessarily a pLSt if π is not infinitely divisible. The infinite divisibility of π_a turns out to be not only necessary for the infinite divisibility of π but also sufficient.

Proposition 6.2. *Let $a > 0$, let π be a pLSt, and define*

$$\pi_a(s) := \frac{\pi(a)\, \pi(s)}{\pi(a+s)}.$$

Then π is infinitely divisible iff π_a is the pLSt of an infinitely divisible distribution.

Proof. Again, we use Theorem 4.1. As noted in the proof of Proposition 6.1, the ρ-functions of π and π_a are related by

$$\rho_a(s) = \rho(s) - \rho(a+s). \tag{6.3}$$

Hence, if ρ is completely monotone, then so is ρ_a. To prove the converse statement, we solve (6.3) for ρ, and find

$$\rho(s) = \sum_{k=0}^{\infty} \rho_a(ka+s) + \ell_F,$$

where we used the fact that $\lim_{s\to\infty} \rho(s) = \ell_F$ if F is the distribution function corresponding to π; cf. Proposition A.3.3. It follows that ρ is completely monotone if ρ_a is completely monotone. ◻

In the characterization of infinite divisibility just proved the condition that π_a is an infinitely divisible pLSt for a fixed $a > 0$, may be replaced by the condition that π_a is just a pLSt for *all* $a > 0$, or for all $a \in (0, \varepsilon)$ for some $\varepsilon > 0$. We formulate this result as a 'self-decomposability' characterization of infinite divisibility; cf. Chapter V.

Theorem 6.3. *A pLSt* π *is infinitely divisible iff for all* $a > 0$ *there exists a pLSt* π_a *such that*

$$(6.4) \qquad \pi(s) = \frac{\pi(a+s)}{\pi(a)}\, \pi_a(s).$$

PROOF. Suppose that for all $a > 0$ the pLSt π can be factorized in two pLSt's as in (6.4). Then for all $a > 0$ the function $s \mapsto \pi(s)/\pi(a+s)$ is completely monotone, and hence so is the function ρ_a defined by

$$\rho_a(s) := \frac{1}{a} \Big\{ \frac{\pi(s)}{\pi(a+s)} - 1 \Big\}.$$

Now let $a \downarrow 0$; since then $\rho_a(s) \to -\pi'(s)/\pi(s)$, it follows that the ρ-function of π is completely monotone. Hence π is infinitely divisible by Theorem 4.1. The converse statement immediately follows from Proposition 6.1 (iv) or Proposition 6.2. ◻◻

Note that for an infinitely divisible pLSt π both factors in (6.4) are infinitely divisible, while the second factor is even compound-Poisson because $\pi_a(s) \to \pi(a)\, \mathrm{e}^{a\ell_F} > 0$ as $s \to \infty$; see Proposition A.3.3 for the formula for ℓ_F that we used here.

The result of Proposition 3.5 can also be viewed as a closure property: If π and π_0 are pLSt's, then

$$(6.5) \qquad \pi,\ \pi_0 \text{ infinitely divisible} \implies \pi \circ (-\log \pi_0) \text{ infinitely divisible.}$$

For instance, taking here π exponential shows that if π_0 is an infinitely divisible pLSt, then so is π with $\pi(s) = 1/\big(1 - \log \pi_0(s)\big)$; π is compound-exponential. On the other hand, taking for π_0 a stable pLSt as considered in Example 4.9, we obtain the following useful special case.

Proposition 6.4. *If* π *is an infinitely divisible pLSt, then so is* $s \mapsto \pi(s^\gamma)$ *for every* $\gamma \in (0, 1]$.

The closure property of Proposition 6.1 (iii) holds for general $a \in \mathbb{R}$ when adapted as follows: If π is a pLSt with $\pi(a) < \infty$, then

$$(6.6)\qquad \pi \text{ infinitely divisible } \Longrightarrow \pi(a+\cdot)/\pi(a) \text{ infinitely divisible;}$$

the same proof can be used, but one can also use Proposition 2.3, as was done for deriving (2.5). Now, combining the closure properties (6.5) and (6.6) yields the following generalization.

Proposition 6.5. *Let π and π_0 be pLSt's, and let $a \in \mathbb{R}$ be such that $\pi(a) < \infty$. Then the following implication holds:*

$$(6.7)\qquad \pi,\, \pi_0 \textit{ infinitely divisible} \Longrightarrow \frac{\pi \circ (a - \log \pi_0)}{\pi(a)} \textit{ infinitely divisible.}$$

For a pLSt π one might consider more general compositions $\pi \circ \sigma$ with σ such that this composition is completely monotone. For this to be the case we only have the sufficient condition that σ' is completely monotone (cf. Proposition A.3.7 (vi)); as in the proof of Theorem 4.3 one shows, however, that σ then necessarily has the form $a - \log \pi_0$ with $a \in \mathbb{R}$ and π_0 an infinitely divisible pLSt.

7. Moments

Let X be an infinitely divisible $\mathbb{R}_+$-valued random variable with distribution function F. We are interested in the *moments* of X of any nonnegative order; for $r \geq 0$ let

$$\mu_r := \mathbb{E}X^r = \int_{\mathbb{R}_+} x^r \, \mathrm{d}F(x) \qquad [\leq \infty].$$

We will relate the μ_r to moments of the canonical function K of X. To this end we use the functional equation of Theorem 4.10:

$$(7.1)\qquad \int_{[0,x]} u \, \mathrm{d}F(u) = \int_{[0,x]} F(x-u) \, \mathrm{d}K(u) \qquad [x \geq 0],$$

and recall that K is an LSt-able function satisfying $\int_{(1,\infty)} (1/x) \, \mathrm{d}K(x) < \infty$. A first result is obtained by letting $x \to \infty$ in (7.1) or, equivalently, $s \downarrow 0$ in (4.13); then one sees that the expectation of X itself is given by

$$(7.2)\qquad \mathbb{E}X = \lim_{x \to \infty} K(x).$$

Another use of (7.1) shows that for $\alpha \geq 0$ the moment of order $r = \alpha + 1$ can be obtained as $\mu_{\alpha+1} = \int_{\mathbb{R}_+} x^\alpha \, \mathrm{d}(F \star K)(x)$, and hence, by (A.2.3),

$$\mu_{\alpha+1} = \int_{\mathbb{R}_+^2} (u+v)^\alpha \, \mathrm{d}F(u) \, \mathrm{d}K(v). \tag{7.3}$$

The moment of K of order $\alpha \geq 0$ is denoted by ν_α, so

$$\nu_\alpha := \int_{\mathbb{R}_+} x^\alpha \, \mathrm{d}K(x) \qquad [\leq \infty].$$

We first take $\alpha = n \in \mathbb{Z}_+$ in (7.3). Then using the binomial formula for $(u+v)^n$ one sees that the moment sequences $(\mu_n)_{n \in \mathbb{Z}_+}$ and $(\nu_n)_{n \in \mathbb{Z}_+}$ are related by

$$\mu_{n+1} = \sum_{j=0}^{n} \binom{n}{j} \mu_j \, \nu_{n-j} \qquad [n \in \mathbb{Z}_+]. \tag{7.4}$$

Since these relations have exactly the same structure as those in (A.2.20) for $(\mu_n)_{n \in \mathbb{Z}_+}$ and the sequence $(\kappa_n)_{n \in \mathbb{N}}$ of *cumulants* of F, one is led to the following result.

Theorem 7.1. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with canonical function K, and let $n \in \mathbb{Z}_+$. Then the $(n+1)$-st order moment μ_{n+1} of F is finite iff the n-th order moment ν_n of K is finite:*

$$\mu_{n+1} < \infty \iff \nu_n < \infty. \tag{7.5}$$

In this case the $(n+1)$-st order cumulant κ_{n+1} of F equals ν_n:

$$\kappa_{n+1} = \nu_n. \tag{7.6}$$

Corollary 7.2. *The cumulants of an infinitely divisible distribution on $\mathbb{R}_+$ (as far as they exist) are nonnegative.*

Next we use the basic formula (7.3) for general $\alpha \geq 0$ and show that the equivalence in (7.5) can be generalized as follows.

Theorem 7.3. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with canonical function K, and let $\alpha \geq 0$. Then the $(\alpha+1)$-st order moment $\mu_{\alpha+1}$ of F is finite iff the α-th order moment ν_α of K is finite:*

$$\mu_{\alpha+1} < \infty \iff \nu_\alpha < \infty. \tag{7.7}$$

PROOF. When $\alpha = 0$, by (7.2) or (7.3) we have $\mu_1 = \nu_0$, so let $\alpha > 0$ and apply (7.3). Since $v^\alpha \le (u+v)^\alpha \le 2^\alpha\,(u^\alpha + v^\alpha)$ for all $(u,v) \in \mathbb{R}_+^2$, it is then seen that

$$\nu_\alpha \le \mu_{\alpha+1} \le 2^\alpha\,(\mu_\alpha\,\nu_0 + \nu_\alpha). \tag{7.8}$$

The implication to the right in (7.7) now immediately follows. To prove the converse, suppose that $\nu_\alpha < \infty$, and let $\lfloor \alpha \rfloor = n$ ($\in \mathbb{Z}_+$). Then $\nu_n < \infty$ and hence, because of (7.5), $\mu_{n+1} < \infty$. But then we also have $\mu_\alpha < \infty$, and so by (7.8) $\mu_{\alpha+1} < \infty$. □

If we let $\alpha \in (-1,0)$ in Theorem 7.3, then we have to adapt the result somewhat. For such an α we set

$$\nu_\alpha := \int_{(0,\infty)} x^\alpha\,\mathrm{d}K(x), \quad \nu'_\alpha := \int_{(1,\infty)} x^\alpha\,\mathrm{d}K(x) \quad [\text{both } \le \infty],$$

and note that ν_α and ν'_α are simultaneously finite in case $\mathbb{P}(X = \ell_X) > 0$, since then $\int_{(0,1)} (1/x)\,\mathrm{d}K(x) < \infty$; see Proposition 4.4. Further, we need the following generalization of (7.3), which is proved in a similar way: For $\alpha > -1$ and $c \ge 0$

$$\mathbb{E}\,X^{\alpha+1}\,1_{\{X>c\}} = \int_{\{(u,v)\in\mathbb{R}_+^2\,:\,u+v>c\}} (u+v)^\alpha\,\mathrm{d}F(u)\,\mathrm{d}K(v). \tag{7.9}$$

Theorem 7.4. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with canonical function K, and let $\alpha \in (-1,0)$. Then the $(\alpha+1)$-st order moment $\mu_{\alpha+1}$ of F is finite iff the α-th order moment ν'_α of K, as defined above, is finite:*

$$\mu_{\alpha+1} < \infty \iff \nu'_\alpha < \infty. \tag{7.10}$$

PROOF. First, apply (7.9) with $c = 0$, and let $b \ge 0$ be such that $F(b) > 0$. Since $(u+v)^\alpha \ge (b+v)^\alpha$ for all $(u,v) \in [0,b] \times (0,\infty)$, it is then seen that

$$\mu_{\alpha+1} \ge F(b) \int_{(0,\infty)} (b+v)^\alpha\,\mathrm{d}K(v).$$

The implication to the right in (7.10) now immediately follows. Next, apply (7.9) with $c = 2$, and observe that

$$\big\{(u,v) \in \mathbb{R}_+^2 : u+v > 2\big\} \subset \big((1,\infty) \times [0,1]\big) \cup \big(\mathbb{R}_+ \times (1,\infty)\big);$$

since we have $(u+v)^\alpha \leq 1$ for all $(u,v) \in (1,\infty) \times [0,1]$ and $(u+v)^\alpha \leq v^\alpha$ for all $(u,v) \in \mathbb{R}_+ \times (1,\infty)$, it follows that

$$\mu_{\alpha+1} \leq \mathbb{E}\, X^{\alpha+1}\, 1_{\{X\leq 2\}} + \bigl\{1 - F(1)\bigr\}\, K(1) + \nu'_\alpha.$$

Clearly, this estimation proves the reverse implication in (7.10). □

As is well known, the $(\alpha+1)$-st order moment $\mu_{\alpha+1}$ of a distribution function F on $\mathbb{R}_+$ is finite iff the α-th order moment of the function with $\overline{F}$ as a density is finite, where $\overline{F} := 1 - F$ is the *tail function* corresponding to F. Hence for an infinitely divisible F Theorems 7.3 and 7.4 give information on the asymptotic behaviour of $\overline{F}(x)$ as $x \to \infty$. In Section 9 we will return to this and give more detailed results. But first, in the next section, we pay attention to the *support* of F.

8. Support

Let F be a distribution function on $\mathbb{R}_+$. As agreed in Section A.2, by the *support* $S(F)$ of F we understand the set of *points of increase* of F; equivalently, $S(F)$ is the smallest *closed* subset S of $\mathbb{R}_+$ with $m_F(S) = 1$. According to (A.3.1) the support of the convolution of two distribution functions on $\mathbb{R}_+$ is equal to the (direct) sum of the supports:

$$S(F \star G) = S(F) \oplus S(G). \tag{8.1}$$

For functions on $\mathbb{R}_+$ more general than distribution functions, such as LSt-able functions, the support is defined similarly and (8.1) still holds.

Let F be infinitely divisible with n-th order factor F_n, say, so $F = F_n^{\star n}$. Then from (8.1) it follows that

$$S(F) = S(F_n)^{\oplus n} \qquad [\,n \in \mathbb{N}\,]. \tag{8.2}$$

Also, if F is non-degenerate, then by Proposition I.2.3 $S(F)$ is *unbounded.* We can say more, and first consider the case where $F(0) > 0$. Then by Theorem 3.2 F is compound-Poisson and can be represented as in (4.15):

$$F(x) = \mathrm{e}^{-\lambda} + \sum_{n=1}^{\infty} \Bigl(\frac{\lambda^n}{n!}\,\mathrm{e}^{-\lambda}\Bigr)\, G^{\star n}(x) \qquad [\,x \geq 0\,], \tag{8.3}$$

where $\lambda > 0$ and G is a distribution function with $G(0) = 0$. From this it is immediately seen that

$$S(F) \supset \{0\} \cup \bigcup_{n=1}^{\infty} S(G^{\star n});$$

we need not have equality here, but we have, if we replace the union over n in the right-hand side by its closure. This is easily verified by showing that a non-zero element outside the closure cannot belong to $S(F)$; cf. the second part of the proof of Proposition A.2.1. Since for $t > 0$ the convolution power $F^{\star t}$ of F satisfies (8.3) with λ replaced by λt, it follows that its support $S(F^{\star t})$ does not depend on t, and hence, by (8.2), the support $S(F)$ of F is *closed under addition*. This can also be seen by applying (8.1): $S(G^{\star n}) = S(G)^{\oplus n}$ for all n, and hence $\bigcup_{n=1}^{\infty} S(G^{\star n})$ is equal to the additive semigroup generated by $S(G)$; notation: $\mathrm{sg}\,\{S(G)\}$. Here we can replace G by the canonical function K of F; since G and K are related as in (4.11), we have $S(G) = S(K)$. We conclude that

$$(8.4) \qquad S(F) = \{0\} \cup \overline{\mathrm{sg}\,\{S(K)\}}.$$

This relation implies that $S(F)$ is a proper subset of $\mathbb{R}_+$ as soon as K has a *positive* left extremity. If $\ell_K = 0$, however, then $S(F) = \mathbb{R}_+$; this is an immediate consequence of the following lemma (and the fact that $K(0) = 0$ if $F(0) > 0$).

Lemma 8.1. *Let K be a nonnegative nondecreasing function on $\mathbb{R}$ with $K(0) = 0$ and $\ell_K = 0$. Then the support $S(K)$ of K satisfies*

$$\overline{sg\,\{S(K)\}} = \mathbb{R}_+.$$

PROOF. Put $S := \mathrm{sg}\,\{S(K)\}$, and suppose that $\mathbb{R}_+ \setminus \overline{S} \neq \emptyset$. Then there exist $c > 0$ and $\varepsilon_0 > 0$ such that $(c - \varepsilon_0, c + \varepsilon_0) \cap \overline{S} = \emptyset$. On the other hand, since $K(0) = 0$ and $\ell_K = \inf S(K) = 0$, there is a sequence $(a_n)_{n \in \mathbb{N}}$ in $S(K)$ satisfying $0 < a_n \downarrow 0$ as $n \to \infty$. Hence we can take n_0 such that $a_{n_0} < 2\varepsilon_0$. This implies, however, that $m\, a_{n_0} \in (c - \varepsilon_0, c + \varepsilon_0)$ for some $m \in \mathbb{N}$, whereas $m\, a_{n_0} \in S$: a contradiction. ◻

This lemma also enables us to easily deal with the case of an infinitely divisible F *without mass at zero*. Supposing, as usual, that $\ell_F = 0$ (and hence $K(0) = 0$), we will show that also in this case the support of F

is all of $\mathbb{R}_+$. In doing so we start from the functional equation given by Theorem 4.10:

$$(8.5) \qquad \int_{[0,x]} u \, \mathrm{d}F(u) = (F \star K)(x) \qquad [x \geq 0],$$

where K is the canonical function of F. Since $F(0) = 0$, this equation implies that $S(F) = S(F \star K)$, so by (8.1) we have $S(F) = S(F) \oplus S(K)$. By iteration it follows that

$$S(F) = S(F) \oplus \bigcup_{n=1}^{\infty} S(K)^{\oplus n} = S(F) \oplus \operatorname{sg}\{S(K)\},$$

and hence, as $0 \in S(F)$ and $S(F)$ is closed, $\overline{\operatorname{sg}\{S(K)\}} \subset S(F)$. Since by Proposition 4.4 $F(0) = 0$ implies that $\int_0^1 (1/x) \, \mathrm{d}K(x) = \infty$, we have $\ell_K = 0$; an application of Lemma 8.1 now shows that $S(F) = \mathbb{R}_+$. We summarize; the first statement in the corollary also follows by applying Proposition 4.5 (i).

Theorem 8.2. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with $\ell_F = 0$ and with canonical function K. Then the support of F is equal to the closure of the additive semigroup generated by the support of K supplemented with 0:*

$$S(F) = \{0\} \cup \overline{\operatorname{sg}\{S(K)\}}.$$

Moreover, $S(F) = \mathbb{R}_+$ iff $\ell_K = 0$, which is the case if $F(0) = 0$.

Corollary 8.3. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with $\ell_F = 0$. Then $S(F^{\star t}) = S(F)$ for all $t > 0$, and $S(F)$ is closed under addition.*

From Theorem 8.2 it follows that an infinitely divisible distribution function F with $\ell_F = 0$ has an *interval of constancy* only when $\ell_K > 0$ (and hence $F(0) > 0$); in this case $F(x) = F(0)$ for all $x \in [0, \ell_K)$ and F may be constant on several other intervals. This is, of course, illustrated by the special case of distributions on $\mathbb{Z}_+$. For (arbitrary) distributions $p = (p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ the support of p reduces to

$$S(p) = \{k \in \mathbb{Z}_+ : p_k > 0\},$$

so the support of p and the set of zeroes of p are complementary sets in $\mathbb{Z}_+$. For densities f on $\mathbb{R}_+$ the situation is not that simple; a problem is that f is not uniquely defined. However, f can always be chosen such that $f(x) = 0$ for all $x \notin S(F)$; we then have

$$S(F) = \overline{\{x > 0 : f(x) > 0\}}.$$

Now, let F be infinitely divisible with $\ell_F = 0$ and with a density f that is *continuous* on $(0, \infty)$. Since by Theorem 8.2 $S(F) = \mathbb{R}_+$, for no interval (a, b) in $\mathbb{R}_+$ f vanishes on all of (a, b); nevertheless, it could have zeroes in $(0, \infty)$. It turns out, however, that this does not happen. To show this, we use the functional equation of Theorem 4.17; there is a set $C \subset (0, \infty)$ with $m\big((0, \infty) \setminus C\big) = 0$ such that

$$x\, f(x) = \int_{[0,x)} f(x - u)\, \mathrm{d}K(u) \qquad [\, x \in C\,]. \tag{8.6}$$

Theorem 8.4. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with $\ell_F = 0$ and with a density f that is continuous on $(0, \infty)$. Then*

$$f(x) > 0 \;\; \text{for all } x > 0.$$

Proof. Take $x_1 > 0$ such that $f(x_1) > 0$. We will show that f has no zeroes in the interval (x_1, ∞). The theorem will then be proved, because by the assumption that $\ell_F = 0$, we can choose x_1 arbitrarily small. Suppose that f does have a zero in (x_1, ∞); by the continuity (from the right) of f there is a smallest one, say x_0. Now use (8.6); then it is seen that

$$x\, f(x) \geq \int_{(x-x_0, x-x_1]} f(x - u)\, \mathrm{d}K(u) \qquad [\, x \in C,\; x > x_0\,].$$

Next let $x \downarrow x_0$ through C; as f is continuous on $(0, \infty)$ and hence bounded on every finite closed interval in $(0, \infty)$, we can then apply the dominated convergence theorem to conclude that

$$0 = x_0 f(x_0) \geq \int_{(0, x_0 - x_1]} f(x_0 - u)\, \mathrm{d}K(u).$$

Since $f(x) > 0$ for all $x \in [\, x_1, x_0)$, it follows that $K(x_0 - x_1) = 0$ and hence by (8.6) that $f(x) = 0$ for all $x \leq x_0 - x_1$. This contradicts our assumption that $\ell_F = 0$; we conclude that $f(x) > 0$ for all $x > x_1$. ◻

From the proof it will be clear that this non-zero result also holds for infinitely divisible densities f that are only right-continuous on $(0, \infty)$ and bounded on every finite closed interval in $(0, \infty)$.

9. Tail behaviour

For infinitely divisible densities f on $\mathbb{R}_+$ we cannot expect a simple asymptotic behaviour, because f does not necessarily satisfy $f(x) \to 0$ as $x \to \infty$; see Section 11 for an example. Therefore, we will not consider these *local* tail functions; we shall concentrate on the *global tail function* $\overline{F}$ of an infinitely divisible random variable X with distribution function F:

$$\overline{F}(x) := \mathbb{P}(X > x) = 1 - F(x) \qquad [x \geq 0].$$

We look at $\overline{F}$ in a rather crude way, and restrict ourselves to the asymptotic behaviour of $-\log \overline{F}(x)$ for $x \to \infty$; it will appear that it is determined by the *right extremity* r_K of the canonical function K of F.

We first take a look at the tails of infinitely divisible distributions on $\mathbb{R}_+$ *with positive mass at zero*; by Theorem 3.2 these are the *compound-Poisson* distributions on $\mathbb{R}_+$. So, let $X \stackrel{d}{=} S_N$, where $(S_n)_{n \in \mathbb{Z}_+}$ is an sii-process generated by a positive random variable Y, say (so $S_1 \stackrel{d}{=} Y$), and N is Poisson distributed and independent of (S_n). Then the tail function $\overline{F}$ of X can be written as

$$\overline{F}(x) = \sum_{n=1}^{\infty} \mathbb{P}(N = n)\, \mathbb{P}(S_n > x) \qquad [x \geq 0].$$

Hence, on the one hand we have

$$\text{(9.1)} \qquad \overline{F}(x) \geq \mathbb{P}(N = 1)\, \mathbb{P}(Y > x) \qquad [x \geq 0];$$

this means that an infinitely divisible distribution can have an arbitrarily thick tail. On the other hand, if $a > 0$ is such that $\mathbb{P}(Y > a) > 0$, then

$$\text{(9.2)} \qquad \overline{F}(ka) \geq \mathbb{P}(N = k)\, \big\{\mathbb{P}(Y > a)\big\}^k \qquad [k \in \mathbb{N}],$$

and hence, as by Lemma II.9.1 $-\log \mathbb{P}(N = k) \sim k \log k$ as $k \to \infty$,

$$\limsup_{k \to \infty} \frac{-\log \overline{F}(ka)}{k \log k} \leq 1.$$

Now, for $x > 0$ let $k_x \in \mathbb{N}$ be such that $(k_x - 1)\, a < x \leq k_x a$, so we have $\overline{F}(x) \geq \overline{F}(k_x a)$. Then observing that $(k_x \log k_x)/(x \log x) \to 1/a$ as $x \to \infty$ and letting a tend to the right extremity r_G of the distribution function G of Y, we conclude that

$$\text{(9.3)} \qquad \limsup_{x \to \infty} \frac{-\log \overline{F}(x)}{x \log x} \leq \frac{1}{r_G},$$

where $1/r_G := 0$ if $r_G = \infty$. When r_G is finite, $\overline{F}$ can also be estimated above. In fact, as $S_n \le n\, r_G$ a.s., we then have for $k \in \mathbb{N}$

$$(9.4)\qquad \overline{F}(k\, r_G) = \sum_{n=k+1}^{\infty} \mathbb{P}(N=n)\, \mathbb{P}(S_n > k\, r_G) \le \mathbb{P}(N > k),$$

and hence, as by Lemma II.9.1 $-\log \mathbb{P}(N > k) \sim k \log k$ as $k \to \infty$,

$$\liminf_{k\to\infty} \frac{-\log \overline{F}(k\, r_G)}{k \log k} \ge 1.$$

Now, similarly to the way (9.3) is obtained, this inequality implies that

$$(9.5)\qquad \liminf_{x\to\infty} \frac{-\log \overline{F}(x)}{x \log x} \ge \frac{1}{r_G}.$$

Finally, we combine the results (9.3) and (9.5), and use the relation between G and the canonical function K of F given by (4.11) to be able to replace r_G by the right extremity r_K of K. Thus we arrive at the following property of the tail function $\overline{F}$ of an infinitely divisible distribution function F on $\mathbb{R}_+$ with $F(0) > 0$ and with canonical function K:

$$(9.6)\qquad \lim_{x\to\infty} \frac{-\log \overline{F}(x)}{x \log x} = \frac{1}{r_K}.$$

This limiting result turns out to hold also for infinitely divisible distribution functions F with $\ell_F = 0$ *without mass at zero*. To show this we take such an F, and split its canonical function K as $K = K_1 + K_2$, where $K_2 = \{K - K(a)\}\, 1_{(a,\infty)}$ for some $a \in (0, r_K)$. Then from Proposition 4.5 (ii) it follows that if F_1 and F_2 are infinitely divisible distribution functions on $\mathbb{R}_+$ with canonical functions K_1 and K_2, respectively, then $F = F_1 \star F_2$ or, in terms of random variables, $X \overset{d}{=} X_1 + X_2$ with X_1 and X_2 independent. Now, as K_2 vanishes on $(0, a)$, F_2 has positive mass at zero, and hence its tail function $\overline{F}_2$ satisfies (9.6) with r_K replaced by r_{K_2}. By construction, however, we have $r_{K_2} = r_K$; moreover,

$$(9.7)\qquad \overline{F}(x) = \mathbb{P}(X_1 + X_2 > x) \ge \mathbb{P}(X_2 > x) = \overline{F}_2(x) \qquad [\,x \ge 0\,].$$

We conclude that (9.6) holds with 'lim' and '=' replaced by 'lim sup' and '≤'. For the reverse inequality we may suppose $r := r_K < \infty$. Then by the functional equation (4.12), for $x > 0$ we can estimate, also when $F(0) > 0$, as follows:

$$
\begin{aligned}
\overline{F}(x) &= \int_{(x,\infty)} \mathrm{d}F(u) = \int_{(x,\infty)} \frac{1}{u}\,\mathrm{d}(F\star K)(u) \le \\
&\le \frac{1}{x}\int_{(x,\infty)} \mathrm{d}(F\star K)(u) = \frac{1}{x}\int_{(0,r]} \overline{F}(x-u)\,\mathrm{d}K(u) \le \\
&\le \frac{1}{x}\,\overline{F}(x-r)\,K(r),
\end{aligned}
$$

and hence, as by (7.2) we have $K(r) = \lim_{x\to\infty} K(x) = \mathbb{E}X$,

$$
\overline{F}(x) \le \frac{\mathbb{E}X}{x}\,\overline{F}(x-r) \qquad [\,x>0\,]. \tag{9.8}
$$

Note that for $x \in (0,r)$ this inequality reduces to Markov's inequality. Now, take $x = kr$ in (9.8) with $k \in \mathbb{N}$; then by iteration it follows that

$$
\overline{F}(kr) \le \frac{(\mathbb{E}X/r)^k}{k!}\,\overline{F}(0) \qquad [\,k\in\mathbb{N}\,], \tag{9.9}
$$

i.e., for $\overline{F}(kr)$ we have obtained a Poisson-type upperbound. Therefore, we can proceed as before, including an application of Lemma II.9.1, and conclude that (9.6) holds with 'lim' and '=' replaced by 'lim inf' and '$\ge$'. Thus we have proved (9.6) as stated, also when $F(0) = 0$ and $\ell_F = 0$. For non-degenerate F the condition that $\ell_F = 0$ can be dropped; this is seen by applying the result to $H := F(\ell_F + \cdot)$ with $\ell_H = 0$. We summarize.

Theorem 9.1. *Let F be a non-degenerate infinitely divisible distribution function on $\mathbb{R}_+$ with canonical function K. Then the tail function $\overline{F}$ satisfies*

$$
\lim_{x\to\infty} \frac{-\log \overline{F}(x)}{x\,\log x} = \frac{1}{r_K}, \tag{9.10}
$$

where r_K is the right extremity of K, possibly ∞, in which case $1/r_K := 0$.

What is a bit surprising is not the *rate* at which the tails of infinitely divisible distributions tend to zero, but the *regularity*, i.e., the existence of a limit in (9.10). When r_K is finite, Theorem 9.1 gives the exact rate with which $-\log \overline{F}$ tends to infinity. In case $r_K = \infty$ we only know that

$$
\lim_{x\to\infty} \frac{-\log \overline{F}(x)}{x\,\log x} = 0. \tag{9.11}
$$

In general not more than this can be said; the convergence in (9.11) can be arbitrarily slow. For *compound-exponential* distributions, which by Proposition 5.2 satisfy (9.11), it can be shown, however, that $\{-\log \overline{F}(x)\}/x$ has a finite limit as $x \to \infty$; we shall not do this.

10. Log-convexity

This section not only parallels Section II.10 on log-convex (and log-concave) distributions on $\mathbb{Z}_+$, but also leans on it rather heavily. Several results for densities (of absolutely continuous distributions on $\mathbb{R}_+$) will be obtained by discretization and taking limits.

In Section II.10 we used the fact that for *any* distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ with $p_0 > 0$ the recurrence relations

$$(10.1)\qquad (n+1)\,p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \qquad [\,n\in\mathbb{Z}_+\,]$$

determine a sequence $(r_k)_{k\in\mathbb{Z}_+}$ of real numbers; in Theorem II.10.1 we showed that if (p_k) is log-convex, then these r_k are nonnegative, so that (p_k) is infinitely divisible. Consider the analogue of (10.1) for a density f on $(0,\infty)$ as given in Theorem 4.17:

$$(10.2)\qquad x\,f(x) = \int_{[0,x)} f(x-u)\,\mathrm{d}K(u) \qquad [\,\text{almost all } x>0\,].$$

Now it is *not* clear that for every f this functional equation determines a function K of bounded variation. Even if it did, it would be very difficult to show, along the lines of the proof of Theorem II.10.1, that if f is log-convex, then K is nondecreasing, so that f is infinitely divisible.

Thus it seems that the only way to deal with log-convex densities, is via discretization. There are, of course, several ways to write a distribution function F on $\mathbb{R}_+$ as the (weak) limit of a sequence of *lattice* distribution functions. The following one turns out to be convenient for our purposes:

$$(10.3)\qquad F(x) = \lim_{h\downarrow 0} F_h(x) \qquad [\,x\in\mathbb{R}\,],$$

where F_h corresponds to the distribution $\big(p_k(h)\big)_{k\in\mathbb{Z}_+}$ on the lattice $h\,\mathbb{Z}_+$ with

$$(10.4)\qquad p_k(h) = \begin{cases} F(h) & \text{, if } k=0,\\ F\big((k+1)h\big) - F(kh) & \text{, if } k\ge 1.\end{cases}$$

If F is absolutely continuous with density f, then we have

$$(10.5)\qquad p_k(h) = \int_{kh}^{(k+1)h} f(x)\,\mathrm{d}x = \int_0^h f(kh+\theta)\,\mathrm{d}\theta \qquad [\,k\in\mathbb{Z}_+\,].$$

Now, suppose that f is *log-convex* on $(0, \infty)$, i.e. (cf. Section A.3), f satisfies the inequalities

$$\text{(10.6)} \qquad \big\{f\big(\tfrac{1}{2}(x+y)\big)\big\}^2 \le f(x)\, f(y) \qquad [\,x > 0,\, y > 0\,].$$

Then for any $h > 0$ the lattice distribution $\big(p_k(h)\big)_{k \in \mathbb{Z}_+}$ is *log-convex* as well:

$$\text{(10.7)} \qquad \big\{p_k(h)\big\}^2 \le p_{k-1}(h)\, p_{k+1}(h) \qquad [\,k \in \mathbb{N}\,];$$

this follows from (10.5) and Proposition II.10.6, as in the proof of Proposition II.10.8. Conversely, if $\big(p_k(h)\big)$ in (10.5) is log-convex for all $h > 0$, then f is log-convex; this can be proved by observing that F is also the weak limit, as $h \downarrow 0$, of the distribution function with (log-convex) density f_h, where f_h is obtained from $\big(p_k(h)\big)$ by log-linear interpolation, i.e., by linear interpolation of $\big(\log p_k(h)\big)$. In fact, one can say somewhat more: If $F_n \to F$ weakly as $n \to \infty$ with $F(0) = 0$, and F_n has a log-convex, and hence nonincreasing, density f_n so that F_n is concave, then F is concave and hence has a nonincreasing density f; moreover, using Helly's selection theorem one shows that the log-convexity of f_n implies the log-convexity of f. Thus we have (briefly) shown the validity of the following useful lemma.

Lemma 10.1. *Let F be a distribution function on $\mathbb{R}_+$ with $F(0) = 0$. Then the following three assertions are equivalent:*

(i) *F has a density f that is log-convex on $(0, \infty)$.*

(ii) *F is the weak limit, as $h \downarrow 0$, of a distribution function F_h corresponding to a log-convex distribution $\big(p_k(h)\big)$ on the lattice $h\,\mathbb{Z}_+$.*

(iii) *F is the weak limit, as $n \to \infty$, of a distribution function F_n having a density f_n that is log-convex on $(0, \infty)$.*

Since results for distributions on $\mathbb{Z}_+$ have an obvious translation to distributions on the lattice $h\,\mathbb{Z}_+$, we can use this lemma to obtain results for log-convex densities on $(0, \infty)$ from their discrete counterparts. We start with a continuous version of Theorem II.10.1.

Theorem 10.2. *If F is a distribution function on $\mathbb{R}_+$ having a density f that is log-convex on $(0, \infty)$, then F is infinitely divisible.*

PROOF. Let F have a log-convex density f. By the lemma, F can then be seen as the weak limit, as $h \downarrow 0$, of a log-convex distribution $\big(p_k(h)\big)$ on the lattice $h\,\mathbb{Z}_+$. Now, on account of Theorem II.10.1 $\big(p_k(h)\big)$ is infinitely divisible on $h\,\mathbb{Z}_+$, and hence on $\mathbb{R}_+$. Since by Proposition 2.2 the set of infinitely divisible distributions on $\mathbb{R}_+$ is closed under weak convergence, we conclude that F is infinitely divisible as well. ◻◻

Next, we will give a sufficient condition in terms of the canonical function K for an infinitely divisible distribution function F to have a log-convex density f. In fact, we will prove a continuous version of Theorem II.10.2; in view of this we will suppose that K has a *log-convex* density k. In this case k is convex $\big($see (A.3.12)$\big)$, and hence nonincreasing because $\int_1^\infty (1/x)\,k(x)\,\mathrm{d}x < \infty$.

We first consider the general situation with a *nonincreasing* canonical density k in more detail; it is also of importance in Chapter V when studying the self-decomposable distributions on $\mathbb{R}_+$. Since $k(0+)$ then exists in $(0,\infty]$, we have $\int_0^\infty (1/x)\,k(x)\,\mathrm{d}x = \infty$, so by Proposition 4.16 F has a unique density f satisfying

$$(10.8)\qquad x\,f(x) = \int_0^x f(x-u)\,k(u)\,\mathrm{d}u \qquad [\,x>0\,].$$

It will become clear that for the behaviour of f the case $k(0+)=1$ is critical. To see why this might be so, we refer to Example 11.15. In the following proposition we state a special case of this example; it is needed in the proof of the subsequent theorem.

Proposition 10.3. *Let $r>0$, and let F be the infinitely divisible distribution function with $\ell_F = 0$ for which the canonical function K has (nonincreasing) density k given by $k = 1_{(0,r)}$. Then F has a density f that is constant on $(0,r)$, and continuous and nonincreasing on $(0,\infty)$; in particular, f is bounded on $(0,\infty)$.*

Theorem 10.4. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with canonical function K having a density k that is nonincreasing on $(0,\infty)$. Then F has a unique density f satisfying (10.8); f is continuous and positive on $(0,\infty)$, and it is bounded on $(0,\infty)$ if $k(0+)>1$. Moreover:*

(i) *If f is nonincreasing near 0 with $f(0+)=\infty$, then $k(0+)\le 1$.*

(ii) *If $f(0+)$ exists in $(0,\infty)$, then $k(0+)=1$.*

(iii) *If f is nondecreasing near 0 with $f(0+) = 0$, then $k(0+) \geq 1$.*

PROOF. Above we saw that F has a unique density f satisfying (10.8). By Theorem 8.4 f is positive on $(0, \infty)$ as soon as f is continuous on $(0, \infty)$. To show the continuity of f we denote the right-hand side of (10.8) by $h(x)$, and start with the case where $k(0+) \leq 1$; take k right-continuous. Then h can be written as

$$h(x) = k(0+) \{F(x) - (G \star F)(x)\} \qquad [x > 0],$$

where G is a distribution function on $\mathbb{R}_+$: $G(x) = 1 - k(x)/k(0+)$ for $x > 0$. From the continuity of F it now follows that h is continuous on $(0, \infty)$. Since $x\, f(x) = h(x)$ for $x > 0$, f is continuous as well.
Next, suppose that $k(0+) > 1$, possibly $k(0+) = \infty$. Then we can write k as the sum of two canonical densities k_1 and k_2 with k_1 as in the preceding proposition, where $r > 0$ is such that $k(x) \geq 1$ for $0 < x < r$. For $i = 1, 2$ let F_i be the infinitely divisible distribution function with canonical density k_i; since $k_i(0+) > 0$, F_i is absolutely continuous with density f_i, say. By Proposition 4.5 (ii) we have $F = F_1 \star F_2$, so, besides f, also the function $\widetilde{f}$ with

$$\widetilde{f}(x) := \int_0^x f_1(x - y)\, f_2(y)\, \mathrm{d}y \qquad [x > 0],$$

is a density of F. Now, by Proposition 10.3 f_1 can be chosen to be bounded and continuous on $(0, \infty)$. Clearly, it follows that $\widetilde{f}$ is bounded as well. It is also continuous. To see this, take $x > 0$ and $\varepsilon > 0$, and write

$$\big|\widetilde{f}(x + \varepsilon) - \widetilde{f}(x)\big| \leq \int_0^x \big|f_1(x + \varepsilon - y) - f_1(x - y)\big|\, f_2(y)\, \mathrm{d}y \,+ \\ + \int_x^{x+\varepsilon} f_1(x + \varepsilon - y)\, f_2(y)\, \mathrm{d}y.$$

Because of the boundedness of f_1 the second integral tends to zero as $\varepsilon \downarrow 0$ and we can apply the dominated convergence theorem, together with the continuity of f_1, to see that also the first integral tends to zero. Thus $\widetilde{f}$ is right-continuous on $(0, \infty)$; left-continuity is proved similarly. We finally return to h, which may be written as $h(x) = \int_0^x \widetilde{f}(x - u)\, k(u)\, \mathrm{d}u$ for $x > 0$. Since $\widetilde{f}$ here is bounded and continuous on $(0, \infty)$, we can proceed as above for $\widetilde{f}$ to show that h is continuous on $(0, \infty)$. It follows that also f is continuous. Hence $f = \widetilde{f}$, so f is bounded.

Finally, we prove (i), (ii) and (iii), and use the fact that $F(x)/x \to f(0+)$ as $x \downarrow 0$, if $f(0+)$ exists; similarly for K and $k(0+)$. First, let $f(0+)$ exist in $(0, \infty)$. Since by (10.8) and the monotonicity of k we have $x\, f(x) \geq k(x)\, F(x)$ for $x > 0$, it is then seen that $k(0+) \leq 1$. Now we can estimate as follows: $x\, f(x) \leq k(0+)\, F(x)$ for $x > 0$, so we also have $k(0+) \geq 1$. This proves (ii). Parts (i) and (iii) are shown similarly; use (10.8) and the monotonicity of f rather than that of k. ◻◻

We are now ready to return to the situation above where k is log-convex. From parts (i) and (ii) of the preceding theorem it then follows that if F has a density f that is log-convex, and hence continuous and nonincreasing, then necessarily $k(0+) \leq 1$. We show that the converse of this holds, too.

Theorem 10.5. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with canonical function K having density k. If k is log-convex, then F has a log-convex density f iff $k(0+) \leq 1$.*

Proof. Let k be log-convex; then k is positive and nonincreasing on $(0, \infty)$. Suppose that $k(0+) \leq 1$; we then have to show that F has a log-convex density f.
We first do so assuming that $k(0+) < 1$, and apply a discretization of K analogous to that of F given in the beginning of this section; for $h > 0$ set

$$r_j(h) := K\big((j+1)h\big) - K(jh) = \int_{jh}^{(j+1)h} k(x)\, \mathrm{d}x \qquad [\, j \in \mathbb{Z}_+\,],$$

and let K_h be the step function with $K_h(x) = 0$ for $x < h$ and with a jump $r_j(h)$ at $(j+1)h$ for $j \in \mathbb{Z}_+$, so K_h is a step function with discontinuities restricted to $h\,\mathbb{N}$. Since

$$\int_0^\infty \frac{1}{x}\, \mathrm{d}K_h(x) = \sum_{j=0}^\infty \frac{r_j(h)}{(j+1)h} \leq \frac{r_0(h)}{h} + \int_h^\infty \frac{1}{x}\, \mathrm{d}K(x) < \infty,$$

from Proposition 4.12 it follows that K_h is the canonical function of an infinitely divisible distribution function F_h that corresponds to a distribution $\big(p_j(h)\big)_{j \in \mathbb{Z}_+}$ on the lattice $h\,\mathbb{Z}_+$. Moreover, $K_h \to K$ pointwise as $h \downarrow 0$, and $\widehat{K}_h \leq \widehat{K}$ for $h > 0$ because $K_h \leq K$; cf. (A.3.2). Since $\int_0^s \widehat{K}(u)\, du = -\log \widehat{F}(s) < \infty$ for $s > 0$, from Proposition 4.6 (ii) it follows that $F_h \to F$ weakly as $h \downarrow 0$. In view of Lemma 10.1 it is therefore sufficient to show that $\big(p_j(h)\big)$ is log-convex for all $h > 0$ sufficiently small. To this end we note that, by the functional equation (4.12) applied to F_h, the sequences $\big(p_j(h)\big)$ and $\big(r_j(h)\big)$ are related by

$$(n+1)h\,p_{n+1}(h) = \sum_{j=0}^{n} p_j(h)\,r_{n-j}(h) \qquad [\,n \in \mathbb{Z}_+\,],$$

so $\big(p_j(h)\big)$ satisfies (10.1) with $r_j := r_j(h)/h$. Now, as k is log-convex, $\big(r_j(h)\big)$ is log-convex $\big($cf. the proof of (10.7)$\big)$, and hence so is (r_j). Applying Theorem II.10.2, we obtain the desired log-convexity of $\big(p_j(h)\big)$ if we can show that $r_1 \geq r_0^2$, i.e.,

$$h\,r_1(h) \geq \big\{r_0(h)\big\}^2.$$

At this point the condition $k(0+)<1$ comes in; it implies $k(h) \geq k(0+)\,k(x)$ for $x \in (0,h)$ and h sufficiently small. Since because of the log-convexity of k the function $k/k(\cdot + h)$ is nonincreasing, we also have $k(0+)\,k(y+h) \geq k(h)\,k(y)$ for $y > 0$. Hence for h sufficiently small

$$k(y+h) \geq k(x)\,k(y) \qquad \big[\,x, y \in (0,h)\,\big].$$

Integration (with respect to x and y) over $(0,h)^2$ yields the desired inequality, so $\big(p_j(h)\big)$ is log-convex. This completes the proof in case $k(0+) < 1$. Finally, consider the boundary case where $k(0+) = 1$. Then we write $K = \lim_{n\to\infty} K_n$ with $K_n := (1-1/n)\,K$. Since K_n has a log-convex density k_n satisfying $k_n(0+) < 1$, from what we just proved above, we conclude that the distribution function F_n with canonical function K_n has a log-convex density f_n. By Proposition 4.5 (i), however, we have $F_n = F^{\star(1-1/n)}$, so $F_n \to F$ weakly as $n \to \infty$; therefore we can apply Lemma 10.1 to conclude that f is log-convex as well. ◫

The *gamma* (r,λ) distribution provides a simple illustration of this theorem. Its canonical function K has a log-convex density k given by $k(x) = r\,\mathrm{e}^{-\lambda x}$ for $x > 0$; see Example 4.8. Since $k(0+) = r$, Theorem 10.5 yields the well-known fact that the gamma (r,λ) density is log-convex iff $r \leq 1$; cf. Example 5.4.

The gamma distribution with shape parameter $r \geq 1$ has a density f that is *log-concave* on $(0,\infty)$, i.e. (cf. Section A.2), f satisfies the inequalities

(10.9) $$\big\{f\big(\tfrac{1}{2}(x+y)\big)\big\}^2 \geq f(x)\,f(y) \qquad [\,x > 0,\ y > 0\,].$$

We note in passing that, contrary to log-convex densities, log-concave densities are not necessarily restricted to a half-line; for example, the normal densities are log-concave. Though many infinitely divisible densities are log-concave, *not* every log-concave density is infinitely divisible. Nevertheless, there is an analogue of Theorem 10.5.

Theorem 10.6. *Let F be an infinitely divisible distribution function on $\mathbb{R}_+$ with canonical function K having density k. If k is log-concave, then F has a log-concave density f iff $k(0+) \geq 1$.*

PROOF. Let k be log-concave. Then $k_1 := \log k$ is concave, so k_1 is monotone near zero with $k_1(0+) < \infty$. Hence k is monotone near zero with $k(0+) < \infty$, and k is continuous and bounded on finite intervals. Now, proceeding as for $\widetilde{f}$ in the proof of Theorem 10.4 one shows that a density f of F that satisfies (10.8), is continuous on $(0, \infty)$; rewrite the right-hand side of (10.8) as $\int_0^x k(x-u)\, f(u)\, \mathrm{d}u$. Moreover, one easily verifies that the assertions in (i), (ii), (iii) of Theorem 10.4 still hold for such an f. It follows that if f is log-concave, and hence monotone near zero with $f(0+) < \infty$ and continuous on $(0, \infty)$, then necessarily $k(0+) \geq 1$. For the converse statement we use the method of proof of Theorem 10.5: Discretize K in case $k(0+) > 1$, use Lemma 10.1 with 'log-convex' replaced by 'log-concave', and apply Theorem II.10.3. We don't give the details. ◫

The exponential density is *log-linear*: both log-convex and log-concave. Moreover, it is a simple example of a completely monotone density. Recall that a density f on $(0, \infty)$ is said to be *completely monotone* if it has alternating derivatives:

$$(10.10) \qquad (-1)^n f^{(n)}(x) \geq 0 \qquad [\, n \in \mathbb{Z}_+;\ x > 0\,].$$

By Bernstein's theorem (Theorem A.3.6) f is completely monotone iff there exists an LSt-able function G such that

$$(10.11) \qquad f(x) = \int_{\mathbb{R}_+} \mathrm{e}^{-\lambda x}\, \mathrm{d}G(\lambda) \qquad [\, x > 0\,].$$

Combining the Bernstein representation and Schwarz's inequality shows that a completely monotone function is log-convex; see Proposition A.3.8. Together with Theorem 10.2 this leads to the following result.

Theorem 10.7. *If a probability density f on $(0,\infty)$ is completely monotone, then it is log-convex, and hence infinitely divisible.*

The completely monotone densities can be identified within the class of infinitely divisible densities. We state the result without proof; it can be obtained from its discrete counterpart, Theorem II.10.5, by discretization again. An alternative proof will be given, however, in Chapter VI.

Theorem 10.8. *A distribution function F with $F(0) = 0$ has a completely monotone density iff it is infinitely divisible having a canonical density k such that $x \mapsto k(x)/x$ is completely monotone with Bernstein representation of the form*

$$\frac{1}{x}\,k(x) = \int_0^\infty e^{-\lambda x}\, v(\lambda)\, d\lambda \qquad [\,x > 0\,], \tag{10.12}$$

where v is a measurable function on $(0,\infty)$ satisfying $0 \le v \le 1$ and necessarily

$$\int_0^1 \frac{1}{\lambda}\, v(\lambda)\, d\lambda < \infty. \tag{10.13}$$

The theorem can be used, for instance, to easily show that the (log-convex) gamma density with shape parameter $r \le 1$ is completely monotone; cf. Example 4.8.

We return to log-convexity, and treat some *closure properties*. Clearly, log-convexity is preserved under *pointwise multiplication*: If f and g are log-convex densities on $(0,\infty)$, then so is $c\, f g$, where $c > 0$ is a norming constant, provided that the product function is integrable. The following consequence of Proposition A.3.10 is less trivial: If f and g are log-convex densities on $(0,\infty)$, then so is $\alpha\, f + (1-\alpha)\, g$ for every $\alpha \in [\,0,1\,]$. This result can be extended to general *mixtures*, because log-convexity is defined in terms of weak inequalities and hence is preserved under taking limits.

Proposition 10.9. *The class of log-convex densities on $(0,\infty)$ is closed under mixing: If, for every θ in the support Θ of a distribution function G, $f(\cdot\,;\theta)$ is a log-convex density on $(0,\infty)$, then so is f given by*

$$f(x) = \int_\Theta f(x;\theta)\, dG(\theta) \qquad [\,x > 0\,]. \tag{10.14}$$

This result can be used to prove the infinite divisibility of a large class of densities that occur in renewal theory.

Proposition 10.10. *Let F be a distribution function on $\mathbb{R}_+$ having a log-convex density f and finite first moment μ. Then the function g defined by*

$$g(x) = \frac{1}{\mu}\{1 - F(x)\} \qquad [x > 0],$$

is a probability density on $(0, \infty)$ that is log-convex and hence infinitely divisible.

PROOF. From (A.2.4) it follows that g is a probability density on $(0, \infty)$. Since g can be written as

$$g(x) = \frac{1}{\mu}\int_0^\infty f(x + \theta)\,\mathrm{d}\theta \qquad [x > 0],$$

it is seen that g is a mixture of log-convex densities on $(0, \infty)$, and hence is log-convex, by Proposition 10.9. □

Because of Proposition A.3.7 (iii) it will be no surprise that the results above also hold with 'log-convexity' replaced by 'complete monotonicity'; we don't give the details.

11. Examples

So far, we have only seen the *gamma* distributions (including *exponential*) and the *stable* distributions as concrete examples of infinitely divisible distributions on $\mathbb{R}_+$; the gamma distribution with shape parameter $r \le 1$ is compound-exponential, and has a log-convex, and even completely monotone, density. Now, in this section some more concrete examples are presented illustrating the scope of the results in the previous sections; the examples are not always in the same order as these results.

We start with some examples around the *exponential* distribution. Its pLSt π is given by $\pi(s) = \lambda/(\lambda + s)$ for some $\lambda > 0$, and has canonical representation

$$\text{(11.1)} \qquad \pi(s) = \exp\Big[-\int_0^\infty (1 - \mathrm{e}^{-sx})\,\frac{1}{x}\,k(x)\,\mathrm{d}x\Big],$$

with canonical density k given by $k(x) = \mathrm{e}^{-\lambda x}$ for $x > 0$; see Example 4.8.

Example 11.1. Let A and Y be independent, where A has a Bernoulli $(\frac{1}{2})$ distribution on $\{0,1\}$ and Y has a standard gamma (r) distribution with shape parameter $r=3$. Consider X such that $X \stackrel{d}{=} AY$. Then the pLSt π of X can be written as

$$\pi(s) = \tfrac{1}{2} + \tfrac{1}{2}\Big(\frac{1}{1+s}\Big)^3 = \\ = \frac{(2+s)(1+s+s^2)}{2(1+s)^3} = \Big(\frac{1}{1+s}\Big)^3 \Big/ \prod_{j=1}^{3} \frac{\mu_j}{\mu_j+s},$$

where $\mu_1 := 2$, $\mu_2 := \mu$ and $\mu_3 := \overline{\mu}$ with $\mu := \frac{1}{2}\{1+i\sqrt{3}\}$. Applying (11.1) for the exponential (λ) distribution, including an extension to complex λ, we conclude that π satisfies (11.1) with k given by

$$k(x) = 3\,\mathrm{e}^{-x} - \sum_{j=1}^{3} \mathrm{e}^{-\mu_j x} = 3\,\mathrm{e}^{-x} - \mathrm{e}^{-2x} - 2\,\mathrm{e}^{-\frac{1}{2}x} \cos\big(\tfrac{1}{2}x\sqrt{3}\big),$$

where $x>0$. For x sufficiently large and such that $\frac{1}{2}x\sqrt{3}$ is a multiple of 2π, however, $k(x)$ is *negative*. Hence, as noted in the comments on the canonical representation of Theorem 4.3, X is *not* infinitely divisible. Alternatively, the zero $-\mu$ of π is inside the half-plane of analyticity given by $\operatorname{Re} z > -1$, so X is *not* infinitely divisible by Theorem 2.8. Thus we have shown that a *mixture* of the form $1-\alpha+\alpha\,\pi_0$, with π_0 an infinitely divisible pLSt, is not necessarily infinitely divisible; cf. (6.1).
We note that changing the shape parameter into $r=2$ does yield *infinite divisibility* of X; as above, one shows that the pLSt π of X then satisfies (11.1) with k given by

$$k(x) = 2\,\mathrm{e}^{-x}\,(1-\cos x) \qquad [\,x>0\,],$$

which is nonnegative for all x. Similarly, when $r=1$, π satisfies (11.1) with

$$k(x) = \mathrm{e}^{-x}\,(1-\mathrm{e}^{-x}) \qquad [\,x>0\,].$$

□□

Example 11.2. Let Y have a standard exponential distribution, and consider X such that $X \stackrel{d}{=} \sqrt{Y}$. Then X has an absolutely continuous distribution with density f given by

$$f(x) = 2\,x\,\mathrm{e}^{-x^2} \qquad [\,x>0\,].$$

Since $\overline{F}(x) = \mathbb{P}(X>x) = \mathrm{e}^{-x^2}$, we have $\big(-\log \overline{F}(x)\big)/(x\log x) \to \infty$, so X is *not* infinitely divisible by Theorem 9.1. In a similar way one shows that if $X \stackrel{d}{=} |Y|$ with Y standard normal, then X is *not* infinitely divisible; its *half-normal* density g is given by

$$g(x) = \frac{2}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \qquad [\, x > 0\,].$$

If, however, $X \stackrel{d}{=} e^Y$ with Y standard normal, then the *log-normal* density h of X is given by

$$h(x) = \frac{1}{\sqrt{2\pi}}\, \frac{1}{x}\, e^{-\frac{1}{2}(\log x)^2} \qquad [\, x > 0\,],$$

and X turns out to be *infinitely divisible*. We cannot show this here, but will give a proof in Chapter VI. ◻◻

Example 11.3. Let Y have a standard exponential distribution, and consider X such that $X \stackrel{d}{=} Y^2$. Then X has an absolutely continuous distribution with density f given by

$$f(x) = \frac{1}{2\sqrt{x}}\, e^{-\sqrt{x}} \qquad [\, x > 0\,].$$

Now, differentiating $\log f$ twice, one sees that f is *log-convex* and hence *infinitely divisible*. Using some elementary properties of completely monotone functions as listed in Proposition A.3.7, one shows that f is even *completely monotone*. In the same way one can show that, more generally, a density f of the following form is *log-convex*, and even *completely monotone*:

$$f(x) = \frac{\alpha}{\Gamma(r)}\, x^{\alpha r - 1} \exp\,[-x^{\alpha}] \qquad [\, x > 0\,],$$

where $\alpha \in (0, 1\,]$ and $r \in (0, 1/\alpha\,]$. The corresponding distribution is known as the *generalized-gamma* (r, α) distribution; for $r = 1$ we get the *Weibull* distribution with parameter α. ◻◻

We proceed with some more examples of completely monotone or log-convex densities. The first one shows, among other things, that moments alone do not decide the infinite divisibility of a distribution. From subsequent examples it will appear that, though two densities may seem rather similar, the infinite divisibility of the one may be much harder to prove than that of the other.

Example 11.4. Consider the probability densities f_α on $(0,\infty)$ given by

$$f_\alpha(x) = \tfrac{1}{24}\,(1 - \alpha\,\sin x^{\frac14})\,\exp\left[-x^{\frac14}\right] \qquad [\,x > 0\,],$$

where $\alpha \in [\,0,1\,]$. Using integration by parts one can show that these densities all have the same moment sequence; the moment μ_n of order $n \in \mathbb{Z}_+$ is given by $\mu_n = \frac{1}{6}\,(4n+3)!$. As noted in Example 11.3, f_0 is *completely monotone* and hence *infinitely divisible*. Because of Theorem 8.4, however, f_1 is *not* infinitely divisible; $f_1(x) = 0$ for $x = (\frac12\pi + 2k\pi)^4$ with $k \in \mathbb{Z}_+$. By Proposition 2.2 it follows that there must be infinitely many α in $(0,1)$ for which f_α is *not* infinitely divisible. ◻◻

Example 11.5. For $r > 1$ consider the probability density f on $(0,\infty)$ given by

$$f(x) = (r-1)\,\frac{1}{(1+x)^r} \qquad [\,x > 0\,].$$

Since f has alternating derivatives, it is *completely monotone* and hence *infinitely divisible*. This distribution is known as the *Pareto* distribution. For $r = 2$ we obtain a density very similar to the *half-Cauchy* density g, which is given by

$$g(x) = \frac{2}{\pi}\,\frac{1}{1+x^2} \qquad [\,x > 0\,].$$

The *infinite divisibility* of this density is much harder to prove; see Chapter VI. ◻◻

Example 11.6. Consider the probability density f on $(0,\infty)$ given by

$$f(x) = \frac{1}{\mathrm{e}-1}\,\exp\bigl[-(x - \mathrm{e}^{-x})\bigr] \qquad [\,x > 0\,].$$

Differentiating $\log f$ twice, one sees that f is *log-convex* and hence *infinitely divisible*. Using some elementary properties of completely monotone functions as listed in Proposition A.3.7, one shows that f is even *completely monotone*.

Now, change a minus-sign into a plus-sign; consider the density g given by

$$g(x) = \frac{\mathrm{e}}{\mathrm{e}-1}\,\exp\bigl[-(x + \mathrm{e}^{-x})\bigr] \qquad [\,x > 0\,].$$

Then g is *log-concave*, and it can be shown that g is *infinitely divisible*; we will do so in Chapter VI, where a similar proof is given of the infinite divisibility of the half-Cauchy distribution. Note that g can be recognized as a *half-Gumbel* density. ◫

Example 11.7. Consider the probability density f on $(0, \infty)$ given by

$$f(x) = \tfrac{9}{14}\, \mathrm{e}^{-x} \sqrt{1 + 3\,\mathrm{e}^{-x}} \qquad [\, x > 0\,].$$

Then f is *log-convex*, and hence *infinitely divisible*, but *not* completely monotone. To see this, we write f as

$$f(x) = \tfrac{9}{7} \sqrt{\tfrac{1}{4}\, \mathrm{e}^{-2x} + \tfrac{3}{4}\, \mathrm{e}^{-3x}} = \tfrac{9}{7} \sqrt{\pi(x)},$$

where π is the pLSt of a Bernoulli distribution on $\{2, 3\}$. Since a pLSt is a log-convex function, f is a log-convex density. One easily verifies, however, that the distribution corresponding to π, is not 2-divisible; so $\sqrt{\pi}$, and hence f, is not completely monotone. ◫

Next we present some functions π of which we do not know in advance whether they are pLSt's or not. Still, most of them will be shown to be pLSt's of infinitely divisible distributions.

Example 11.8. Let π be the quotient of the pLSt's of two different exponential distributions, so π is of the form

$$\pi(s) = \frac{\lambda}{\lambda + s} \Big/ \frac{\lambda + a}{\lambda + a + s} \qquad [\, s > 0\,],$$

where $\lambda > 0$ and $a > -\lambda$, $a \neq 0$. We further take $a > 0$, since otherwise $\pi(s) > 1$ for large s so that π is not a pLSt. Then π is an *infinitely divisible* pLSt. To show this one can compute the ρ-function of π and apply Theorem 4.1. Alternatively, one can use Proposition 6.1 (iv); π can be written as

$$\pi(s) = \frac{\pi_1(a)\, \pi_1(s)}{\pi_1(a + s)} \quad \text{with} \;\; \pi_1(s) = \frac{\lambda}{\lambda + s}.$$

A third way is writing π as

$$\pi(s) = \frac{\lambda}{\lambda + a} + \frac{a}{\lambda + a}\, \frac{\lambda}{\lambda + s},$$

so π is a mixture of a distribution degenerate at zero and an exponential distribution; now proceed as in the last part of Example 11.1. By computing the σ-function of π and applying Theorem 5.5 one sees that π is even *compound-geometric.* ◻◻

Example 11.9. Let π be the function on $\mathbb{R}_+$ given by

$$\pi(s) = 1 + s - \sqrt{(1+s)^2 - 1} \qquad [\, s \geq 0\,].$$

In order to determine whether π is an infinitely divisible pLSt, we compute its ρ-function and find for $s > 0$

$$\rho(s) := -\frac{\mathrm{d}}{\mathrm{d}s} \log \pi(s) = \frac{1}{\sqrt{(1+s)^2 - 1}} = \frac{1}{\sqrt{s}}\, \frac{1}{\sqrt{2+s}}.$$

Clearly, ρ is completely monotone, so by Theorem 4.1 the function π is an *infinitely divisible pLSt.* An alternative way of showing this is noting that π can be written as

$$\pi(s) = \frac{1}{1+s}\, P\Big(\Big\{\frac{1}{1+s}\Big\}^2\Big) \quad \text{with} \quad P(z) := \frac{1 - \sqrt{1-z}}{z};$$

now use Propositions 2.1 and 3.1, and the fact (cf. Example II.11.11) that P is an infinitely divisible pgf. The function P is the pgf of the 'reduced' first-passage time $\frac{1}{2}\,(T_1 - 1)$ from 0 to 1 in the symmetric Bernoulli walk; see the beginning of Section VII.2. In Example VII.7.1 we shall see that π can be interpreted as the transform of a first-passage time as well. Finally, let I_r be the modified Bessel function of the first kind of order r; cf. Section A.5. Then it can be shown (see Notes) that π is the Lt of the density f given by

$$f(x) = \frac{1}{x}\, \mathrm{e}^{-x}\, I_1(x) = \frac{1}{\pi} \int_0^2 \mathrm{e}^{-\lambda x} \sqrt{\lambda\,(2-\lambda)}\, \mathrm{d}\lambda \qquad [\, x > 0\,],$$

and that ρ is the Lt of the (canonical) density k given by

$$k(x) = \mathrm{e}^{-x}\, I_0(x) = \frac{1}{\pi} \int_0^2 \mathrm{e}^{-\lambda x}\, \frac{1}{\sqrt{\lambda\,(2-\lambda)}}\, \mathrm{d}\lambda \qquad [\, x > 0\,].$$

It follows that both f and k are *completely monotone.* ◻◻

Example 11.10. Let π be the function on $(0, \infty)$ given by

$$\pi(s) = 2\,\frac{\sqrt{1+s}-1}{s} = \frac{2}{1+\sqrt{1+s}} \qquad [\,s > 0\,].$$

In order to decide whether or not π is an infinitely divisible pLSt, one is tempted to compute the ρ-function of π:

$$\rho(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log \pi(s) = \tfrac{1}{4}\,\frac{1}{\sqrt{1+s}}\,\pi(s).$$

Clearly, $s \mapsto 1/\sqrt{1+s}$ is completely monotone; we do not know yet, however, that π is completely monotone. But we don't need this, because in the ρ_0-function of π the factor π disappears:

$$\rho_0(s) = \frac{\mathrm{d}}{\mathrm{d}s}\,\frac{1}{\pi(s)} = \tfrac{1}{4}\,\frac{1}{\sqrt{1+s}}.$$

From Theorem 5.1 we now conclude that π is a pLSt that is *compound-exponential* and hence *infinitely divisible.* With a little more effort one can say more. Expansion of $\pi(s)$ in positive powers of $(1+s)^{-\frac{1}{2}}$ and Laplace inversion (together with use of the dominated convergence theorem) shows that π is the Lt of a probability density f with

$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{1}{\Gamma(\frac{1}{2}n)}\,x^{\frac{1}{2}n-1}\,\mathrm{e}^{-x} =$$

$$= 2\,\mathrm{e}^{-x}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\frac{1}{2})}\,x^{k-\frac{1}{2}} - 2.$$

Using the facts that $\Gamma(k+\frac{1}{2}) = (k-\frac{1}{2})\,\Gamma(k-\frac{1}{2})$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, for f' we find

$$f'(x) = -\frac{1}{\sqrt{\pi}}\,x^{-\frac{3}{2}}\,\mathrm{e}^{-x} \qquad [\,x > 0\,].$$

This means that $\lim_{x\to\infty} f(x) = 0$; it follows that f can be written as

$$f(x) = \frac{1}{\sqrt{\pi}}\int_x^{\infty} y^{-\frac{3}{2}}\,\mathrm{e}^{-y}\,\mathrm{d}y \qquad [\,x > 0\,].$$

Since $-f'$ is a completely monotone function, we conclude that f is *completely monotone.* We return to this example in Chapter VI, where we will represent π as the pLSt of a mixture of gamma $(\frac{1}{2})$ distributions. □

Example 11.11. Let π be the function on $(0, \infty)$ given by

$$\pi(s) = \frac{1}{1 + s^\gamma} \qquad [\, s > 0\,],$$

where $\gamma \in (0, 1\,]$. Since $\pi(s) = \pi_0(s^\gamma)$ with π_0 the pLSt of the standard exponential distribution, from Proposition 6.4 it follows that π is an *infinitely divisible pLSt*. Use of Example 4.9 shows that π is even *compound-exponential*. Note that the pLSt π_1 with $\pi_1(s) = \pi(1 + s)/\pi(1)$, which by Proposition 6.1 (iii) is *infinitely divisible*, reduces to the pLSt of Example 11.10 if one takes $\gamma = \frac{1}{2}$. □

Example 11.12. Let π be the function on $(0, \infty)$ given by

$$\pi(s) = \frac{1}{\cosh \sqrt{s}} = \frac{2}{\mathrm{e}^{\sqrt{s}} + \mathrm{e}^{-\sqrt{s}}} \qquad [\, s > 0\,].$$

Then π is an *infinitely divisible pLSt*. To show this one is tempted to use Proposition 6.4 and consider π_1 given by

$$\pi_1(s) = \frac{1}{\cosh s} = \frac{2}{\mathrm{e}^{s} + \mathrm{e}^{-s}} \qquad [\, s > 0\,];$$

then $\pi(s) = \pi_1(\sqrt{s})$. One easily verifies, however, that the ρ-function of π_1 is *not* completely monotone, so if π_1 is a pLSt, then it is *not* infinitely divisible. But π_1 is not even a pLSt because

$$\lim_{n\to\infty} \big\{\pi_1(s/\sqrt{n})\big\}^n = \exp\big[-\tfrac{1}{2}s^2\big],$$

and the limit function is *not* completely monotone. That π is indeed an infinitely divisible pLSt, will be shown only in Chapter VII. □

Example 11.13. For $r > 1$ let π be the function on $\mathbb{R}_+$ given by

$$\pi(s) = \frac{\zeta(r + s)}{\zeta(r)} \qquad [\, s \geq 0\,],$$

where ζ is the Riemann zeta function: $\zeta(r) := \sum_{n=1}^{\infty} 1/n^r$. Then one easily shows that π is the pLSt of a random variable X with a *discrete* distribution (*Riemann zeta* distribution) given by

$$\mathbb{P}(X = \log n) = c_r\, \frac{1}{n^r} \qquad [\, n \in \mathbb{N};\ c_r := 1/\zeta(r)\,].$$

Note that $X \stackrel{\mathrm{d}}{=} \log{(Y + 1)}$ with Y *discrete Pareto* (r) distributed on $\mathbb{Z}_+$; in Example II.11.6 we showed that Y is infinitely divisible. Now, we prove that X also is infinitely divisible; note that the support of X is closed under addition (cf. Corollary 8.3). We use the well-known fact that $\zeta(r) = \prod_{j\in\mathcal{P}}(1-j^{-r})^{-1}$ where $\mathcal{P}$ is the set of primes. Then it follows that π can be written as

$$\pi(s) = \prod_{j\in\mathcal{P}} \frac{1-j^{-r}}{1-j^{-r}\,\mathrm{e}^{-s\,\log j}} = \prod_{j\in\mathcal{P}} \pi_j(s),$$

where π_j is the pLSt of the geometric (j^{-r}) distribution on the lattice $(\log j)\,\mathbb{Z}_+$. Since π_j is infinitely divisible for every j, from Propositions 2.1 and 2.2 we conclude that π is *infinitely divisible* as well. ◻

Finally we present some examples that are obtained in a more constructive way. The first one shows that infinitely divisible densities can be rather wild.

Example 11.14. Consider the probability density f on $(0,\infty)$ given by

$$f(x) = \frac{1}{2\sqrt{\pi}}\,\mathrm{e}^{-x} \sum_{k=0}^{\lfloor x\rfloor} \frac{1}{\sqrt{x-k}}\,\big(\tfrac{1}{2}\mathrm{e}\big)^k \qquad [\,x>0\,].$$

Though f has poles at all nonnegative integers $\big(f(n+)=\infty$ for all $n\in\mathbb{Z}_+\big)$, it is an *infinitely divisible* density. In fact, it is the density of $X{+}Y$, where X and Y are independent, X has a *geometric* $(\frac{1}{2})$ distribution and Y has a standard *gamma* $(\frac{1}{2})$ distribution. This example also shows that for an infinitely divisible density f we need not have $\lim_{x\to\infty} f(x) = 0$. ◻

Example 11.15. Let $a>0$ and $r>0$, and let F be the *infinitely divisible* distribution function with $\ell_F = 0$ for which the canonical function K is absolutely continuous with density k given by

$$k(x) = \begin{cases} a & \text{, if } 0<x<r, \\ 0 & \text{, otherwise.} \end{cases}$$

Since $\int_0^\infty (1/x)\,k(x)\,\mathrm{d}x = \infty$, from Proposition 4.16 it follows that F is absolutely continuous with unique density f satisfying the functional equation (4.19), which in our case reduces to

$$x\,f(x) = \begin{cases} a\,F(x) & \text{, if } 0<x\le r, \\ a\,\big\{F(x)-F(x-r)\big\} & \text{, if } x>r. \end{cases}$$

By the continuity of F it follows that f is continuous on $(0,\infty)$; hence F is differentiable on $(0,\infty)$ with $F' = f$. The equation above then implies that f is differentiable on $(0,\infty)\setminus\{r\}$ with

$$x\, f'(x) = \begin{cases} (a-1)\, f(x) & \text{, if } 0 < x < r, \\ (a-1)\, f(x) - a\, f(x-r) & \text{, if } x > r. \end{cases}$$

Solving the differential equation on $(0,r)$ we find that f satisfies

$$f(x) = c\, x^{a-1} \qquad [\, 0 < x < r\,],$$

for some $c > 0$. It follows that $f(0+)$ and $k(0+)\,(= a)$ are related by

$$f(0+) = \begin{cases} \infty & \text{, if } k(0+) < 1, \\ c & \text{, if } k(0+) = 1, \\ 0 & \text{, if } k(0+) > 1. \end{cases}$$

We further consider the special case where $a = 1$. It shows that *an infinitely divisible density may be constant on an arbitrarily long interval* $(0,r)$. The constant value c can be determined from the pLSt $\pi = \widehat{F}$; by Proposition A.3.4 we have

$$c = f(0+) = \lim_{s\to\infty} s\,\pi(s) = \lim_{s\to\infty} s\,\exp\Big[-\int_0^r \frac{1-\mathrm{e}^{-sx}}{x}\,\mathrm{d}x\Big] =$$
$$= \frac{1}{r}\exp\Big[-\lim_{t\to\infty}\Big(\int_0^t \frac{1-\mathrm{e}^{-y}}{y}\,\mathrm{d}y - \log t\Big)\Big] = \frac{1}{r}\,\mathrm{e}^{-\gamma},$$

where γ is Euler's constant (see Section A.5). Finally, we note that from the differential equation on (r,∞) it is seen that $f' < 0$ on (r,∞), so f is nonincreasing on $(0,\infty)$; in particular, f is bounded on $(0,\infty)$. □

12. Notes

Infinitely divisible random variables with values in $\mathbb{R}_+$ were first considered separately and in detail in Feller (1971), where the subject is linked to complete monotonicity. The information about the zeroes of pLSt's in Theorem 2.8 can also be deduced from results on zeroes of characteristic functions as given in Lukacs (1970). A representation similar to that in Theorem 3.9 can be found in Vervaat (1979).

The canonical representation of Theorem 4.3 is given in Feller (1971). The functional equation in Theorem 4.10 was first given and used by Steutel (1970). It would be nice to know whether this equation yields a direct proof of Theorem 4.14. Important work in the area of Lebesgue properties has been done by Blum and Rosenblatt (1959), and by Tucker (1962, 1964, 1965); we come back to this in Chapter IV. The functional equation in Theorem 4.17 seems to have no analogue for compound-exponential densities; cf. the remark following Theorem 5.6. Such an analogue does exist for generalized compound-exponential densities as considered in van Harn (1978). Here also the 'self-decomposability' result of Theorem 6.3 can be found.

The theorems on moments in Section 7 are basically due to Wolfe (1971b). Relations (7.6) and (7.7) make it possible to introduce 'fractional cumulants' by defining $\kappa_{\alpha+1} := \nu_\alpha$ for $\alpha \geq 0$. Embrechts et al. (1979), and Embrechts and Goldie (1981) give relations between the tails of an infinitely divisible distribution and its canonical function.

Theorem 8.4 about zeroes of infinitely divisible densities is given by Steutel (1970); a similar theorem for infinitely divisible densities on $\mathbb{R}$ is given by Sharpe (1969a) under rather severe restrictions. The tail behaviour of infinitely divisible distributions on $\mathbb{R}$ has been studied by many authors; here we only name Kruglov (1970) and Sato (1973). As in the case of supports, the functional equation of Theorem 4.10 makes our proofs on $\mathbb{R}_+$ in Section 9 essentially simpler.

The infinite divisibility of log-convex densities occurs in Steutel (1970). For continuously differentiable densities a condition, somewhat more general than log-convexity, is given in Bondesson (1987). Theorem 10.6 is proved by Yamazato (1982) and, in a different way, by Hansen (1988b), from which we also took Theorem 10.5. Here Lemma 10.1 is used without proof; the analogue of this lemma for log-concave densities is given in Dharmadhikari and Joag-Dev (1988).

Example 11.4 is taken from Feller (1971). The integral representations of f and k in Example 11.9 follow from formulas in Abramowitz and Stegun (1992). Example 11.13 was first given by Khintchine (1938); see also Gnedenko and Kolmogorov (1968); generalizations are considered by Lin and Hu (2001). Example 11.14 was suggested by J. Keilson many years ago (personal communication). Example 11.15 gives the distribution of the random variable $r\sum_{k=1}^{\infty}(X_1X_2\cdots X_k)^{1/a}$, where the X_j are independent and uniformly distributed on $(0,1)$; the example is studied in Vervaat (1979) and occurs in various applications.

Chapter IV

INFINITELY DIVISIBLE DISTRIBUTIONS ON THE REAL LINE

1. Introduction

Most properties proved in this chapter can be specialized to properties for distributions on $\mathbb{R}_+$ or $\mathbb{Z}_+$. Quite often the properties thus obtained will be less informative than the ones proved earlier in Chapters II and III of this book. On the other hand, many results obtained for distributions on $\mathbb{R}_+$ or $\mathbb{Z}_+$ will not be available for distributions on $\mathbb{R}$, or less generally true, or harder to prove. Since in the literature one usually deals with results for general distributions on $\mathbb{R}$ (the books by Feller are exceptions), these results are more easily available than those for distributions on $\mathbb{R}_+$ or $\mathbb{Z}_+$. With this in mind we will only give an outline of the rather technical proof of the Lévy-Khintchine representation of an infinitely divisible characteristic function. It is convenient, and occasionally inevitable, to use some results from Chapters II and III; but if one is willing to accept these few results, then the present chapter can be read independent of the preceding ones.

We repeat the definition of infinite divisibility, now for general $\mathbb{R}$-valued random variables. A random variable X is said to be *infinitely divisible* if for every $n \in \mathbb{N}$ a random variable X_n exists, the *n-th order factor* of X, such that

$$X \stackrel{\mathrm{d}}{=} X_{n,1} + \cdots + X_{n,n}, \tag{1.1}$$

where $X_{n,1}, \ldots, X_{n,n}$ are independent and distributed as X_n. Note that if X takes its values in $W \subset \mathbb{R}$, then the factors X_n of X are *not* necessarily W-valued. We have seen this in Chapter II for $W = \mathbb{Z}_+$; the factors have their values in $\mathbb{Z}_+$ iff $\mathbb{P}(X = 0) > 0$. In the case where $W = \mathbb{Z}$, which we will encounter in the present chapter, the condition $\mathbb{P}(X=0) > 0$ is necessary but not sufficient for the X_n to be $\mathbb{Z}$-valued; this will be shown in the next section. Mostly we tacitly exclude the trivial case of a distribution degenerate at zero, so we then assume that $\mathbb{P}(X=0) < 1$.

In Section I.2 we agreed that the distribution and transform of an infinitely divisible random variable X will be called *infinitely divisible* as well. The distribution of X will mostly be represented by its distribution function F. It follows that a distribution function F on $\mathbb{R}$ is infinitely divisible iff for every $n \in \mathbb{N}$ there is a distribution function F_n on $\mathbb{R}$, the *n-th order factor* of F, such that F is the n-fold convolution of F_n with itself:

$$(1.2) \qquad F(x) = F_n^{\star n}(x) \qquad [\, n \in \mathbb{N}\,].$$

As $\{\max\{Y_1, .., Y_n\} \le x\} \subset \{Y_1 + \cdots + Y_n \le nx\} \subset \{\min\{Y_1, .., Y_n\} \le x\}$, we have the following useful inequalities for F and its factor F_n:

$$(1.3) \qquad \{F_n(x)\}^n \le F(nx) \le 1 - \{1 - F_n(x)\}^n \qquad [\, x \in \mathbb{R}\,].$$

A reformulation similar to (1.2) can be given for a density f of F in case of absolute continuity (with respect to Lebesgue measure) of all factors. As a tool we use characteristic functions and, more generally, Fourier-Stieltjes transforms (FSt's). A characteristic function ϕ is infinitely divisible iff for every $n \in \mathbb{N}$ there is a characteristic function ϕ_n, the *n-th order factor* of ϕ, such that

$$(1.4) \qquad \phi(u) = \{\phi_n(u)\}^n \qquad [\, n \in \mathbb{N}\,].$$

The correspondence between a distribution function F on $\mathbb{R}$ and its characteristic function ϕ will be expressed by $\phi = \widetilde{F}$. For further conventions and notations concerning distributions on $\mathbb{R}$ and FSt's we refer to Section A.2.

In this chapter we review the basic properties of infinitely divisible distributions on $\mathbb{R}$ more or less parallel to Chapters II and III. For reasons made clear above there will be some deviations. In Section 2 we state some elementary properties and give the normal, Cauchy and Laplace distributions as first examples of infinitely divisible distributions on $\mathbb{R}$. The important classes of compound-Poisson, compound-geometric and compound-exponential distributions are introduced in Section 3. The Lévy-Khintchine and Lévy canonical representations for an infinitely divisible characteristic function are presented in Section 4; several special cases are considered, and important sufficient conditions are given in terms of the canonical quantities for an infinitely divisible distribution function F to be continuous or absolutely continuous. In Section 5 we briefly return to the compound-exponential distributions on $\mathbb{R}$; Section 6 is devoted to closure properties. Relations between moments of an infinitely divisible distribution and those of the corresponding canonical function are given in Section 7. The structure of the support of infinitely divisible distributions is determined in Section 8, and their tail behaviour is considered in Section 9. Log-convexity is the subject of Section 10; since log-convex densities on $\mathbb{R}$ do not exist, we restrict ourselves to densities that are symmetric around zero and log-convex on $(0, \infty)$. It is also of some interest, however, to look at log-convex (real) characteristic functions. As usual, the final two sections, 11 and 12, contain examples and bibliographical remarks.

Finally we note that this chapter only treats the basic properties of infinitely divisible distributions on $\mathbb{R}$. Results for self-decomposable and stable distributions on $\mathbb{R}$ can be found in Sections V.6 and V.7, for mixtures of zero-mean normal distributions and of sym-gamma distributions in Sections VI.9 and VI.10, and for generalized sym-gamma convolutions in Section VI.11. Also Section VII.6 on shot noise contains information on infinitely divisible distributions on $\mathbb{R}$.

2. Elementary properties

We start by giving some simple closure properties. The first two of them, stated in the following proposition, follow directly from (1.1) or (1.4).

Proposition 2.1.

(i) *If X is an infinitely divisible random variable, then so is aX for every $a \in \mathbb{R}$. Equivalently, if ϕ is an infinitely divisible characteristic function, then so is ϕ_a with $\phi_a(u) := \phi(au)$ for every $a \in \mathbb{R}$. In particular, if ϕ is an infinitely divisible characteristic function, then so is $\overline{\phi}$, the complex conjugate of ϕ.*

(ii) *If X and Y are independent infinitely divisible random variables, then $X + Y$ is an infinitely divisible random variable. Equivalently, if ϕ and ψ are infinitely divisible characteristic functions, then their pointwise product $\phi\,\psi$ is an infinitely divisible characteristic function.*

Let ϕ be a characteristic function. Then so is $|\phi|^2 = \phi\,\overline{\phi}$; it is the characteristic function of $X - X'$ if X and X' are independent with characteristic function ϕ. Combining both parts of Proposition 2.1 we see that if ϕ is infinitely divisible, then so is $|\phi|^2$. Now, consider $|\phi|$ with ϕ infinitely divisible; by (1.4) we can write $|\phi|$ as

$$(2.1) \qquad \big|\phi(u)\big| = \big\{\big|\phi_{2n}(u)\big|^2\big\}^n \qquad [n \in \mathbb{N}],$$

where $\phi_2, \phi_4, \ldots$ are characteristic functions. Taking $n = 1$ we see that $|\phi| = |\phi_2|^2$, so $|\phi|$ is a characteristic function; this is generally not true if ϕ is not infinitely divisible. Further, (2.1) implies that $|\phi|$ is even infinitely divisible. Thus we have proved the following result.

Proposition 2.2. *If ϕ is an infinitely divisible characteristic function, then so is $|\phi|$.*

The last closure property we mention here, states that infinite divisibility is preserved under weak convergence; its proof takes more effort than in the $\mathbb{R}_+$-case.

Proposition 2.3. *If a sequence $(X^{(m)})$ of infinitely divisible random variables converges in distribution to X, then X is infinitely divisible. Equivalently, if a sequence $(\phi^{(m)})$ of infinitely divisible characteristic functions converges (pointwise) to a characteristic function ϕ, then ϕ is infinitely divisible.*

PROOF. Let $\phi^{(m)}$ be an infinitely divisible characteristic function with $\phi^{(m)} = \widetilde{F}^{(m)}$, and suppose that $\phi^{(m)} \to \phi$ as $m \to \infty$, where $\phi = \widetilde{F}$ is a characteristic function. Then, by (1.4), for every $m \in \mathbb{N}$ there exist characteristic functions $\phi_n^{(m)} = \widetilde{F}_n^{(m)}$ with $n \in \mathbb{N}$ such that $\phi^{(m)} = \big\{\phi_n^{(m)}\big\}^n$. Moreover, from (1.3) it is seen that

$$\big\{F_n^{(m)}(x)\big\}^n \le F^{(m)}(nx) \le 1 - \big\{1 - F_n^{(m)}(x)\big\}^n \qquad [x \in \mathbb{R}].$$

Consider the sequence $(F_n^{(m)})_{m\in\mathbb{N}}$ with n fixed. According to Helly's selection theorem there is a subsequence $(F_n^{(m_k)})_{k\in\mathbb{N}}$ with the property that $F_n^{(m_k)} \to F_n$ in continuity points of F_n as $k \to \infty$, where F_n is a sub-distribution function. In order to show that F_n is a distribution function, in the inequalities above we take $m = m_k$ and let $k \to \infty$; since by the continuity theorem $F^{(m)} \to F$ weakly as $m \to \infty$, it then follows that for every $x \in \mathbb{R}$

that is a continuity point of F_n and for which nx is a continuity point of F, we have

$$\{F_n(x)\}^n \leq F(nx) \leq 1 - \{1 - F_n(x)\}^n.$$

Letting $x \to \infty$ and $\to -\infty$ now shows that F_n is a distribution function. It follows that, as $k \to \infty$, $F_n^{(m_k)} \to F_n$ weakly, and hence $\phi_n^{(m_k)} \to \phi_n$, where $\phi_n := \widetilde{F}_n$ is a characteristic function. Since ϕ can now be written as

$$\phi(u) = \lim_{k\to\infty} \phi^{(m_k)}(u) = \lim_{k\to\infty} \{\phi_n^{(m_k)}(u)\}^n = \{\phi_n(u)\}^n,$$

we conclude that ϕ is infinitely divisible with n-th order factor ϕ_n. □

One may ask whether the factors ϕ_n of an infinitely divisible characteristic function ϕ are uniquely determined by ϕ. Whereas pgf's and pLSt's are trivially non-zero on $(0,1]$ and $[0,\infty)$, respectively, characteristic functions can be zero for real values of the argument. Therefore, it will be clear that if ϕ is just n-divisible, i.e., satisfies the equation in (1.4) for a fixed n, then the factor ϕ_n will generally *not* be unique. On the other hand, $|\phi_n|$ is indeed determined by ϕ because $|\phi_n| = |\phi|^{1/n}$. In case of *infinite* divisibility of ϕ, however, there are no problems because of the following important property.

Proposition 2.4. *Let ϕ be an infinitely divisible characteristic function. Then ϕ has no real zeroes: $\phi(u) \neq 0$ for $u \in \mathbb{R}$.*

Proof. From Proposition 2.2 it follows that $|\phi|^{1/n}$ is a characteristic function for every $n \in \mathbb{N}$. Now, consider the pointwise limit of these characteristic functions:

$$\psi(u) := \lim_{n\to\infty} |\phi(u)|^{1/n} = \begin{cases} 1 & \text{, if } \phi(u) \neq 0, \\ 0 & \text{, if } \phi(u) = 0. \end{cases}$$

As ϕ is continuous and $\phi(0) = 1$, ψ is continuous at 0. Hence, by the continuity theorem ψ is a characteristic function and therefore continuous. This forces ψ to be identically one. Consequently, $\phi(u) \neq 0$ for all u. □

On account of this result we can uniquely define the logarithm of an infinitely divisible characteristic function ϕ in such a way that $\log \phi$ is continuous with $\log\phi(0)=0$; see Section A.2. Defining then $\phi^t := \exp[\,t\log\phi\,]$ for $t>0$ with $\log\phi$ as indicated, we can rewrite (1.4) as

$$\{\phi(u)\}^{1/n} = \phi_n(u) \qquad [\,n\in\mathbb{N}\,]. \tag{2.2}$$

It follows that the factors ϕ_n are uniquely determined by ϕ. Moreover, we can formulate a first criterion for infinite divisibility. Recall that by Bochner's theorem (see Section A.2) the set of characteristic functions equals the set of *nonnegative definite* ($\mathbb{C}$-valued) functions ϕ on $\mathbb{R}$ with $\phi(0)=1$.

Proposition 2.5. *A nonvanishing characteristic function ϕ is infinitely divisible iff ϕ^t is a characteristic function for all $t\in T$, where $T=(0,\infty)$, $T=\{1/n : n\in\mathbb{N}\}$ or $T=\{a^{-k} : k\in\mathbb{N}\}$ for any fixed integer $a\geq 2$. Equivalently, ϕ is infinitely divisible iff ϕ^t is nonnegative definite for all $t\in T$ with T as above.*

Proof. Let ϕ be infinitely divisible. Then by (2.2) $\phi^{1/n}$ is a characteristic function for all $n\in\mathbb{N}$, and hence $\phi^{m/n}$ is a characteristic function for all $m,n\in\mathbb{N}$. It follows that ϕ^t is a characteristic function for all positive $t\in\mathbb{Q}$ and hence, by the continuity theorem, for all $t>0$.
Conversely, from (2.2) we know that ϕ is infinitely divisible if $\phi^{1/n}$ is a characteristic function for every $n\in\mathbb{N}$. We can even be more restrictive because, for a given integer $a\geq 2$, any $t\in(0,1)$ can be represented as $t=\sum_{k=1}^{\infty} t_k a^{-k}$ with $t_k\in\{0,\ldots,a-1\}$ for all k, and hence for these t

$$\{\phi(u)\}^t = \lim_{m\to\infty} \prod_{k=1}^{m} \Big(\{\phi(u)\}^{1/a^k}\Big)^{t_k}.$$

By the continuity theorem it follows that if ϕ^t is a characteristic function for all t of the form a^{-k} with $k\in\mathbb{N}$, then so is ϕ^t for all $t\in(0,1)$. □

Corollary 2.6. *If ϕ is an infinitely divisible characteristic function, then so is ϕ^t for all $t>0$. In particular, the factors X_n of an infinitely divisible random variable X are infinitely divisible.*

The continuous multiplicative semigroup $(\phi^t)_{t\geq 0}$ of characteristic functions generated by an infinitely divisible characteristic function ϕ, corresponds to the set of one-dimensional marginal distributions of an *sii-process*, i.e., a process $X(\cdot)$ with stationary independent increments, started at zero and continuous in probability; see Section I.3. If $X(1)$, with characteristic function ϕ, has distribution function F, then for $t > 0$ the distribution function of $X(t)$, with characteristic function ϕ^t, will be denoted by $F^{\star t}$, so:

$$(2.3) \qquad F^{\star t}(x) = \mathbb{P}\big(X(t) \le x\big), \qquad \int_{\mathbb{R}} \mathrm{e}^{iux}\, \mathrm{d}F^{\star t}(x) = \big\{\phi(u)\big\}^t.$$

Of course, examples of infinitely divisible distributions on $\mathbb{R}$, and hence of $\mathbb{R}$-valued sii-processes, are provided by the basic examples from Chapter II, viz. the *negative-binomial* distributions and the *stable* distributions on $\mathbb{Z}_+$, and those from Chapter III, viz. the *gamma* distributions and the *stable* distributions on $\mathbb{R}_+$. We now add three important distributions not supported by $\mathbb{R}_+$. They can be viewed as symmetric counterparts of the distributions just mentioned; see, however, also Example 4.9.

Example 2.7. For $\sigma^2 > 0$, let X have the *normal* $(0, \sigma^2)$ distribution, so its density f and characteristic function ϕ are given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}x^2/\sigma^2}, \qquad \phi(u) = \mathrm{e}^{-\frac{1}{2}\sigma^2 u^2}.$$

Then for $t > 0$ the t-th power ϕ^t of ϕ is recognized as the characteristic function of $\sqrt{t}\, X$ with normal $(0, t\sigma^2)$ distribution. From Proposition 2.5 we conclude that the normal (μ, σ^2) distribution is *infinitely divisible* for $\mu = 0$ and hence for $\mu \in \mathbb{R}$. The corresponding sii-process is *Brownian motion.* It is also a *stable process* with exponent 2; see Chapter V. ◻◻

Example 2.8. For $\lambda > 0$, let X have the *Cauchy* (λ) distribution, so its density f and characteristic function ϕ are given by

$$f(x) = \frac{1}{\pi}\, \frac{\lambda}{\lambda^2 + x^2}, \qquad \phi(u) = \mathrm{e}^{-\lambda |u|}.$$

Then for $t > 0$ the t-th power ϕ^t of ϕ is recognized as the characteristic function of tX with Cauchy $(t\lambda)$ distribution. We conclude that the Cauchy (λ) distribution is *infinitely divisible.* The corresponding sii-process is a *Cauchy process.* It is also a *stable process* with exponent 1; see Chapter V. ◻◻

Example 2.9. For $\lambda > 0$, let X have the *Laplace* (λ) distribution, so its density f and characteristic function ϕ are given by

$$f(x) = \tfrac{1}{2}\lambda\, \mathrm{e}^{-\lambda|x|}, \qquad \phi(u) = \frac{\lambda^2}{\lambda^2 + u^2} = \frac{\lambda}{\lambda - iu}\, \frac{\lambda}{\lambda + iu}.$$

From the second representation of ϕ it follows that $X \stackrel{d}{=} Y - Y'$, where Y and Y' are independent and have the same exponential distribution. Since this distribution is infinitely divisible (see Example III.2.6), from Proposition 2.1 it immediately follows that the Laplace (λ) distribution is *infinitely divisible.* Now, for $r > 0$ consider the r-th power of ϕ or, equivalently, let Y and Y' above have a gamma (r, λ) distribution. Then it follows that the *sym-gamma* (r, λ) distribution with characteristic function ϕ_r given by

$$\phi_r(u) = \Big(\frac{\lambda^2}{\lambda^2 + u^2}\Big)^r,$$

is *infinitely divisible* as well; here 'sym-gamma' stands for 'symmetrized-gamma'. Let f_r be a corresponding density. Then, for example,

$$f_2(x) = \tfrac{1}{4}\lambda\,\big(1 + \lambda|x|\big)\,\mathrm{e}^{-\lambda|x|};$$

for general r there is an expression for f_r in terms of a modified Bessel function of the second kind. □□

Sometimes we want to consider a characteristic function ϕ for *complex* values of its argument. If ϕ corresponds to an $\mathbb{R}_+$-valued random variable X, then $\phi(z)$ is well defined at least for al $z \in \mathbb{C}$ with $\operatorname{Im} z \geq 0$. In Proposition III.2.7 we have seen that if such an X is infinitely divisible with $p_0 := \mathbb{P}(X = 0) > 0$, then

$$(2.4) \qquad \big|\phi(z)\big| \geq p_0^2 \qquad [\,\operatorname{Im} z \geq 0\,];$$

hence ϕ has no zeroes in the closed upper half-plane. The characteristic function ϕ of a general ($\mathbb{R}$-valued) infinitely divisible random variable X with $p_0 > 0$ may be well defined on the real line only, so one may only hope to have the inequality in (2.4) for $z = u \in \mathbb{R}$. Unfortunately, the technique used in the $\mathbb{R}_+$-case does not easily lead to such a result; it can be used, however, to prove a somewhat different inequality in case X is $\mathbb{Z}$-valued and has finite variance.

Before showing this we consider the factors X_n of an infinitely divisible random variable X that has its *values in* $\mathbb{Z}$. Letting $n \to \infty$ in (2.2) and using the continuity theorem, we see that necessarily $X_n \stackrel{d}{\longrightarrow} 0$ as $n \to \infty$. Hence, if the X_n are $\mathbb{Z}$-valued, too, then $\mathbb{P}(X_n = 0) > 0$ for all n sufficiently large, so by (1.1) $\mathbb{P}(X = 0) > 0$. The converse statement is not true in general; the factor X_n of $X := Y - 1$, with Y Poisson distributed, has values in $-1/n + \mathbb{Z}_+$. When we start from a *symmetric* distribution on $\mathbb{Z}$, we do have a converse.

Proposition 2.10. *Let X have a symmetric infinitely divisible distribution. Then the factors X_n of X satisfy*

$$\mathbb{P}(X_n = 0) \geq \mathbb{P}(X = 0) \qquad [n \in \mathbb{N}], \tag{2.5}$$

and if X is $\mathbb{Z}$-valued, then the X_n are also $\mathbb{Z}$-valued iff $\mathbb{P}(X = 0) > 0$.

PROOF. Let ϕ be the characteristic function of X. Since ϕ is continuous and real, and by Proposition 2.4 has no zeroes, we have $\phi(u) > 0$ for all $u \in \mathbb{R}$. Since the characteristic function ϕ_n of the factor X_n is given by $\phi_n = \phi^{1/n}$, it follows that $\phi_n(u) \geq \phi(u)$ for all $u \in \mathbb{R}$. Now, use the inversion formula (A.2.14) implying that

$$\mathbb{P}(X = 0) = \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} \phi(u) \, \mathrm{d}u,$$

and similarly for X_n and ϕ_n; this immediately yields inequality (2.5). Next let X be $\mathbb{Z}$-valued. Above we saw that if the X_n are also $\mathbb{Z}$-valued, then $\mathbb{P}(X = 0) > 0$. For the converse we suppose that $\mathbb{P}(X = 0) > 0$ and let $n \in \mathbb{N}$; then by (2.5) we have $\mathbb{P}(X_n = 0) > 0$. As by (1.1)

$$\mathbb{P}(X = x) \geq n \, \mathbb{P}(X_n = x) \left\{ \mathbb{P}(X_n = 0) \right\}^{n-1} \qquad [x \neq 0],$$

it follows that any possible value of X_n is also a possible value of X. We conclude that X_n is $\mathbb{Z}$-valued, too. □

Proposition 2.11. *Let ϕ be the characteristic function of an infinitely divisible, $\mathbb{Z}$-valued random variable X with $\sigma^2 := \mathbb{V}\mathrm{ar}\, X < \infty$. Then:*

$$\left| \phi(u) \right| \geq e^{-2\sigma^2} \qquad [u \in \mathbb{R}]. \tag{2.6}$$

PROOF. First, consider the special case where X has a symmetric distribution with $\mathbb{P}(X = 0) > 0$. Then we can apply the preceding proposition to conclude that the factors X_n of X are $\mathbb{Z}$-valued. Since by (1.1) $\mathbb{E} X_n = 0$ and $\mathbb{V}\mathrm{ar}\, X_n = \sigma^2/n$, by Chebyshev's inequality it follows that

$$\mathbb{P}(X_n = 0) = 1 - \mathbb{P}\big(|X_n| \geq 1\big) \geq 1 - \sigma^2/n.$$

Now, we note that any characteristic function $\psi = \widetilde{F}_Y$ can be estimated as follows:

$$\big|\psi(u)\big| \geq \mathbb{P}(Y=0) - \Big|\int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{iuy}\,\mathrm{d}F_Y(y)\Big| \geq 2\mathbb{P}(Y=0) - 1.$$

Taking here $Y = X_n$ with characteristic function $\phi^{1/n}$, we then see that for n sufficiently large

$$\big|\phi(u)\big| = \big|\phi^{1/n}(u)\big|^n \geq \big(2\mathbb{P}(X_n=0) - 1\big)^n \geq (1 - 2\sigma^2/n)^n,$$

from which by letting $n \to \infty$ we obtain (2.6). Finally, consider the general case. Then the random variable $X_0 := X - X'$, where X and X' are independent with characteristic function ϕ, satisfies the conditions of the special case we considered first. Hence we have (2.6) with ϕ replaced by $|\phi|^2$ and σ^2 replaced by $2\sigma^2$. Taking square roots yields (2.6) as stated. □□

It is not hard to find examples of infinitely divisible distributions with infinite second moment for which there exists $\delta > 0$ such that its characteristic function ϕ satisfies $\big|\phi(u)\big| \geq \delta$ for all $u \in \mathbb{R}$; see Section 11.

We briefly return to considering a characteristic function ϕ for complex values of its argument, and recall some well-known facts from Section A.2. Any ϕ has two abscissas of convergence; these are the largest numbers u_ϕ and v_ϕ in $[0,\infty]$ such that $\phi(z)$ is well-defined for all $z \in \mathbb{C}$ with $-u_\phi < \operatorname{Im} z < v_\phi$. The set of these values of z (if non-empty) is called the *strip of analyticity* of ϕ, because ϕ can be shown to be analytic on this set. Moreover, u_ϕ and v_ϕ are determined by the (left- and right-) tails of the corresponding distribution function F in the following way:

$$u_\phi = \liminf_{x\to\infty} \frac{-\log\big\{1 - F(x)\big\}}{x}, \quad v_\phi = \liminf_{x\to\infty} \frac{-\log F(-x)}{x}. \tag{2.7}$$

Using these results we can easily prove the following generalization of Proposition 2.4.

Theorem 2.12. *Let ϕ be an infinitely divisible characteristic function. Then:*

(i) *ϕ has no zeroes in $\mathbb{R}$.*

(ii) *ϕ has no zeroes in its strip of analyticity.*

(iii) *If ϕ is an entire function, i.e., if ϕ is analytic on all of $\mathbb{C}$, then ϕ has no zeroes in $\mathbb{C}$.*

PROOF. We only need to prove part (ii). Let S be the strip of analyticity of ϕ; assume that $S \neq \emptyset$. Then the factors ϕ_n of ϕ are also well defined on S, and hence analytic there; this immediately follows from (2.7) and the inequalities in (1.3). Now, suppose that $\phi(z_0) = 0$ for some $z_0 \in S$; on some neighbourhood B of z_0, ϕ would then have an expansion of the form

$$\phi(z) = \sum_{k=1}^{\infty} a_k\,(z - z_0)^k \qquad [\, z \in B\,].$$

Since also $\phi_n(z_0) = 0$, ϕ_n can be represented similarly. By equating coefficients in the identity $\phi = (\phi_n)^n$ one then sees that $a_1 = \cdots = a_{n-1} = 0$. Since this would hold for any $n \in \mathbb{N}$, it would follow that ϕ is zero on B and hence on all of S, which is impossible. ◻◻

This theorem may be used to show that a given characteristic function is not infinitely divisible; an example is given in Section 11. It also implies that if ϕ is an infinitely divisible characteristic function, then $\log \phi(z)$, and hence $\phi^t(z)$ with $t > 0$, can be defined for z in the strip of analyticity of ϕ as a continuous function with $\log \phi(0) = 0$; see also Section A.2.

3. Compound distributions

Important classes of infinitely divisible distributions can be obtained by random stopping of processes with stationary independent increments. We will show this first in the *discrete-time* case. To this end we recall some facts from Section I.3. Let $(S_n)_{n \in \mathbb{Z}_+}$ be an sii-process generated by Y (so $S_n = Y_1 + \cdots + Y_n$ for all n with $Y_1, Y_2, \ldots$ independent and distributed as Y), let N be $\mathbb{Z}_+$-valued and independent of (S_n), and consider X such that

(3.1) $$X \overset{\mathrm{d}}{=} S_N \quad (\text{so } X \overset{\mathrm{d}}{=} Y_1 + \cdots + Y_N).$$

Then X is said to have a *compound-N* distribution, and from (I.3.10) one sees that its distribution function and characteristic function can be expressed in similar characteristics of Y and N as follows:

(3.2) $$F_X(x) = \sum_{n=0}^{\infty} \mathbb{P}(N = n)\, F_Y^{\star n}(x), \quad \phi_X(u) = P_N\big(\phi_Y(u)\big).$$

In particular, it follows that the composition of a pgf with a characteristic function is a characteristic function. Also, note that if N is infinitely divisible, then by Theorem II.2.8 its pgf P_N has no zeroes in the closed disk $|z| \leq 1$, so that ϕ_X in (3.2) does not vanish anywhere on $\mathbb{R}$. Therefore, we can apply Proposition 2.5 and its discrete counterpart in Chapter II to obtain the following result.

Proposition 3.1. *A random variable X that has a compound-N distribution with N $\mathbb{Z}_+$-valued and infinitely divisible $\big($in the discrete sense, so $\mathbb{P}(N=0)>0\big)$, is infinitely divisible. Equivalently, the composition $P \circ \widetilde{G}$ of an infinitely divisible pgf P with an arbitrary characteristic function $\widetilde{G}$ is an infinitely divisible characteristic function.*

The best-known compound distributions that are infinitely divisible are the *compound-Poisson* distributions, for which N above is Poisson distributed. By (3.2) their characteristic functions have the form

$$\phi(u) = \exp\big[-\lambda\big\{1-\widetilde{G}(u)\big\}\big], \tag{3.3}$$

where $\lambda > 0$ and G is a distribution function. Here G can always be chosen to be continuous at zero. The pair (λ, G) is then uniquely determined by ϕ; this is easily verified from the uniqueness theorem for FSt's. Of course, the infinite divisibility of the compound-Poisson distributions immediately follows from Proposition 3.1 or from (3.3). We formally state this result.

Theorem 3.2. *The compound-Poisson distributions, with characteristic functions given by (3.3), are infinitely divisible.*

From the first part of (3.2) with $F_Y = G$ it is clear that a compound-Poisson distribution is a distribution with positive mass at zero and, if G is continuous, with no mass at any other point. Now, for such distributions infinite divisibility turns out to be *equivalent* to being compound-Poisson. We formulate this in terms of random variables.

Theorem 3.3. *Let X be a random variable satisfying $\mathbb{P}(X=0)>0$ and $\mathbb{P}(X=x)=0$ for all $x \neq 0$. Then X is infinitely divisible iff it has a compound-Poisson distribution.*

PROOF. Let X be infinitely divisible with characteristic function ϕ, and set $p_0 := \mathbb{P}(X = 0)$. By (1.1) the n-th order factor X_n of X, with characteristic function $\phi^{1/n}$, satisfies $\mathbb{P}(X_n = x) = 0$ for all $x \neq 0$, and hence $\mathbb{P}(X_n = 0) = p_0^{1/n}$. So there exists a distribution function G_n such that

$$\widetilde{G}_n(u) = \frac{\phi^{1/n}(u) - p_0^{1/n}}{1 - p_0^{1/n}} = 1 - \frac{1 - \phi^{1/n}(u)}{1 - p_0^{1/n}}.$$

Now, let $n \to \infty$ and use the fact that $\lim_{n\to\infty} n\left(1 - \alpha^{1/n}\right) = -\log \alpha$ for $\alpha \in \mathbb{C}$ with $\alpha \neq 0$. Then from the continuity theorem we conclude that there exists a distribution function G such that

$$\widetilde{G}(u) = 1 - \frac{1}{\lambda}\left\{-\log \phi(u)\right\} \text{ with } \lambda := -\log p_0. \tag{3.4}$$

It follows that ϕ can be written in the form (3.3) with, necessarily, G continuous. The converse statement is contained in Theorem 3.2. □

In the next section we will generalize this result to distributions with at least one atom. The compound-Poisson distributions are basic among the infinitely divisible distributions on $\mathbb{R}$. This is also apparent from the following limit result; it is obtained by applying Proposition 2.3 and using the fact that, as in the proof of Theorem 3.3, an infinitely divisible characteristic function ϕ can be written as

$$\phi(u) = \lim_{n\to\infty} \exp\left[-n\left\{1 - \phi^{1/n}(u)\right\}\right]. \tag{3.5}$$

Theorem 3.4. *A distribution on $\mathbb{R}$ is infinitely divisible iff it is the weak limit of compound-Poisson distributions.*

The compound-N distributions with N geometrically distributed will be called *compound-geometric*. Their characteristic functions have the form

$$\phi(u) = \frac{1-p}{1 - p\,\widetilde{G}(u)}, \tag{3.6}$$

where $p \in (0,1)$ and G is a distribution function. As for the compound-Poisson distributions one can show that the pair (p, G) is uniquely determined by ϕ if we choose G to be continuous at zero, which can always be done. Since the geometric distribution is infinitely divisible, from Proposition 3.1 it is immediately clear that the compound-geometric distributions are infinitely divisible. They are even compound-Poisson. This follows from the fact that if N has a geometric (p) distribution, then N is compound-Poisson: $P_N = P_M \circ Q$, where M is Poisson (λ) distributed with $\lambda := -\log(1-p)$ and Q is the pgf of the logarithmic-series (p) distribution on $\mathbb{N}$, so $Q(z) = \{-\log(1-pz)\}/\lambda$. Hence $\phi := P_N \circ \widetilde{G}$ can be written as $\phi = P_M \circ (Q \circ \widetilde{G})$, so ϕ is compound-Poisson. We summarize.

Theorem 3.5. *A compound-geometric distribution, with characteristic function of the form (3.6), is compound-Poisson, and hence infinitely divisible.*

Next we randomly stop *continuous-time* processes with stationary independent increments, and again recall some facts from Section I.3. Let $S(\cdot)$ be a continuous-time sii-process generated by Y $\big($so $S(1) \stackrel{\mathrm{d}}{=} Y$ and Y is infinitely divisible$\big)$, let T be $\mathbb{R}_+$-valued and independent of $S(\cdot)$, and consider X such that

$$X \stackrel{\mathrm{d}}{=} S(T). \tag{3.7}$$

Then X is said to have a *compound-T* distribution, and from (I.3.8) one sees that the distribution function and characteristic function of X can be expressed in similar characteristics of Y and T by

$$F_X(x) = \int_{\mathbb{R}_+} F_Y^{\star t}(x)\, \mathrm{d}F_T(t), \quad \phi_X(u) = \pi_T\big(-\log \phi_Y(u)\big). \tag{3.8}$$

Note that if T is infinitely divisible, then by Theorem III.2.8 its pLSt π_T has no zeroes in the closed half-plane $\mathrm{Re}\, z \geq 0$, so that ϕ_X in (3.8) does not vanish anywhere on $\mathbb{R}$. Therefore, we can apply Proposition 2.5 and its $\mathbb{R}_+$-counterpart in Chapter III to obtain the following general result.

Proposition 3.6. *A random variable X that has a compound-T distribution with T $\mathbb{R}_+$-valued and infinitely divisible, is infinitely divisible. Equivalently, the composition $\pi \circ (-\log \phi_0)$ where π is an infinitely divisible pLSt and ϕ_0 is an infinitely divisible characteristic function, is an infinitely divisible characteristic function.*

The compound-T distributions with T degenerate at one constitute precisely the set of all infinitely divisible distributions. It is more interesting to take T standard exponentially distributed. Since then $\pi_T(s) = 1/(1+s)$, the resulting *compound-exponential* distributions have characteristic functions of the form

(3.9) $$\phi(u) = \frac{1}{1 - \log\, \phi_0(u)},$$

where ϕ_0 is an *infinitely divisible* characteristic function. The function ϕ_0, which is sometimes called the *underlying* (infinitely divisible) characteristic function of ϕ, is uniquely determined by ϕ because

(3.10) $$\phi_0(u) = \exp\bigl[1 - 1/\phi(u)\bigr].$$

Note that taking T exponential$\,(\lambda)$ with $\lambda \neq 1$ leads to the same class of distributions; just use Corollary 2.6. The following theorem is an immediate consequence of Proposition 3.6 and the infinite divisibility of the exponential distribution.

Theorem 3.7. *A compound-exponential distribution, with characteristic function of the form (3.9), is infinitely divisible.*

It turns out that within the class of compound-exponential distributions the compound-geometric distributions play the same role as do the compound-Poisson distributions in the class of all infinitely divisible distributions. In fact, there are analogues to Theorems 3.2, 3.3 and 3.4; we will show this by using these theorems.

Theorem 3.8. *A distribution on $\mathbb{R}$ is compound-geometric iff it is compound-exponential having an underlying infinitely divisible distribution that is compound-Poisson.*

Proof. First, let ϕ be a characteristic function of the form (3.9) with ϕ_0 compound-Poisson (and hence infinitely divisible, because of Theorem 3.2). Then ϕ_0 has the form (3.3), so ϕ can be written as

$$\phi(u) = \frac{1}{1 + \lambda\bigl\{1 - \widetilde{G}(u)\bigr\}} = \frac{1 - \lambda/(1+\lambda)}{1 - \bigl\{\lambda/(1+\lambda)\bigr\}\, \widetilde{G}(u)},$$

which is of the compound-geometric form (3.6). The converse statement is proved similarly: If ϕ has the form (3.6), then computing the right-hand side of (3.10) shows that ϕ can be written as in (3.9) with ϕ_0 of the form (3.3); take $\lambda = p/(1-p)$. ◻

Theorem 3.9. *Let X be a random variable satisfying $\mathbb{P}(X=0) > 0$ and $\mathbb{P}(X=x) = 0$ for all $x \neq 0$. Then X is compound-exponential iff it is compound-geometric.*

PROOF. For both implications we use Theorem 3.8; the implication to the left is then immediate, so we consider the one to the right. Let X have a compound-exponential distribution. Then $X \stackrel{d}{=} S(T)$ where $S(\cdot)$ is an sii-process generated by Y $\big($so $Y \stackrel{d}{=} S(1)\big)$, and T is standard exponentially distributed and independent of $S(\cdot)$. We will show that for $x \in \mathbb{R}$

$$(3.11) \qquad A_x := \{t \geq 0 : \mathbb{P}\big(S(t) = x\big) > 0\} = \begin{cases} \mathbb{R}_+ & , \text{if } x = 0, \\ \emptyset & , \text{if } x \neq 0, \end{cases}$$

which implies that Y, like X, has a single atom at zero. Then we are done because by Theorem 3.3 Y then is compound-Poisson, so that X is compound-geometric. To prove (3.11) we note that, similar to the first part of (3.8),

$$\mathbb{P}(X = x) = \int_0^\infty \mathbb{P}\big(S(t) = x\big)\, \mathrm{e}^{-t}\, \mathrm{d}t \qquad [x \in \mathbb{R}],$$

so in view of the conditions put on X for $x \in \mathbb{R}$ we have (m being Lebesgue measure)

$$(3.12) \qquad m(A_x) \begin{cases} > 0 & , \text{if } x = 0, \\ = 0 & , \text{if } x \neq 0. \end{cases}$$

Now, using general properties of sii-processes one easily verifies that

$$(3.13) \qquad t_1 \in A_0,\ t_2 \in A_x \implies t_1 + t_2 \in A_x \qquad [x \in \mathbb{R}].$$

In particular, A_0 is closed under addition. Since we also have $m(A_0) > 0$, there then exists, as is well known, $a_0 \geq 0$ such that $A_0 \supset (a_0, \infty)$. A second combination of (3.12) and (3.13) now shows that $A_x = \emptyset$ for $x \neq 0$. This means that for every $t_0 \geq 0$ we have $\mathbb{P}\big(S(t_0) = x\big) = 0$ for all $x \neq 0$. Suppose that $t_0 \notin A_0$. Then $S(t_0)$ would have a continuous distribution, and hence so has $S(mt_0)$, having the same distribution as the sum of m independent copies of $S(t_0)$; so $mt_0 \notin A_0$ for all $m \in \mathbb{N}$. For $t_0 > 0$, however, this contradicts the fact that $A_0 \supset (a_0, \infty)$. We conclude that we may take $a_0 = 0$, so $A_0 = \mathbb{R}_+$, and (3.11) is proved. □

Theorem 3.10. *A distribution on $\mathbb{R}$ is compound-exponential iff it is the weak limit of compound-geometric distributions.*

PROOF. Use Theorem 3.4 for the underlying infinitely divisible distribution of a compound-exponential distribution, and then apply Theorem 3.8. □

In Section 5 we briefly return to the class of compound-exponential distributions. Since they correspond one-to-one to the infinitely divisible distributions (via the underlying distributions), the results of the next section are also of importance for this class.

4. Canonical representation

In the preceding section we found several classes of infinitely divisible characteristic functions. We now want to characterize *all* such functions. This is achieved by the so-called *Lévy-Khintchine canonical representation.* Since a full proof of this formula is rather technical and can be found in several textbooks, we do not give all the details.

Let $\phi = \widetilde{F}$ be an infinitely divisible characteristic function. Then ϕ is the limit of characteristic functions that are compound-Poisson; by (3.5) we have

$$\log \phi(u) = \lim_{n\to\infty} n\left\{\phi^{1/n}(u) - 1\right\} = \lim_{n\to\infty} \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1)\,\mu_n(\mathrm{d}x), \tag{4.1}$$

where μ_n is the Stieltjes measure induced by the function $n\,F^{\star(1/n)}$. To get a canonical representation we would like to actually take the limit in (4.1). To this end we use, without further comment, some well-known results on weak convergence as mentioned in Section A.2. First, note that for large n the measure μ_n will be concentrated near zero. In fact, it can be proved that

$$\sup_n \int_{[-1,1]} x^2\,\mu_n(\mathrm{d}x) < \infty, \quad \sup_n \int_{\mathbb{R}\setminus[-1,1]} \mu_n(\mathrm{d}x) < \infty.$$

We therefore rewrite (4.1) as

$$\log \phi(u) = \lim_{n\to\infty} \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1)\,\frac{1+x^2}{x^2}\,\mathrm{d}G_n(x), \tag{4.2}$$

where G_n defined by

$$G_n(x) := \int_{(-\infty,x]} \frac{y^2}{1+y^2}\,\mu_n(\mathrm{d}y) \qquad [\,x \in \mathbb{R}\,],$$

is a nondecreasing, bounded function satisfying $\sup_n G_n(\infty) < \infty$; here and in similar cases $G_n(\infty)$ is the short-hand notation for $\lim_{x\to\infty} G_n(x)$. By Helly's selection theorem it follows that there exists a subsequence $(G_{n_k})_{k\in\mathbb{N}}$ with the property that $G_{n_k} \to G$ in continuity points of G as $k \to \infty$, where G is a nondecreasing, right-continuous function on $\mathbb{R}$. Since one can show that the Stieltjes measure m_{G_n} induced by G_n satisfies

$$\lim_{t\to\infty} \sup_n m_{G_n}\big(\mathbb{R}\backslash(-t,t]\big) = 0,$$

we also have $G_{n_k}(\infty) \to G(\infty)$. Hence we could apply Helly's theorem on convergence of integrals of continuous bounded functions if the integrand in (4.2) were not unbounded near zero. To remedy this we rewrite (4.2) as

$$\text{(4.3)} \qquad \log\phi(u) = \lim_{n\to\infty} \Big\{ iua_n + \int_{\mathbb{R}} \Big(\mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2}\Big)\, \frac{1+x^2}{x^2}\, \mathrm{d}G_n(x) \Big\},$$

where $a_n := \int_{\mathbb{R}} x/(1+x^2)\, \mu_n(\mathrm{d}x)$. Now, the integrand is continuous and bounded on $\mathbb{R}\backslash\{0\}$ with limit $-\frac{1}{2}u^2$ as $x \to 0$. It follows that along the subsequence (n_k) the integral in (4.3) converges to the integral with G_n replaced by G. Since then necessarily also $a_{n_k} \to a$ for some $a \in \mathbb{R}$, we conclude that

$$\text{(4.4)} \qquad \log\phi(u) = iua + \int_{\mathbb{R}} \Big(\mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2}\Big)\, \frac{1+x^2}{x^2}\, \mathrm{d}G(x).$$

This is the so-called *Lévy-Khintchine representation* for ϕ. Note that if G has a positive jump σ^2 at zero, then in the right-hand side of (4.4) we get a term $-\frac{1}{2}u^2\sigma^2$, so ϕ has a *normal factor.*

Conversely, any (nonvanishing) function ϕ that satisfies (4.4) for some $a \in \mathbb{R}$ and some bounded, nondecreasing function G, is the characteristic function of an infinitely divisible distribution on $\mathbb{R}$. This can be shown by approximating the integral in (4.4) by Riemann sums. After some rearrangement of terms one then sees that ϕ is the pointwise limit of a product of infinitely divisible characteristic functions, viz. of a (possibly degenerate) normal distribution and of Poisson distributions on different lattices. Letting $u \to 0$ in (4.4) and using the bounded convergence theorem shows that ϕ is continuous at zero. One can now apply Proposition 2.3 to conclude that ϕ is an infinitely divisible characteristic function. Thus we have obtained the following celebrated result.

Theorem 4.1 (Lévy-Khintchine representation). *A $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of an infinitely divisible distribution iff ϕ has the form*

$$\text{(4.5)} \qquad \phi(u) = \exp\Big[\, iua + \int_{\mathbb{R}} \Big(\mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2}\Big)\, \frac{1+x^2}{x^2}\, dG(x) \Big],$$

where $a \in \mathbb{R}$ and G is a bounded, nondecreasing, right-continuous function on $\mathbb{R}$ with $G(x) \to 0$ as $x \to -\infty$ (for $x = 0$ the integrand is defined by continuity: $-\frac{1}{2}u^2$).

As it should be in a canonical representation, the *canonical pair* (a, G) is uniquely determined by ϕ. We will prove a bit more and allow G to be *of finite total variation*, so G can be written as the difference of two bounded, *nondecreasing* functions.

Proposition 4.2. *Let ϕ be a function of the form (4.5) with $a \in \mathbb{R}$ and G a right-continuous function of finite total variation with $G(x) \to 0$ as $x \to -\infty$. Then a and G are uniquely determined by ϕ.*

Proof. Let $k(u, x)$ be the integrand in (4.5); then $\log \phi$ can be written as

$$\log \phi(u) = iua + \int_{\mathbb{R}} k(u, x)\, \mathrm{d}G(x).$$

First, consider the (infinitely divisible) case where G is nondecreasing. We evaluate the following function ψ, using Fubini's theorem:

$$\psi(u) := \int_0^1 \Big[\log \phi(u) - \tfrac{1}{2} \big\{ \log \phi(u+h) + \log \phi(u-h) \big\} \Big]\, \mathrm{d}h =$$

$$= \int_0^1 \Big(\int_{\mathbb{R}} \big[k(u, x) - \tfrac{1}{2} \big\{ k(u+h, x) + k(u-h, x) \big\} \big]\, \mathrm{d}G(x) \Big)\, \mathrm{d}h =$$

$$= \int_0^1 \Big(\int_{\mathbb{R}} \mathrm{e}^{iux} (1 - \cos hx)\, \frac{1+x^2}{x^2}\, \mathrm{d}G(x) \Big)\, \mathrm{d}h =$$

$$= \int_{\mathbb{R}} \mathrm{e}^{iux} \Big(\int_0^1 (1 - \cos hx)\, \mathrm{d}h \Big)\, \frac{1+x^2}{x^2}\, \mathrm{d}G(x).$$

So ψ is the Fourier-Stieltjes transform $\widetilde{L}$ of the function L defined by

$$L(y) := \int_{(-\infty, y]} \ell(x)\, \mathrm{d}G(x), \quad \text{with } \ell(x) := \Big(1 - \frac{\sin x}{x}\Big)\, \frac{1+x^2}{x^2};$$

note that ℓ is bounded. By the uniqueness theorem for Fourier-Stieltjes transforms it follows that L is determined by ψ and hence by ϕ. Since the function ℓ is bounded away from 0, we can write

$$G(x) = \int_{(-\infty, x]} \frac{1}{\ell(y)}\, \mathrm{d}L(y),$$

so G also is determined by ϕ; the constant a in (4.5) is then unique as well. Next, consider the general case. Suppose that ϕ has two representations of the form (4.5) determined by the pairs (a, G), (b, H), with G and H of finite total variation. Write $G = G_1 - G_2$ and $H = H_1 - H_2$ with G_1, G_2, H_1 and H_2 bounded, nondecreasing and right-continuous. Then it follows that

$$iua + \int_{\mathbb{R}} k(u,x)\,\mathrm{d}\big\{G_1(x) + H_2(x)\big\} = \\ = iub + \int_{\mathbb{R}} k(u,x)\,\mathrm{d}\big\{H_1(x) + G_2(x)\big\}.$$

Now, since $G_1 + H_2$ and $H_1 + G_2$ are nondecreasing, we are in the case that was considered first. Hence $a = b$ and $G_1 + H_2 = H_1 + G_2$, so $G = H$. □□

This proposition can be used to show that a given characteristic function is *not* infinitely divisible; cf. Example III.11.1. Also, the uniqueness of the canonical pair (a, G) in the infinitely divisible case implies, together with the continuity theorem, the following not surprising limit result; we do not give details.

Proposition 4.3. *Let F and F_n with $n \in \mathbb{N}$ be infinitely divisible distribution functions with canonical pairs (a, G) and (a_n, G_n), respectively. Then, as $n \to \infty$, $F_n \to F$ weakly iff $a_n \to a$ and $G_n \to G$ at continuity points of G.*

Several characteristics of an infinitely divisible distribution can be expressed in terms of its canonical quantities. Before showing this and giving some examples, we prefer to switch from the Lévy-Khintchine representation to the so-called *Lévy representation*, because the latter is more directly related to the corresponding distribution. In fact, we switch from the Lévy-Khintchine canonical pair (a, G) to the *Lévy canonical triple* (a, σ^2, M) where

$$(4.6) \qquad \sigma^2 = m_G\big(\{0\}\big), \quad M(x) = \begin{cases} \displaystyle\int_{(-\infty,x]} \frac{1+y^2}{y^2}\,\mathrm{d}G(y) & \text{, if } x < 0, \\ \displaystyle -\int_{(x,\infty)} \frac{1+y^2}{y^2}\,\mathrm{d}G(y) & \text{, if } x > 0. \end{cases}$$

Theorem 4.1 can now be reformulated as follows; the uniqueness assertion is, of course, a consequence of Proposition 4.2.

Theorem 4.4 (Lévy representation). *A $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of an infinitely divisible distribution iff ϕ has the form*

$$\phi(u) = \exp\Big[\, iua - \tfrac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(e^{iux} - 1 - \frac{iux}{1+x^2}\Big)\, dM(x)\Big], \tag{4.7}$$

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and M is a right-continuous function that is nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$ with $M(x) \to 0$ as $x \to -\infty$ or $x \to \infty$ and

$$\int_{[-1,1]\setminus\{0\}} x^2\, dM(x) < \infty. \tag{4.8}$$

The canonical triple (a, σ^2, M) is uniquely determined by ϕ.

Clearly, the first two quantities in the canonical triple (a, σ^2, M) of an infinitely divisible characteristic function $\phi = \widetilde{F}$ represent a shift and a normal component, respectively, in F. The third quantity is often called the *Lévy function* of ϕ and of F. From (4.6) and the discussion in the beginning of the present section it follows that (a, σ^2, M) can be obtained from F in the following way (the expression for M only holds for continuity points x of M):

$$\begin{cases} a = \lim_{n\to\infty} n \displaystyle\int_{\mathbb{R}} \frac{x}{1+x^2}\, dF^{\star(1/n)}(x), \\ \sigma^2 = \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} n \displaystyle\int_{(-\varepsilon,\varepsilon]} \frac{x^2}{1+x^2}\, dF^{\star(1/n)}(x), \\ M(x) = \begin{cases} \lim_{n\to\infty} n\, F^{\star(1/n)}(x) & \text{, if } x < 0, \\ -\lim_{n\to\infty} n\,\{1 - F^{\star(1/n)}(x)\} & \text{, if } x > 0. \end{cases} \end{cases} \tag{4.9}$$

By the uniqueness of the canonical representation we easily obtain the following simple, but useful relations.

Proposition 4.5. *Let X be an infinitely divisible random variable with distribution function F and canonical triple (a, σ^2, M). Then:*

(i) *The canonical triple of $-X$ is $(-a, \sigma^2, M^-)$ where M^- is given by $M^-(x) := -M\big((-x)-\big)$.*

(ii) *For $\alpha > 0$ the canonical triple of αX is $\big(a_\alpha, \alpha^2\sigma^2, M(\cdot/\alpha)\big)$ with a_α given by $a_\alpha := \alpha a + \alpha(1-\alpha^2) \int_{\mathbb{R}\setminus\{0\}} x^3/\{(1+x^2)(1+\alpha^2x^2)\}\, dM(x)$.*

(iii) *For* $t > 0$ *the canonical triple of* $F^{\star t}$ *is* $(ta, t\sigma^2, tM)$.

(iv) *If* F_1, F_2 *are infinitely divisible distribution functions with canonical triples* (a_1, σ_1^2, M_1), (a_2, σ_2^2, M_2), *then* $F = F_1 \star F_2$ *iff* $a = a_1 + a_2$, $\sigma^2 = \sigma_1^2 + \sigma_2^2$ *and* $M = M_1 + M_2$.

We determine the canonical representations of the three examples from Section 2, and add a fourth basic example which generalizes the first two.

Example 4.6. The *normal* $(0, \sigma^2)$ distribution of Example 2.7 with characteristic function ϕ given by

$$\phi(u) = \mathrm{e}^{-\frac{1}{2}\sigma^2 u^2},$$

has canonical triple $(0, \sigma^2, 0)$, so with Lévy function M that vanishes everywhere. This immediately follows by comparing the expression for ϕ with representation (4.7). □

Example 4.7. The *Cauchy* (λ) distribution of Example 2.8 with characteristic function ϕ given by

$$\phi(u) = \mathrm{e}^{-\lambda|u|},$$

has canonical triple $(0, 0, M)$ with M absolutely continuous with density

$$m(x) = \frac{\lambda}{\pi x^2} \qquad [x \neq 0].$$

This can be verified directly by integration from (4.7). It can be found from the relations in (4.9) by using the dominated convergence theorem and the well-known fact, which easily follows from the expression for ϕ, that the Cauchy distribution function F has the (stability) property that $F^{\star(1/n)}(x) = F(nx)$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Note that by (4.6) the Lévy-Khintchine canonical function G is given by $G(x) = \lambda\, F(\lambda x)$, so the standard Cauchy distribution has $G = F$. □

Example 4.8. The *sym-gamma* (r, λ) distribution of Example 2.9 with characteristic function ϕ given by

$$\phi(u) = \Big(\frac{\lambda^2}{\lambda^2 + u^2}\Big)^r,$$

has canonical triple $(0, 0, M)$ with M absolutely continuous with density

$$m(x) = \frac{r}{|x|}\,\mathrm{e}^{-\lambda|x|} \qquad [x \neq 0].$$

To show this we recall that ϕ can be viewed as the characteristic function of $Y - Y'$ with Y and Y' independent and gamma (r, λ) distributed. Now, in Example III.4.8 we have seen that Y has ($\mathbb{R}_+$-) canonical density k given by $k(x) = r\,\mathrm{e}^{-\lambda x}$; for the characteristic function ϕ_Y of Y this implies:

$$\phi_Y(u) = \exp\Big[\int_0^\infty (\mathrm{e}^{iux} - 1)\,\frac{r}{x}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}x\Big].$$

By comparing this expression with (4.7) we conclude that Y has canonical triple $(a_1, 0, M_1)$ with M_1 absolutely continuous, where a_1 and a density m_1 of M_1 are given by

$$a_1 = \int_0^\infty \frac{r}{1+x^2}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}x, \quad m_1(x) = \begin{cases} 0 & \text{, if } x < 0, \\ \dfrac{r}{x}\,\mathrm{e}^{-\lambda x} & \text{, if } x > 0. \end{cases}$$

Finally, applying Proposition 4.5 (i), (iv) yields the result stated above. □

Example 4.9. For $\lambda > 0$, $\gamma > 0$, let ϕ be the function on $\mathbb{R}$ given by

$$\phi(u) = \exp\big[-\lambda\,|u|^\gamma\big] \qquad [\,u \in \mathbb{R}\,].$$

In case $\gamma > 2$ this function is *not* a characteristic function because $\phi'(0) = \phi''(0) = 0$, so if $\phi = \phi_X$, then we would have $\mathbb{E}X = \mathbb{E}X^2 = 0$. For $\gamma = 2$ we get Example 4.6, and for $\gamma = 1$ Example 4.7. Now take general $\gamma \le 2$. Then $\gamma = 2\delta$ with $\delta \in (0, 1\,]$, so ϕ can be written as

$$\phi(u) = \pi(u^2), \quad \text{with} \quad \pi(s) := \exp\big[-\lambda\,s^\delta\big].$$

Since by Example III.4.9 the function π is an infinitely divisible pLSt, we can apply Proposition 3.6 with ϕ_0 normal to conclude that ϕ is an *infinitely divisible characteristic function.* When $\gamma < 2$, ϕ has canonical triple $(0, 0, M)$ with M absolutely continuous with density

$$m(x) = \frac{c}{|x|^{1+\gamma}} \qquad [\,x \ne 0\,],$$

where $c > 0$ is a constant depending on λ and γ. This can be verified directly by integration from (4.7). For more details, including the value of c, we refer to Section V.7; in view of the results given there, for $\lambda > 0$ and $\gamma \in (0, 2\,]$ the characteristic function ϕ, and the corresponding distribution, will be called *symmetric stable* (λ) *with exponent* γ. □

Generally, it is not so easy to determine the canonical triple (a, σ^2, M) for a given infinitely divisible characteristic function ϕ. The proof of Proposition 4.2 shows a way to obtain σ^2 and M (via G) from ϕ, but in practice this will not be very helpful. For σ^2 there is an alternative; it can be used in the preceding examples.

Proposition 4.10. *Let ϕ be an infinitely divisible characteristic function with canonical triple (a, σ^2, M). Then σ^2 can be obtained as*

$$\sigma^2 = \lim_{u \to \infty} \frac{-2 \log \phi(u)}{u^2}. \tag{4.10}$$

PROOF. By (4.7) it suffices to show that as $u \to \infty$

$$\int_{\mathbb{R}\setminus\{0\}} g_u(x) \, \mathrm{d}M(x) \to 0, \quad \text{if } g_u(x) := \frac{1}{u^2} \Big(\mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2} \Big).$$

Obviously, we have $\lim_{u \to \infty} g_u(x) = 0$ for $x \neq 0$. Moreover, using the fact that $|\mathrm{e}^{iy} - 1 - iy| \leq \frac{1}{2} y^2$ for $y \in \mathbb{R}$, we see that $|g_u(x)|$ is bounded by 3 for $|x| > 1$ and by $\frac{3}{2} x^2$ for $|x| \leq 1$, uniformly in $u \geq 1$. Because of (4.8) we can now apply the dominated convergence theorem. □

For distributions on $\mathbb{R}_+$ things are less complicated. As shown in Section III.4, the ($\mathbb{R}_+$-) canonical function K of an infinitely divisible distribution function F on $\mathbb{R}_+$ with pLSt $\pi = \widehat{F}$ can be obtained from π via the relation $\widehat{K} = \rho$, where $\rho := (-\log \pi)' = -\pi'/\pi$. Moreover, the relation $-\pi' = \pi \widehat{K}$ yields a useful functional equation for F. Returning to distributions on $\mathbb{R}$, one is tempted to consider the function $(\log \phi)'$ if ϕ is given by (4.7). Such a derivative need not exist, however, and if it does, it has no simple relation with the canonical function M.

Still, some conclusions can be drawn from (4.7). We will characterize several interesting classes of infinitely divisible distributions in terms of the canonical quantities and give simplified representations for the corresponding characteristic functions. In doing so we will focus on a precise formulation of the results; the proofs, which mainly consist of manipulating representations and verifying integrability conditions, will be brief. We start with considering infinitely divisible distributions that are *symmetric*. They are closely related to distributions on $\mathbb{R}_+$, and occur frequently in practice. Moreover, they have the pleasant property that because of Proposition 2.4 their characteristic functions are *positive* on $\mathbb{R}$.

Theorem 4.11. *Let F be a distribution function on $\mathbb{R}$ with characteristic function ϕ. Then the following three assertions are equivalent:*

(i) *F is symmetric and infinitely divisible.*

(ii) *F is infinitely divisible with canonical triple $(0, \sigma^2, M)$ with $\sigma^2 \geq 0$ and M satisfying $M = M^-$, where $M^-(x) := -M\big((-x)-\big)$.*

(iii) *ϕ has the form*

$$\phi(u) = \exp\Big[-\tfrac{1}{2}u^2\sigma^2 + 2\int_{(0,\infty)} (\cos\, ux - 1)\, dM(x)\Big],$$

where $\sigma^2 \geq 0$ and M is a right-continuous, nondecreasing function on $(0,\infty)$ with $\lim_{x\to\infty} M(x) = 0$ and $\int_{(0,1]} x^2\, dM(x) < \infty$.

In this case σ^2 in (iii) equals σ^2 in (ii) and M in (iii) is the Lévy function restricted to $(0,\infty)$.

Proof. The equivalence of (i) and (ii) immediately follows from Proposition 4.5 (i). The equivalence of (ii) and (iii) is obtained by rewriting (4.7); one can also start from (i) and use the fact that ϕ then satisfies $\phi = |\phi| = \exp\,[\mathrm{Re}\log\,\phi]$. □

There is a simple 1–1 correspondence between the symmetric distributions on $\mathbb{R}$ and the distributions on $\mathbb{R}_+$. In fact, by taking $Y := |X|$ one sees that a random variable X has a *symmetric* distribution iff

(4.11) $\quad X \overset{\mathrm{d}}{=} A\,Y,\;$ with Y $\mathbb{R}_+$-valued,

where A has a Bernoulli $(\frac{1}{2})$ distribution on $\{-1, 1\}$ and is independent of Y. In terms of distribution functions and, in case of absolute continuity, densities this relation reads as follows:

(4.12) $\quad F(x) = \frac{1}{2}\,\{1 + G(x)\}$ for $x \geq 0$, $\quad f(x) = \frac{1}{2}\,g\big(|x|\big)$ for $x \in \mathbb{R}$.

In view of these simple relations one might think that if Y is infinitely divisible, then so is X, but this is *not* so; for a counter-example we refer to Section 11 and for some positive results of this type to Section 10. On the other hand, *not* every (symmetric) infinitely divisible X has the property that Y is infinitely divisible; in Example III.11.2 it is noted that the *half-normal* distribution is *not* infinitely divisible (whereas the *half-Cauchy* distribution and, of course, the *half-Laplace* distribution are infinitely divisible). It would be interesting to know which symmetric infinitely divisible distributions stem from infinitely divisible distributions on $\mathbb{R}_+$ as above.

We do not go further into this, and turn to a similar problem that is easier to solve. To this end, in stead of (4.11), we consider random variables X such that

$$(4.13) \qquad X \stackrel{\mathrm{d}}{=} Y - Y', \ \text{ with } Y \ \mathbb{R}_+\text{-valued},$$

where Y' is independent of Y with $Y' \stackrel{\mathrm{d}}{=} Y$. Clearly, such an X has a *symmetric* distribution, and if Y is infinitely divisible, then so is X; cf. Example 4.8. Note that, conversely, a symmetric infinitely divisible X can be written as in (4.13) with Y infinitely divisible, but *not* necessarily $\mathbb{R}_+$-valued; one can take $Y \stackrel{\mathrm{d}}{=} X_2$, the 2-nd order factor of X, in which case Y has a symmetric distribution, too. Therefore, it is not trivial to ask for the (symmetric infinitely divisible) distributions that can be obtained from infinitely divisible distributions on $\mathbb{R}_+$ as in (4.13).

To answer this question we first consider the class of infinitely divisible distributions *with finite left extremity*; they share in the advantages of distributions on the positive half-line. At this point it is easiest to make use of some results on $\mathbb{R}_+$.

Lemma 4.12. *Let F be an infinitely divisible distribution function with characteristic function ϕ. Then $\ell_F > -\infty$ iff ϕ has the form*

$$(4.14) \qquad \phi(u) = \exp\Big[iu\gamma + \int_{(0,\infty)} (\mathrm{e}^{iux} - 1)\, \frac{1}{x}\, dK(x)\Big],$$

where $\gamma \in \mathbb{R}$ and K is an LSt-able function with $K(0) = 0$ and satisfying $\int_{(1,\infty)} (1/x)\, dK(x) < \infty$. In this case the pair (γ, K) is uniquely determined by F, and $\ell_F = \gamma$ and $F(\ell_F) = \mathrm{e}^{-\lambda}$, where $\lambda := \int_{(0,\infty)} (1/x)\, dK(x)$ $(\leq \infty)$. If $0 < \lambda < \infty$, then F is shifted compound-Poisson on $\mathbb{R}_+$:

$$(4.15) \qquad \phi(u) = \exp\big[iu\ell_F + \lambda \big\{\widetilde{G}(u) - 1\big\}\big],$$

where G is the distribution function with $G(x) = (1/\lambda) \int_{(0,x]} (1/y)\, dK(y)$ for $x > 0$ and $G(0) = 0$.

Proof. All the assertions immediately follow from Theorem III.4.3 and Proposition III.4.4; when $\ell_F > -\infty$, these are applied for the infinitely divisible distribution function $H := F(\ell_F + \cdot)$ for which $\ell_H = 0$. ◻

The first part of this lemma yields the following result on the Lévy representation of an infinitely divisible distribution with finite left extremity.

The subsequent corollary characterizes the *infinitely divisible distributions on* $\mathbb{R}_+$, which are considered separately in the preceding chapter; recall that $K(0) = \ell_F$.

Theorem 4.13. *Let F be a distribution function on $\mathbb{R}$ with characteristic function ϕ. Then the following three assertions are equivalent:*

(i) *F is infinitely divisible with $\ell_F > -\infty$.*

(ii) *F is infinitely divisible with canonical triple $(a, 0, M)$ with $a \in \mathbb{R}$, $M(0-) = 0$ and $\int_{(0,1]} x\, dM(x) < \infty$.*

(iii) *ϕ has the form*

$$\phi(u) = \exp\Big[iu\gamma + \int_{(0,\infty)} (e^{iux} - 1)\, dM(x)\Big],$$

where $\gamma \in \mathbb{R}$ and M is a right-continuous, nondecreasing function on $(0, \infty)$ with $\lim_{x\to\infty} M(x) = 0$ and $\int_{(0,1]} x\, dM(x) < \infty$.

In this case $\gamma = \ell_F = a - \int_{(0,\infty)} x/(1 + x^2)\, dM(x)$ and M in (iii) is the Lévy function restricted to $(0, \infty)$.

Proof. Let F satisfy (i). Then by Lemma 4.12 ϕ has the form (4.14) with $\gamma = \ell_F$. Since $b := \int_{(0,\infty)} 1/(1 + x^2)\, \mathrm{d}K(x) < \infty$, ϕ can be rewritten as

$$\phi(u) = \exp\Big[\, iu(\gamma + b) + \int_{(0,\infty)} \Big(\mathrm{e}^{iux} - 1 - \frac{iux}{1 + x^2}\Big) \frac{1}{x}\, \mathrm{d}K(x)\Big].$$

Comparing with (4.7) one sees that F has canonical triple $(a, 0, M)$ with $a = \gamma + b$, $M(0-) = 0$ and $M(x) = -\int_{(x,\infty)} (1/y)\, \mathrm{d}K(y)$ for $x > 0$, so we have $\int_{(0,1]} x\, \mathrm{d}M(x) = K(1)$.
The rest of the proof is similar. If F satisfies (ii), then (4.7) can be rewritten as in (iii), and if ϕ has the form given in (iii), then we have (4.14) with $K(x) = \int_{(0,x]} y\, \mathrm{d}M(y)$ for $x > 0$. The various integrability conditions are easily verified. □

Corollary 4.14. *Let F be an infinitely divisible distribution function on $\mathbb{R}$ with canonical triple (a, σ^2, M). Then F is a distribution function on $\mathbb{R}_+$ iff $\sigma^2 = 0$, $\gamma := a - \int_{(0,\infty)} x/(1 + x^2)\, dM(x) \geq 0$, $\int_{(0,1]} x\, dM(x) < \infty$ and $M(0-) = 0$. In this case the ($\mathbb{R}_+$-) canonical function K and the Lévy canonical quantities a and M are related by*

$$(4.16) \quad \begin{cases} K(x) = \gamma + \int_{(0,x]} y\, dM(y) \;\; \text{for } x \geq 0; \\ a = \int_{\mathbb{R}_+} \frac{1}{1+x^2}\, dK(x),\;\; M(x) = -\int_{(x,\infty)} \frac{1}{y}\, dK(y) \;\; \text{for } x > 0. \end{cases}$$

From the theorem it follows that the condition $M = 0$ on $(-\infty, 0)$ is *not* sufficient for an infinitely divisible distribution without normal component $(\sigma^2 = 0)$ to have a finite left extremity; if $\int_{(0,1]} x\, \mathrm{d}M(x) = \infty$ (and, of course, still $\int_{(0,1]} x^2\, \mathrm{d}M(x) < \infty$), then the support of the distribution is unbounded below. Clearly, Theorem 4.13 has an obvious counterpart for the *right extremity* r_F of F; we do not make it explicit.

The relations in (4.16) can sometimes be used to determine the canonical triple of a given infinitely divisible distribution with finite left extremity, like the *gamma* distribution; cf. Example 4.8. There we used Proposition 4.5 (i), (iv) in order to handle also the *sym-gamma* distribution. Now, in a similar way this proposition leads to the following result.

Theorem 4.15. *Let X be a random variable with characteristic function ϕ. Then the following three assertions are equivalent:*

(i) *$X \stackrel{\mathrm{d}}{=} Y - Z$ with Y and Z independent and infinitely divisible with $\ell_Y > -\infty$ and $\ell_Z > -\infty$.*

(ii) *X is infinitely divisible with canonical triple $(a, 0, M)$ with $a \in \mathbb{R}$ and M satisfying $\int_{[-1,1]\setminus\{0\}} |x|\, dM(x) < \infty$.*

(iii) *ϕ has the form*

$$\phi(u) = \exp\Big[iu\gamma + \int_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1)\, dM(x)\Big],$$

where $\gamma \in \mathbb{R}$ and M is right-continuous, nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$ with $M(x) \to 0$ as $x \to -\infty$ or $x \to \infty$ and satisfying $\int_{[-1,1]\setminus\{0\}} |x|\, dM(x) < \infty$.

In this case $\gamma = \ell_Y - \ell_Z = a - \int_{\mathbb{R}\setminus\{0\}} x/(1+x^2)\, dM(x)$ and M in (iii) is the Lévy function.

Proof. As suggested above, combining Proposition 4.5 (i), (iv) with Theorem 4.13 shows that (i) implies (ii). The converse statement follows by writing $M = M_1 + M_2$ with $M_1 = 0$ on $(-\infty, 0)$ and $M_2 = 0$ on $(0, \infty)$. The equivalence of (i) and (iii), or (ii) and (iii), is proved similarly. □

The quantity γ in this theorem is sometimes called the *shift parameter* of the infinitely divisible distributions having Lévy functions M satisfying $\int_{[-1,1]\setminus\{0\}} |x| \, \mathrm{d}M(x) < \infty$. We further note that Y and Z in (i) are *not* uniquely determined by X; only (the distributions of) $Y - \ell_Y$ and $Z - \ell_Z$, and the value of $\ell_Y - \ell_Z$, are unique. It follows that Y and Z can be chosen to be $\mathbb{R}_+$*-valued*. Requiring $Z \stackrel{\mathrm{d}}{=} Y$ in this case leads, together with Theorem 4.11, to the following answer to the question on *symmetric infinitely divisible* distributions that was raised in connection with (4.13).

Corollary 4.16. *A random variable X can be written (in distribution) as*

$$X \stackrel{\mathrm{d}}{=} Y - Y', \quad \text{with } Y \text{ infinitely divisible and } \mathbb{R}_+\text{-valued},$$

where Y' is independent of Y with $Y' \stackrel{\mathrm{d}}{=} Y$, iff X is infinitely divisible having canonical triple $(0, 0, M)$ with M satisfying $\int_{(0,1]} x \, \mathrm{d}M(x) < \infty$ and $M = M^-$, where $M^-(x) := -M\big((-x)-\big)$.

The Lévy function M of a general infinitely divisible distribution has the property that $\int_{[-1,1]\setminus\{0\}} x^2 \, \mathrm{d}M(x) < \infty$. In Theorem 4.15 we considered the special case where M satisfies $\int_{[-1,1]\setminus\{0\}} |x| \, \mathrm{d}M(x) < \infty$. Now, we will be even more specific and require that $\int_{[-1,1]\setminus\{0\}} \mathrm{d}M(x) < \infty$ or, equivalently, $M(0-) < \infty$ and $M(0+) > -\infty$, i.e., M *bounded*. As before we start with the one-sided case, and can then use the second part of Lemma 4.12.

Theorem 4.17. *Let F be a non-degenerate distribution function on $\mathbb{R}$ with characteristic function ϕ. Then the following three assertions are equivalent:*

(i) *F is infinitely divisible with $\ell_F > -\infty$ and $F(\ell_F) > 0$.*

(ii) *F is infinitely divisible with canonical triple $(a, 0, M)$ with $a \in \mathbb{R}$, $M(0-) = 0$ and M bounded.*

(iii) *ϕ is shifted compound-Poisson on $\mathbb{R}_+$; it has the form*

$$\phi(u) = \exp\big[iu\gamma + \lambda\big\{\widetilde{G}(u) - 1\big\}\big],$$

where $\gamma \in \mathbb{R}$, $\lambda > 0$ and G is a distribution function with $G(0) = 0$.

In this case $\gamma = \ell_F = a - \int_{(0,\infty)} x/(1 + x^2) \, \mathrm{d}M(x)$, $\lambda = -\log F(\ell_F) = -M(0+)$ and $G = 1 + M/\lambda$ on $(0, \infty)$.

Proof. Use Lemma 4.12 and (the proof of) Theorem 4.13; M and K are related as in (4.16). ◻

Theorem 4.18. *Let X be a non-degenerate random variable with characteristic function ϕ. Then the following three assertions are equivalent:*

(i) $X \stackrel{\rm d}{=} Y - Z$ *with Y and Z independent and infinitely divisible with $\ell_Y > -\infty$, $\ell_Z > -\infty$, $\mathbb{P}(Y = \ell_Y) > 0$ and $\mathbb{P}(Z = \ell_Z) > 0$.*

(ii) *X is infinitely divisible with canonical triple $(a, 0, M)$ with $a \in \mathbb{R}$ and with M bounded.*

(iii) *ϕ is shifted compound-Poisson; it has the form*

$$\phi(u) = \exp\left[iu\gamma + \lambda\left\{\widetilde{G}(u) - 1\right\}\right],$$

where $\gamma \in \mathbb{R}$, $\lambda > 0$ and G is a distribution function that is continuous at zero.

In this case (γ, λ, G) can be obtained as follows:

$$\text{(4.17)} \qquad \begin{cases} \gamma = \ell_Y - \ell_Z = a - \int_{\mathbb{R}\setminus\{0\}} x/(1+x^2)\, dM(x), \\ \lambda = -\log \mathbb{P}(Y = \ell_Y; Z = \ell_Z) = M(0-) - M(0+), \\ G = M/\lambda \text{ on } (-\infty, 0),\ G = 1 + M/\lambda \text{ on } (0, \infty). \end{cases}$$

PROOF. Similar to the proof of Theorem 4.15: Combine Proposition 4.5 (i), (iv) with Theorem 4.17, and write $M = M_1 + M_2$ with $M_1(0-) = 0$ and $M_2(0+) = 0$. ◻◻

The preceding theorem leads us to consider *Lebesgue properties* of an infinitely divisible distribution function F. Clearly, if F has a normal component, then F is absolutely continuous. Therefore, we further suppose that F has canonical triple $(a, 0, M)$. According to Theorem 4.18 we then have

$$\text{(4.18)} \qquad M \text{ bounded} \iff F \text{ shifted compound-Poisson.}$$

In this case F has at least one discontinuity point; the following partial converse is implied by Theorem 3.3:

$$\text{(4.19)} \qquad \begin{aligned} &F \text{ has exactly one discontinuity point } \Longrightarrow \\ &\qquad \Longrightarrow F \text{ shifted compound-Poisson.} \end{aligned}$$

By using some elementary properties of continuous and absolutely continuous functions, as mentioned in Section A.2, we can say more.

Proposition 4.19. *Let F be an infinitely divisible distribution function with canonical triple $(a, 0, M)$. Then:*

(i) *F has exactly one discontinuity point iff M is bounded and continuous.*

(ii) *F has exactly one discontinuity point γ, say, and is absolutely continuous on $\mathbb{R}\setminus\{\gamma\}$ iff M is bounded and absolutely continuous.*

PROOF. Combine (4.18) and (4.19), and use the fact that if F is compound-Poisson, shifted over γ, then for Borel sets B

$$m_F(B+\gamma) = \sum_{n=1}^{\infty} \Big(\frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}\Big)\, m_{G^{\star n}}(B) \qquad \big[\, B \subset \mathbb{R}\setminus\{0\}\,\big], \tag{4.20}$$

where $\lambda > 0$ and G is a distribution function that is continuous at zero; G is related to M as stated in Theorem 4.18. Now (i) follows by taking $B = \{b\}$ with $b \neq 0$, and (ii) by taking $B \subset \mathbb{R}\setminus\{0\}$ having Lebesgue measure zero: $m(B) = 0$. ◻

In addition to (i) and (ii) in this proposition we have the following properties:

(iii) *F has at least one discontinuity point iff M is bounded.*

(iv) *F has at least two discontinuity points iff M is bounded and not continuous.*

(v) *F is discrete iff M is bounded and discrete.*

(vi) *F corresponds to a distribution on $\mathbb{Z}$ iff M is bounded and discrete with discontinuities restricted to $\mathbb{Z}\setminus\{0\}$, and the shift parameter γ belongs to $\mathbb{Z}$.*

Here (the direct part of) (iii) is the crucial property; it will be proved as Theorem 4.20 below. The remaining properties then easily follow, as in the proof of Proposition 4.19. Note that (iii), because of (4.18), can be reformulated so as to improve (4.19):

$$\begin{aligned} &F \text{ has at least one discontinuity point } \iff \\ &\qquad\qquad \iff F \text{ shifted compound-Poisson.} \end{aligned} \tag{4.21}$$

Another reformulation is given in the following theorem. To prove it is not so easy; we will use a result on random walks that can be found in Section A.5.

Theorem 4.20. *Let F be an infinitely divisible distribution function with canonical triple $(a, 0, M)$. Then F is continuous iff M is unbounded.*

PROOF. From (4.18) it is seen that if F is continuous, then M is unbounded. To prove the converse it suffices to show that a *symmetric* infinitely divisible distribution with *unbounded* Lévy function has *no mass at zero*. To see this, observe that if X is a random variable with $F_X = F$ and unbounded M, and if $\overline{X} := X - X'$ with X and X' independent and $X' \overset{\mathrm{d}}{=} X$, then by Proposition 4.5 (i), (iv) $\overline{X}$ has a symmetric infinitely divisible distribution with unbounded Lévy function $\overline{M}$ and

$$\mathbb{P}(\overline{X} = 0) \geq \{\mathbb{P}(X = x)\}^2 \qquad [x \in \mathbb{R}].$$

So, let F be symmetric with M unbounded; note that by Theorem 4.11 $a = 0$ and $M = M^-$ where $M^-(x) := -M\big((-x)-\big)$. Take X such that $F_X = F$; we have to show that $\mathbb{P}(X = 0) = 0$. To do so, for $n \in \mathbb{N}$ we write $M = M_n + N_n$ where

$$M_n(x) := \begin{cases} M(x) & \text{, if } x < -1/n \text{ or } x \geq 1/n, \\ M(-1/n) & \text{, if } -1/n \leq x < 0, \\ M\big((1/n)-\big) & \text{, if } 0 < x < 1/n. \end{cases}$$

The functions M_n and N_n can be viewed as canonical functions, so we can take independent infinitely divisible random variables X_n and Y_n with canonical triples $(0, 0, M_n)$ and $(0, 0, N_n)$, respectively; then $X \overset{\mathrm{d}}{=} X_n + Y_n$ because of Proposition 4.5 (iv). Also, from the first part of this proposition it will be clear that M_n and N_n are constructed in such a way that X_n and Y_n inherit the symmetry of X. Therefore, we can proceed as in the proof of Proposition 2.10 to show that

$$\mathbb{P}(X = 0) \leq \mathbb{P}(X_n = 0) \qquad [n \in \mathbb{N}].$$

Now, take n so large that M_n is not identically zero; then X_n is non-degenerate. Since M_n is bounded, Theorem 4.18 guarantees that X_n has a compound-Poisson distribution, so

$$\mathbb{P}(X_n = 0) = \sum_{k=0}^{\infty} \Big(\frac{\lambda_n^k}{k!}\, \mathrm{e}^{-\lambda_n}\Big)\, \mathbb{P}(S_k^{(n)} = 0),$$

where $\lambda_n := M_n(0-) - M_n(0+) = 2M(-1/n) > 0$ and $(S_k^{(n)})_{k \in \mathbb{Z}_+}$ is a random walk with a step-size distribution that is symmetric and has no mass at zero. For such random walks it can be shown that for all $\ell \in \mathbb{Z}_+$, respectively $\ell \to \infty$:

$$\mathbb{P}(S_{2\ell+1}^{(n)} = 0) \le \mathbb{P}(S_{2\ell}^{(n)} = 0) \le$$
$$\le \mathbb{P}(T_{2\ell} = 0) = u_{2\ell} := \binom{2\ell}{\ell} \left(\tfrac{1}{2}\right)^{2\ell} \sim \frac{1}{\sqrt{\pi \ell}},$$

where $(T_k)_{k \in \mathbb{Z}_+}$ is the symmetric Bernoulli walk; see Proposition A.5.1. Hence we can estimate as follows:

$$\mathbb{P}(X_n = 0) \le \sum_{k=0}^{\infty} \Bigl(\frac{\lambda_n^k}{k!}\, \mathrm{e}^{-\lambda_n}\Bigr)\, t_k,$$

where $t_k := u_k$ if k is even, and $:= u_{k-1}$ if k is odd. Finally, use the fact that M is unbounded; it implies that $\lambda_n \to \infty$ as $n \to \infty$. Since $t_k \to 0$ as $k \to \infty$, it follows that the upperbound for $\mathbb{P}(X_n = 0)$ tends to zero. We conclude that $\mathbb{P}(X = 0) = 0$. ◻

We state some consequences of this result; cf. Theorems 4.13 and 4.17.

Corollary 4.21. *Let F be an infinitely divisible distribution function with $\ell_F = -\infty$ and with Lévy function M satisfying $M(0-) = 0$. Then F is continuous.*

Corollary 4.22. *Let F be an infinitely divisible distribution function with $\ell_F > -\infty$. Then F is continuous iff it is continuous at ℓ_F.*

There seems to be no simple necessary and sufficient condition on M for an infinitely divisible distribution function F without normal component to be *absolutely continuous*. We restrict ourselves to the following sufficient condition.

Theorem 4.23. *Let F be an infinitely divisible distribution function with canonical triple $(a, 0, M)$ such that the absolutely continuous part of M is unbounded. Then F is absolutely continuous.*

PROOF. We take $a = 0$; this is no essential restriction. By Proposition 4.5 (iv) we can write $F = F_1 \star F_2$, where F_1 and F_2 are infinitely divisible distribution functions with canonical functions given by the singular part M_{s} and the absolutely continuous part M_{ac} of M, respectively. Evidently, F will be absolutely continuous if F_2 is absolutely continuous. Therefore, without loss of generality, we may assume that M itself is absolutely continuous (and unbounded). For similar reasons we may suppose that M vanishes everywhere on $(-\infty, 0)$.

In order to show that such an M yields an absolutely continuous F, we write M as $M = \sum_{n=1}^{\infty} M_n$ with

$$M_n(x) := \begin{cases} M(x) - M(\varepsilon_{n-1}) & , \text{ if } \varepsilon_n \le x < \varepsilon_{n-1}, \\ M(\varepsilon_n) - M(\varepsilon_{n-1}) & , \text{ if } 0 < x < \varepsilon_n, \\ 0 & , \text{ elsewhere,} \end{cases}$$

where $\varepsilon_0 := \infty$ $\big($and $M(\infty) := 0\big)$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ is a sequence that decreases to zero. For $n \in \mathbb{N}$ let X_n be an infinitely divisible random variable with canonical triple $(0, 0, M_n)$, and take the X_n independent; then by Proposition 4.5 (iv) $\sum_{n=1}^{k} X_n$ has Lévy function $\sum_{n=1}^{k} M_n$. Since this function tends to M as $k \to \infty$, and X_n and X have no normal components, from (4.6) and Helly's theorem it follows that $\sum_{n=1}^{k} G_n \to G$ at continuity points of G, where G_n and G are the Lévy-Khintchine canonical functions corresponding to M_n and M. Hence we can apply Proposition 4.3 to conclude that the series $\sum_{n \ge 1} X_n$ converges in distribution and hence, as is well known, almost surely, and its sum $\sum_{n=1}^{\infty} X_n$ has distribution function F. Now, observe that M_n is absolutely continuous and bounded. Hence by Theorem 4.18 there exist $\gamma_n \in \mathbb{R}$ and independent random variables $Y_{n,1}, Y_{n,2}, \ldots$ with the same absolutely continuous distribution such that

$$X_n \stackrel{\mathrm{d}}{=} \gamma_n + (Y_{n,1} + \cdots + Y_{n,N_n}) =: \widehat{X}_n,$$

where N_n is independent of the $Y_{n,i}$ and has a Poisson (λ_n) distribution with λ_n given by $\lambda_n := -M_n(0+) = M(\varepsilon_{n-1}) - M(\varepsilon_n)$; the unboundedness of M makes it possible to choose (ε_n) such that $\lambda_n > 0$ for all n. Of course, in view of the independence of the X_n we take the sequences $(N_n, Y_{n,1}, Y_{n,2}, \ldots)$ with $n \in \mathbb{N}$ independent as well. Then the unboundedness of M also ensures that $K := \inf\{n \in \mathbb{N} : N_n \ne 0\}$ is not defective:

$$\mathbb{P}(K = \infty) = \mathbb{P}\Big(\bigcap_{n=1}^{\infty} \{N_n = 0\}\Big) = \exp\Big[-\sum_{n=1}^{\infty} \lambda_n\Big] = \mathrm{e}^{M(0+)} = 0.$$

With $p_{k,\ell} := \mathbb{P}(K = k;\, N_k = \ell) = \mathbb{P}(N_1 = 0; \cdots; N_{k-1} = 0;\, N_k = \ell)$ for $k, \ell \in \mathbb{N}$ we can now write F as

$$F(x) = \mathbb{P}\big(\textstyle\sum_{n=1}^{\infty} X_n \le x\big) = \displaystyle\sum_{k=1}^{\infty} \mathbb{P}\big(\textstyle\sum_{n=1}^{\infty} \widehat{X}_n \le x;\, K = k\big) =$$

$$= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} p_{k,\ell}\,\mathbb{P}\big(\sum_{n=1}^{k}\gamma_n + \sum_{j=1}^{\ell} Y_{k,j} + \sum_{n=k+1}^{\infty}\widehat{X}_n \le x\big) =$$

$$= \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} p_{k,\ell}\,\mathbb{P}\big(Y_{k,1} + U_{k,\ell} \le x\big),$$

where $U_{k,\ell}$ is independent of $Y_{k,1}$ for all k and ℓ. As $Y_{k,1}$ has an absolutely continuous distribution, we conclude that F has been represented as a mixture of absolutely continuous distribution functions and hence is itself absolutely continuous. ◫

Finally, we note that in Section 7 we will characterize the infinitely divisible distributions with finite variances. The resulting representation for the corresponding characteristic functions is known as the *Kolmogorov canonical representation.*

5. Compound-exponential distributions

The *compound-exponential* distributions, introduced in Section 3, form an interesting class of infinitely divisible distributions. They occur frequently in practice, and contain the distributions with densities that are completely monotone on both sides of zero; the infinite divisibility of such densities will be proved in Section 10 (symmetric case) and Chapter VI (general case). In the present section we prove some special properties of the Lévy canonical triples of compound-exponential distributions, we give some illustrative examples, and we characterize the *compound-geometric* distributions among the compound-exponential ones.

Recall from Section 3 that a random variable X and its distribution function F are called *compound-exponential* if

$$(5.1)\qquad X \overset{\mathrm{d}}{=} S(T), \quad F(x) = \int_0^{\infty} F_0^{\star t}(x)\,\mathrm{e}^{-t}\,\mathrm{d}t,$$

where $S(\cdot)$ is an sii-process generated by an infinitely divisible distribution function F_0, and T is standard exponentially distributed and independent of $S(\cdot)$. The corresponding characteristic function ϕ can be written in terms of $\phi_0 = \widetilde{F}_0$ as

$$(5.2)\qquad \phi(u) = \frac{1}{1-\log\,\phi_0(u)}, \quad \text{so } \phi_0(u) = \exp\big[1 - 1/\phi(u)\big].$$

Hence F_0 is uniquely determined by F; it is called the *underlying* (infinitely divisible) distribution function of F. In this section we will use (5.1) rather than (5.2); properties of F_0 from Section 4 are shown to imply special properties of F. We first consider the left extremity.

Proposition 5.1. *The left extremity ℓ_F of a compound-exponential distribution function F is either $-\infty$ or zero. In fact, $\ell_F = -\infty$ if $\ell_{F_0} < 0$, and $\ell_F = 0$ if $\ell_{F_0} \geq 0$; here F_0 is the underlying distribution function of F.*

PROOF. Let F satisfy (5.1). If (a_0, σ_0^2, M_0) is the Lévy triple of F_0, then by Proposition 4.5 (iii) for $t > 0$ the canonical triple of $F_0^{\star t}$ is given by $(ta_0, t\sigma_0^2, tM_0)$. From Theorem 4.13 it follows that

$$\ell_{F_0^{\star t}} = t\,\ell_{F_0} \qquad [\,t > 0\,], \tag{5.3}$$

also when $\ell_{F_0} = -\infty$. Now apply (5.1); we conclude that if $\ell_{F_0} < 0$ then $\ell_F = -\infty$, and if $\ell_{F_0} \geq 0$ then $\ell_F = 0$. □

This proposition rules out analogues of results in Section 4 with finite $\ell_F \neq 0$, whereas results with $\ell_F = 0$ are covered in Chapter III. Therefore, the subsequent statements are only interesting in the case where $\ell_F = -\infty$. According to Theorem 4.13 for the canonical triple (a, σ^2, M) of F this means that $\sigma^2 > 0$ or $M(0-) > 0$ or $\int_{(0,1]} x \,\mathrm{d}M(x) = \infty$. It turns out, however, that both the first and the third of these conditions cannot be satisfied by any compound-exponential distribution.

To show this we extend (5.1) as in Section I.3 as follows: If $X(\cdot)$ is the sii-process generated by X and $T(\cdot)$ is the one generated by T, then for $r > 0$

$$X(r) \stackrel{\mathrm{d}}{=} S\big(T(r)\big), \quad F^{\star r}(x) = \int_0^\infty F_0^{\star t}(x)\, \frac{1}{\Gamma(r)}\, t^{r-1}\, \mathrm{e}^{-t}\, \mathrm{d}t. \tag{5.4}$$

Now use (4.9), which expresses M in terms of $F^{\star r}$ for $r \downarrow 0$. By the monotone convergence theorem and the fact that $(1/n)\,\Gamma(1/n) = \Gamma(1/n + 1) \to 1$ as $n \to \infty$, it then follows that for continuity points $x < 0$ of M:

$$\begin{aligned} M(x) &= \lim_{n\to\infty} n\, F^{\star(1/n)}(x) = \\ &= \lim_{n\to\infty} \frac{n}{\Gamma(1/n)} \int_0^\infty F_0^{\star t}(x)\, t^{1/n-1}\, \mathrm{e}^{-t}\, \mathrm{d}t = \\ &= \int_0^\infty F_0^{\star t}(x)\, t^{-1}\, \mathrm{e}^{-t}\, \mathrm{d}t, \end{aligned}$$

and similarly for continuity points $x > 0$ of M:

$$M(x) = \lim_{n\to\infty} n\,\{1 - F^{\star(1/n)}(x)\} = -\int_0^\infty \{1 - F_0^{\star t}(x)\}\, t^{-1}\, \mathrm{e}^{-t}\, \mathrm{d}t,$$

expressions that are quite similar to that in (5.1) for $F(x)$. Therefore, as in the proof of Proposition 5.1 one shows that $\ell_{F_0} < 0$ iff $\ell_M = -\infty$, and that this condition on M (here) is equivalent to $M(0-) > 0$. From Proposition 5.1 itself it now follows that

$$\ell_F = -\infty \iff M(0-) > 0. \tag{5.5}$$

We can say more; note that in case $M(0-) = 0$ the integrability condition in (ii) and parts (iii) and (iv) of the following theorem are implied by (5.5), Theorem 4.13 and Proposition 5.1.

Theorem 5.2. *Let F be a compound-exponential distribution function. Then the Lévy canonical triple (a, σ^2, M) of F has the following properties:*

(i) *The Lévy function M can be expressed, for continuity points x, in terms of the underlying infinitely divisible distribution function F_0 as follows:*

$$M(x) = \begin{cases} \int_0^\infty F_0^{\star t}(x)\, t^{-1}\, \mathrm{e}^{-t}\, dt & , \text{ if } x < 0, \\ -\int_0^\infty \{1 - F_0^{\star t}(x)\}\, t^{-1}\, \mathrm{e}^{-t}\, dt & , \text{ if } x > 0. \end{cases}$$

(ii) *The Lévy function M has unbounded support and satisfies*

$$\int_{[-1,1]\setminus\{0\}} |x|\, dM(x) < \infty.$$

(iii) *The shift parameter is zero: $\gamma := a - \int_{\mathbb{R}\setminus\{0\}} x/(1+x^2)\, dM(x) = 0$.*

(iv) *There is no normal component: $\sigma^2 = 0$.*

Moreover, the characteristic function ϕ of F can be given the following simple form where M is the Lévy function of F:

$$\phi(u) = \exp\Big[\int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{iux} - 1)\, dM(x)\Big]. \tag{5.6}$$

Proof. We use relation (4.9) which expresses (a, σ^2, M) in terms of $F^{\star r}$ for $r \downarrow 0$. Part (i) has already been proved in this way. We also saw that from (i) it follows that $\ell_M = -\infty$ if $\ell_{F_0} < 0$. Of course, (i) implies a similar property for right extremities: $r_M = \infty$ if $r_{F_0} > 0$. Hence $\ell_M = -\infty$ or

$r_M = \infty$, so M has unbounded support. To prove the second part of (ii) we first compute a and find by using (5.4) and (i):

$$a = \lim_{n\to\infty} n \int_{\mathbb{R}} \frac{x}{1+x^2}\, \mathrm{d}F^{\star(1/n)}(x) =$$
$$= \lim_{n\to\infty} \frac{n}{\Gamma(1/n)} \int_0^\infty \Bigl(\int_{\mathbb{R}} \frac{x}{1+x^2}\, \mathrm{d}F_0^{\star t}(x)\Bigr)\, t^{1/n-1}\, \mathrm{e}^{-t}\, \mathrm{d}t =$$
$$= \int_0^\infty \Bigl(\int_{\mathbb{R}} \frac{x}{1+x^2}\, \mathrm{d}F_0^{\star t}(x)\Bigr)\, t^{-1}\, \mathrm{e}^{-t}\, \mathrm{d}t = \int_{\mathbb{R}\setminus\{0\}} \frac{x}{1+x^2}\, \mathrm{d}M(x).$$

Hence M satisfies the integrability condition in (ii). Moreover, it follows that the shift parameter γ is zero, so we also have (iii). Part (iv) follows from the following computation:

$$\sigma^2 = \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} n \int_{(-\varepsilon,\varepsilon]} \frac{x^2}{1+x^2}\, \mathrm{d}F^{\star(1/n)}(x) =$$
$$= \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \frac{n}{\Gamma(1/n)} \int_0^\infty \Bigl(\int_{(-\varepsilon,\varepsilon]} \frac{x^2}{1+x^2}\, \mathrm{d}F_0^{\star t}(x)\Bigr)\, t^{1/n-1}\, \mathrm{e}^{-t}\, \mathrm{d}t =$$
$$= \lim_{\varepsilon\downarrow 0} \int_0^\infty \Bigl(\int_{(-\varepsilon,\varepsilon]} \frac{x^2}{1+x^2}\, \mathrm{d}F_0^{\star t}(x)\Bigr)\, t^{-1}\, \mathrm{e}^{-t}\, \mathrm{d}t = 0.$$

That $\sigma^2 = 0$ can also be proved by using Proposition 4.10. Finally, apply Theorem 4.15 to obtain the representation (5.6) for ϕ. ◻◻

This theorem can be used to show that a given distribution is not compound-exponential. For instance, part (iv) implies that the *normal* distribution of Example 4.6 is *not* compound-exponential, and from part (ii) it is seen that the *Cauchy* distribution of Example 4.7 is *not* compound-exponential either. The *sym-gamma* distributions of Example 4.8 do all satisfy the (necessary) conditions (ii), (iii) and (iv) of the theorem.

Example 5.3. The *sym-gamma* (r, λ) distribution with characteristic function ϕ given by

$$\phi(u) = \Bigl(\frac{\lambda^2}{\lambda^2 + u^2}\Bigr)^r,$$

is *compound-exponential* iff $r \le 1$. To show this we consider the function ϕ_0 in (5.2) and take $\lambda = 1$ (which is no essential restriction):

$$\phi_0(u) = \exp\bigl[1 - (1+u^2)^r\bigr].$$

First, let $r \leq 1$, and observe that ϕ_0 can be written as

$$\phi_0(u) = \pi_0(u^2), \quad \text{where } \pi_0(s) := \frac{\pi(1+s)}{\pi(1)}, \; \pi(s) := \exp\,[-s^r].$$

Now note that by Example III.4.9 π is an infinitely divisible pLSt; hence so is π_0 because of Proposition III.6.1 (iii). From Proposition 3.6 it follows that ϕ_0 is an infinitely divisible characteristic function, so ϕ is compound-exponential. Next, let $r > 1$, and suppose that ϕ is compound-exponential; then ϕ_0 would be an infinitely divisible characteristic function. But one easily shows that the 4-th cumulant κ_4 of ϕ_0 is then given by $\kappa_4 = -12r(r-1)$, which is negative; this contradicts the result to be given in Corollary 7.5, so ϕ is not compound-exponential. ◻

The compound-exponential distributions with underlying distributions that are *compound-Poisson*, form the class of *compound-geometric* distributions with characteristic functions of the form

(5.7) $$\phi(u) = \frac{1-p}{1-p\,\widetilde{G}(u)},$$

where $p \in (0,1)$ and G is a distribution function that is continuous at zero. This has been shown in Theorem 3.8; the underlying distribution function F_0 has characteristic function ϕ_0 given by

(5.8) $$\phi_0(u) = \exp\big[-\lambda\,\{1-\widetilde{G}(u)\}\big],$$

where $\lambda = p/(1-p)$. Using this in Theorem 5.2 (i) one can express the Lévy function M of ϕ in terms of (p, G). We give an alternative proof that is more direct.

Theorem 5.4. *Let ϕ be a compound-geometric characteristic function, so ϕ has the form (5.7) for some (p, G) as indicated. Then the Lévy function M of ϕ is given by*

$$M(x) = \begin{cases} \sum_{k=1}^{\infty} (p^k/k)\, G^{\star k}(x) & , \text{ if } x < 0, \\ -\sum_{k=1}^{\infty} (p^k/k)\,\{1 - G^{\star k}(x)\} & , \text{ if } x > 0. \end{cases}$$

Proof. According to (the proof of) Theorem 3.5 ϕ is of the compound-Poisson form (5.8) with $\lambda = -\log(1-p)$ and $\widetilde{G}$ replaced by $Q \circ \widetilde{G}$, where Q is the pgf of the logarithmic-series (p) distribution on $\mathbb{N}$. Now use Theorem 4.18. ◻

Compound-geometric distributions are compound-Poisson. Are there other compound-exponential distributions with this property? Such a distribution necessarily has positive mass at zero, and if it has no mass at any other point, then by Theorem 3.9 the distribution is compound-geometric. By using the important Theorem 4.20 we can say more. Let F be a compound-exponential distribution function with underlying distribution function F_0, so we have (5.1). We first note that if F_0 is continuous, then so is $F_0^{\star t}$ for all $t > 0$. In fact, if $F_0^{\star t_0}$ were not continuous for some $t_0 > 0$, then $F_0^{\star m t_0}$ would be not continuous for every $m \in \mathbb{N}$; now choose m such that $mt_0 > 1$ and observe that then $F_0^{\star m t_0} = F_0 \star F_0^{\star(mt_0 - 1)}$. Thus by (5.1) we have

$$(5.9) \qquad F_0 \text{ continuous} \implies F \text{ continuous.}$$

Now suppose that F is *not* continuous; then F_0 is not continuous either. From (4.21), which is essentially Theorem 4.20, it then follows that F_0 is shifted compound-Poisson, so up to a factor $e^{iu\gamma}$ with $\gamma \in \mathbb{R}$ the characteristic function ϕ_0 of F_0 has the form (5.8). By (5.2), for $\phi = \widetilde{F}$ this means that

$$\phi(u) = \frac{1-p}{1 - iu\delta - p\,\widetilde{G}(u)} = \frac{1}{1 - iu\delta}\,\frac{1-p}{1 - p\,\{1/(1 - iu\delta)\}\,\widetilde{G}(u)},$$

where $p := \lambda/(1+\lambda)$ and $\delta := \gamma/(1+\lambda)$. From the second representation of ϕ it is seen that if γ were non-zero, then F would have an exponential factor and hence would be continuous. So $\gamma = 0$, and F is compound-geometric. Thus we have obtained the following analogue of (4.21).

Theorem 5.5. *Let F be a compound-exponential distribution function. Then F has at least one discontinuity point iff F is compound-geometric.*

Clearly, this result is an improvement of Theorem 3.9. Moreover, it settles the question above; we state the answer as a first corollary. The second corollary contains another remarkable result.

Corollary 5.6. *A compound-exponential distribution is compound-Poisson iff it is compound-geometric.*

Corollary 5.7. *A compound-exponential distribution function F is continuous iff it is continuous at zero.*

In a similar way some further results on Lebesgue properties of compound-exponential distributions can be given; we will not do so.

6. Closure properties

The simplest closure property is given by Proposition 2.1 (i): If X is an infinitely divisible random variable, then so is aX for every $a \in \mathbb{R}$. Consider the more general multiple AX, where A is independent of X and has values in $\{0, a\}$. It need *not* be infinitely divisible; take X degenerate, for instance. A more interesting counter-example is given in Section 11. Thus in terms of characteristic functions we have for $\alpha \in (0, 1)$:

$$(6.1) \qquad \phi \text{ infinitely divisible } \not\Longrightarrow \; 1 - \alpha + \alpha\,\phi \text{ infinitely divisible.}$$

In particular, it follows that the class of infinitely divisible distributions is *not* closed under mixing; see, however, (6.5) below and other mixtures from Chapter VI.

From further results in Section 2 it is clear that the class of infinitely divisible characteristic functions is closed under taking pointwise products, absolute values, positive powers and limits, provided these limits are characteristic functions. In Section III.6, for a positive function π and $a > 0$ we considered, apart from $\pi(a\,\cdot)$ and π^a, the functions π_a defined by

$$(6.2) \qquad \pi_a(s) = \frac{\pi(a+s)}{\pi(a)}, \quad \pi_a(s) = \frac{\pi(a)\,\pi(s)}{\pi(a+s)},$$

and showed in both cases that if π is an infinitely divisible pLSt, then so is π_a. Now, for an infinitely divisible characteristic function ϕ and $a > 0$ one might consider the functions ϕ_a on $\mathbb{R}$ with $\phi_a := \phi(a + \cdot)/\phi(a)$ and $\phi_a := \phi(a)\,\phi/\phi(a + \cdot)$, but in general these functions are *not* even characteristic functions; the standard normal distribution provides a counter-example in both cases, because the necessary condition $|\phi_a| \leq 1$ is violated. As a consequence there is no obvious analogue of the 'self-decomposability' characterization of infinite divisibility given in Theorem III.6.3. On the other hand, the first part of (6.2) suggests looking at distribution functions F_a of the form

$$(6.3) \qquad F_a(x) = c_a \int_{(-\infty, x]} \mathrm{e}^{-a|y|}\, \mathrm{d}F(y),$$

where F is an infinitely divisible distribution function, $a > 0$ and c_a is a norming constant. For F not concentrated on a half-line, however, F_a is in general *not* infinitely divisible; as is shown in Section 11, the normal distribution provides a counter-example again.

The result of Proposition 3.6 can also be viewed as a closure property: If π is an infinitely divisible pLSt, then for characteristic functions ϕ_0 and ϕ we have

$$\text{(6.4)} \qquad \phi_0 \text{ infinitely divisible} \implies \phi := \pi \circ (-\log \phi_0) \text{ infinitely divisible.}$$

For instance, taking here for ϕ_0 a symmetric stable characteristic function as considered in Example 4.9, we obtain the following useful special case.

Proposition 6.1. *If π is an infinitely divisible pLSt, then $u \mapsto \pi\big(|u|^\gamma\big)$ is an infinitely divisible characteristic function for every $\gamma \in (0, 2\,]$.*

In Section 10 we show that in case $\gamma \in (0, 1\,]$ the function $u \mapsto \pi\big(|u|^\gamma\big)$ is an infinitely divisible characteristic function for *every* pLSt π (not necessarily infinitely divisible). Taking π exponential in (6.4) shows that if ϕ_0 is an infinitely divisible characteristic function, then so is ϕ given by $\phi(u) = 1/\big(1 - \log\, \phi_0(u)\big)$; ϕ is compound-exponential. In Chapter VI, on mixtures, we generalize this considerably as follows: If ϕ_0 is an infinitely divisible characteristic function, then so is ϕ with

$$\text{(6.5)} \qquad \phi(u) = 1 - \alpha + \alpha \int_{(0,\infty)} \Big(\frac{\lambda}{\lambda - \log\, \phi_0(u)}\Big)^r \,\mathrm{d}G(\lambda),$$

where $\alpha \in [\,0, 1\,]$, $r \in (0, 2\,]$ and G is a distribution function on $(0, \infty)$.

Finally we note that there are several useful closure properties for subclasses of infinitely divisible distributions, like the compound-Poisson and the compound-exponential distributions.

7. Moments

Let X be an infinitely divisible random variable with distribution function F. We are interested in the *moment* μ_n of X of order $n \in \mathbb{Z}_+$:

$$\mu_n := \mathbb{E}X^n = \int_{\mathbb{R}} x^n \,\mathrm{d}F(x).$$

We look for a necessary and sufficient condition in terms of the canonical triple (a, σ^2, M) of X for μ_n to exist and be finite, and we want to know how μ_n can then be obtained from the triple. Of course, μ_n exists and is finite iff $\mathbb{E}|X|^n < \infty$. Now consider, more generally, the *absolute moment* $\mathbb{E}|X|^r$ of order $r > 0$ (so r not necessarily in $\mathbb{N}$). We start with looking at a special case; it suggests that $\mathbb{E}|X|^r$ being finite or not does not depend on the behaviour of M near zero.

Proposition 7.1. *Let X be an infinitely divisible random variable with Lévy function M vanishing everywhere on $\mathbb{R} \setminus [-1, 1]$. Then $\mathbb{E}|X|^r < \infty$ for all $r > 0$.*

PROOF. Let ϕ be the characteristic function of X. Because of the Lévy representation (4.7) the function $\psi := \log \phi$ can be written as

$$\psi(u) = iua - \tfrac{1}{2}u^2\sigma^2 + \int_{[-1,1]\setminus\{0\}} \Big(\sum_{n=2}^{\infty} \frac{(iux)^n}{n!} + \frac{iux^3}{1+x^2} \Big)\, \mathrm{d}M(x).$$

Now, using (4.8) and the fact that $|x|^n \le x^2$ for $|x| \le 1$ and $n \ge 2$, we can apply Fubini's theorem to conclude that ψ, and hence $\phi = \mathrm{e}^{\psi}$, can be expanded as a power series (everywhere on $\mathbb{R}$). It follows that ϕ has derivatives of all orders. This implies that X has moments of all orders, so $\mathbb{E}|X|^r < \infty$ for all $r > 0$. □

Thus in determining finiteness of absolute moments it seems to be no restriction to suppose that the Lévy function M is bounded. Therefore, in view of Theorem 4.18 we next look at *compound-Poisson* distributions.

Proposition 7.2. *Let X have a compound-Poisson distribution: $X \stackrel{\mathrm{d}}{=} S_N$, where $(S_n)_{n\in\mathbb{Z}_+}$ is an sii-process generated by Y with $\mathbb{P}(Y = 0) = 0$, and N is Poisson distributed and independent of (S_n). Then for all $r > 0$:*

$$\mathbb{E}|X|^r < \infty \iff \mathbb{E}|Y|^r < \infty. \tag{7.1}$$

PROOF. Let $r > 0$. Then conditioning on the values of N we can write

$$\mathbb{E}|X|^r = \sum_{n=1}^{\infty} \mathbb{P}(N = n)\, \mathbb{E}|S_n|^r.$$

Now use Minkowski's inequality, i.e., the triangle inequality, for the function space $\mathcal{L}^r$; see (A.2.9). Then one sees that $\mathbb{E}|S_n|^r \le n^r\, \mathbb{E}|Y|^r$. Hence $\mathbb{E}|X|^r$ can be estimated in the following way:

$$\mathbb{P}(N = 1)\,\big(\mathbb{E}|Y|^r\big) \le \mathbb{E}|X|^r \le (\mathbb{E}N^r)\,\big(\mathbb{E}|Y|^r\big). \tag{7.2}$$

Since $\mathbb{E}N^r < \infty$, the equivalence in (7.1) now immediately follows. □

The general case can be dealt with by splitting the Lévy function, as was done several times in Section 4. In this way Propositions 7.1 and 7.2 can be combined to obtain the following general result on finiteness of absolute moments.

Theorem 7.3. *Let X be an infinitely divisible random variable with Lévy function M. Then for all $r > 0$:*

$$(7.3) \qquad \mathbb{E}|X|^r < \infty \iff c_r := \int_{\mathbb{R}\setminus[-1,1]} |x|^r \, dM(x) < \infty.$$

PROOF. Let (a, σ^2, M) be the canonical triple of X, and take $r > 0$. We write M as $M = M_1 + M_2$ with M_2 given by

$$M_2(x) = \begin{cases} M(x) & \text{, if } x < -1 \text{ or } x \geq 1, \\ M(-1) & \text{, if } -1 \leq x < 0, \\ M(1-) & \text{, if } 0 < x < 1. \end{cases}$$

Then by Proposition 4.5 (iv) we have $X \stackrel{d}{=} X_1 + X_2$, where X_1 and X_2 are independent and infinitely divisible with canonical triples (a, σ^2, M_1) and $(0, 0, M_2)$, respectively. Now, M_1 vanishes outside $[-1, 1]$, so by Proposition 7.1 the absolute moments of X_1 are all finite. Since by applying the inequality $|a+b|^r \leq 2^r\{|a|^r + |b|^r\}$ twice it is seen that

$$(7.4) \qquad \left(\tfrac{1}{2}\right)^r \mathbb{E}|X_2|^r - \mathbb{E}|X_1|^r \leq \mathbb{E}|X|^r \leq 2^r\{\mathbb{E}|X_1|^r + \mathbb{E}|X_2|^r\},$$

we conclude that $\mathbb{E}|X|^r < \infty$ iff $\mathbb{E}|X_2|^r < \infty$. Finally, as M_2 is bounded, X_2 is shifted compound-Poisson, so by Proposition 7.2 we have $\mathbb{E}|X_2|^r < \infty$ iff $\mathbb{E}|Y_2|^r < \infty$, where the distribution function G_2 of Y_2 is related to M_2 as G and M in Theorem 4.18. This means that $\mathbb{E}|Y_2|^r < \infty$ iff $c_r < \infty$ with c_r as defined in (7.3). ◻

We return to the questions from the beginning of this section. Let X, with characteristic function ϕ, be infinitely divisible with canonical triple (a, σ^2, M). Taking $r = n \in \mathbb{N}$ in Theorem 7.3, we get a necessary and sufficient condition in terms of M for $\mu_n := \mathbb{E}X^n$ to (exist and) be finite. Note that because of (4.8) in case $n \geq 2$ the condition can be reformulated in terms of the *moment* α_n of M of order n:

$$(7.5) \qquad \mu_n := \mathbb{E}X^n \text{ finite} \iff \alpha_n := \int_{\mathbb{R}\setminus\{0\}} x^n \, \mathrm{d}M(x) \text{ finite}.$$

We next want to know whether the μ_n can be obtained from the α_n. Return to the proof of Proposition 7.1, where in a special case a power series expansion is given of $\psi := \log \phi$ in which the α_n occur in a simple way. Since in more general cases such an expansion can be shown to hold on a neighbourhood of zero, one may expect a close relation between the α_n and the *cumulants* κ_n of X (cf. Section A.2), at least when $n \geq 2$. This is indeed the case. Of course, the moments μ_n themselves can be obtained from the cumulants κ_n by using (A.2.20):

$$(7.6) \qquad \mu_{n+1} = \sum_{j=0}^{n} \binom{n}{j} \mu_j \, \kappa_{n+1-j} \qquad [\, n \in \mathbb{Z}_+ \,],$$

so $\kappa_1 = \mathbb{E}X$ and $\kappa_2 = \mathbb{V}\mathrm{ar}\, X$.

Theorem 7.4. *Let X be an infinitely divisible random variable with canonical triple (a, σ^2, M), let $n \in \mathbb{N}$, and let X have a finite moment μ_n of order n. Then the n-th order cumulant κ_n of X is given by*

$$(7.7) \qquad \kappa_1 = a + \int_{\mathbb{R}\setminus\{0\}} \frac{x^3}{1+x^2}\, dM(x), \;\; \kappa_2 = \sigma^2 + \alpha_2, \;\; \kappa_n = \alpha_n \text{ if } n \geq 3,$$

where α_n is the n-th order moment of M as defined in (7.5). When M satisfies the condition $\int_{[-1,1]\setminus\{0\}} |x|\, dM(x) < \infty$, the first cumulant κ_1 can be rewritten as

$$(7.8) \qquad \kappa_1 = \gamma + \alpha_1,$$

where γ is the shift parameter of X and α_1 is the first moment of M.

PROOF. Let ϕ be the characteristic function of X; since μ_n is finite, ϕ is differentiable n times. The cumulant κ_n can be obtained from the relation $i^n \kappa_n = \psi^{(n)}(0)$ with, as before, $\psi := \log \phi$; by (4.7) ψ can be written as $\psi(u) = iua - \frac{1}{2}u^2\sigma^2 + \omega(u)$, where

$$\omega(u) := \int_{\mathbb{R}\setminus\{0\}} \Big(\mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2} \Big)\, \mathrm{d}M(x).$$

First, let $n = 1$. Then we get $i\kappa_1 = ia + \omega'(0)$. To compute $\omega'(u)$ we consider the difference quotient of ω at u:

$$\frac{\omega(u+h) - \omega(u)}{h} = \int_{\mathbb{R}\setminus\{0\}} \Big(\mathrm{e}^{iux} \frac{\mathrm{e}^{ihx} - 1}{ihx} - \frac{1}{1+x^2} \Big)\, ix\, \mathrm{d}M(x).$$

Now, using the fact that $|\mathrm{e}^{iy} - 1| \leq |y|$ and $|\mathrm{e}^{iy} - 1 - iy| \leq \frac{1}{2}y^2$ for $y \in \mathbb{R}$, we see that the absolute value of the integrand is bounded by $2|x|$ for $|x| > 1$ and by $\big(|u| + \frac{3}{2}\big)\, x^2$ for $|x| \leq 1$, uniformly in $|h| < 1$. Moreover, because of

Theorem 7.3 we have $\int_{\mathbb{R}\setminus[-1,1]} |x| \, \mathrm{d}M(x) < \infty$. Hence, letting $h \to 0$, we can use dominated convergence to conclude that

$$(7.9) \qquad \omega'(u) = \int_{\mathbb{R}\setminus\{0\}} \Big(\mathrm{e}^{iux} - \frac{1}{1+x^2} \Big) \, ix \, \mathrm{d}M(x).$$

Taking $u = 0$ here yields the formula for κ_1 in (7.7); the alternative (7.8) follows from Theorem 4.15.

Next, let $n \geq 2$. Then we get $i^2 \kappa_2 = -\sigma^2 + \omega''(0)$ if $n = 2$, and $i^n \kappa_n = \omega^{(n)}(0)$ if $n \geq 3$. Since by (7.5) we now have $\int_{\mathbb{R}\setminus\{0\}} |x|^n \, \mathrm{d}M(x) < \infty$, we can proceed as above and conclude that $\omega^{(n)}(u)$ may be obtained from $\omega'(u)$ above by differentiating under the integral sign, so:

$$(7.10) \qquad \omega^{(n)}(u) = \int_{\mathbb{R}\setminus\{0\}} \mathrm{e}^{iux} \, (ix)^n \, \mathrm{d}M(x) \qquad [\, n \geq 2 \,].$$

Taking $u = 0$ yields again the formula for κ_n in (7.7). □

Corollary 7.5. *For an infinitely divisible distribution the cumulants of even order, as far as they exist, are nonnegative.*

The characteristic function ϕ of any random variable X with finite n-th moment has a Taylor expansion as in (A.2.18). Similarly, the function $\psi := \log \phi$ satisfies

$$(7.11) \qquad \psi(u) = \sum_{j=1}^{n} \kappa_j \, \frac{(iu)^j}{j!} + R_n(u), \quad \text{with } R_n(u) = o(u^n) \text{ as } u \to 0,$$

where $\kappa_1, \ldots, \kappa_n$ are the first n cumulants of X. Now, if X is infinitely divisible, then something more can be said on the remainder function R_n; rewriting the Lévy representation for ϕ by use of Theorem 7.4 easily leads to the following result.

Theorem 7.6. *Let X be an infinitely divisible random variable with canonical triple (a, σ^2, M), and let $\psi := \log \phi$ with ϕ the characteristic function of X. If X has a finite first moment, then ψ can be written as*

$$(7.12) \qquad \psi(u) = iu\kappa_1 - \tfrac{1}{2} u^2 \sigma^2 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{iux} - 1 - iux) \, dM(x),$$

and if X has a finite moment of order $n \geq 2$, then

$$(7.13) \qquad \psi(u) = \sum_{j=1}^{n} \kappa_j \, \frac{(iu)^j}{j!} + \int_{\mathbb{R}\setminus\{0\}} \Big(\sum_{j=n+1}^{\infty} \frac{(iux)^j}{j!} \Big) \, dM(x);$$

here κ_j is the cumulant of X of order j.

This theorem can be used to obtain, for every $n \in \mathbb{N}$, a *canonical representation* for the infinitely divisible characteristic functions with finite n-th moments. We will only do so for $n = 2$, and will formulate the result in such a way that Kolmogorov's representation as stated in Theorem I.4.3 appears. In fact, starting from (7.12), or from (7.13) with $n = 2$, using the middle equality of (7.7), and defining the function H on $\mathbb{R}$ by

$$H(x) = \frac{1}{\kappa_2}\Big(\sigma^2\, 1_{[0,\infty)}(x) + \int_{(-\infty,x]\setminus\{0\}} y^2\, \mathrm{d}M(y)\Big) \qquad [x \in \mathbb{R}], \tag{7.14}$$

one is easily led to the following classical result.

Theorem 7.7 (Kolmogorov representation). *A $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of an infinitely divisible distribution with finite non-zero variance iff ϕ has the form*

$$\phi(u) = \exp\Big[iu\mu + \kappa \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux)\, \frac{1}{x^2}\, dH(x)\Big] \qquad [u \in \mathbb{R}], \tag{7.15}$$

where $\mu \in \mathbb{R}$, $\kappa > 0$ and H is a distribution function; for $x = 0$ the integrand is defined by continuity: $-\frac{1}{2}u^2$. The canonical triple (μ, κ, H) is unique; in fact, μ is the mean and κ the variance of the distribution (so $\mu = \mu_1 = \kappa_1$ and $\kappa = \kappa_2$).

From this theorem one easily obtains a *cumulant inequality* that illustrates the special character of the *normal* and *Poisson* distributions.

Proposition 7.8. *The cumulants κ_2, κ_3 and κ_4 of an infinitely divisible distribution with a finite fourth moment satisfy*

$$\kappa_2 \kappa_4 \geq \kappa_3^2. \tag{7.16}$$

Moreover, equality holds in (7.16) iff the distribution is normal or of Poisson type.

Proof. Let X be an infinitely divisible random variable with $\mathbb{E}X^4 < \infty$ and with canonical triple (a, σ^2, M). Consider the Kolmogorov canonical function H for X; it can be viewed as the distribution function of a random variable Y, say. Now, use the fact that $\mathbb{V}\mathrm{ar}\, Y = \mathbb{E}\, Y^2 - (\mathbb{E}\, Y)^2 \geq 0$, and combine (7.7) and (7.14) to see that

$$\kappa_2\, \mathbb{E}\, Y = \kappa_3, \qquad \kappa_2\, \mathbb{E}\, Y^2 = \kappa_4.$$

This immediately yields (7.16). Equality here means that $\mathbb{V}\mathrm{ar}\, Y = 0$, i.e., Y is degenerate. Degeneracy at zero implies that $\sigma^2 > 0$ and $M = 0$, so X is *normal*. Degeneracy away from zero means that $\sigma^2 = 0$ and M has one point of increase, so X is of *Poisson* type. □

In view of (7.13) one might ask when $\psi = \log \phi$ has a power series expansion in a neighbourhood of zero. Of course, for this to be the case the quantities c_r in (7.3) must be finite and become large not too fast. We do not go further into this, and restrict ourselves to a special case which is needed in Section 9.

Proposition 7.9. *Let X be an infinitely divisible random variable with characteristic function ϕ and with Lévy function M vanishing everywhere on $\mathbb{R} \setminus [-r, r]$, for some $r > 0$. Then X has finite moments of all orders, and $\psi := \log \phi$, and hence $\phi = e^{\psi}$, has a power series expansion on $\mathbb{R}$:*

$$(7.17) \qquad \psi(u) = \sum_{n=1}^{\infty} \kappa_n \frac{(iu)^n}{n!} \qquad [u \in \mathbb{R}],$$

where κ_n is the cumulant of X of order n.

PROOF. Proceed as in the proof of Proposition 7.1; note that now Fubini's theorem can be applied because of (4.8) and the fact that $|x|^n \le r^{n-2} x^2$ for $|x| \le r$ and $n \ge 2$. Finally, use (7.7). □

Of course, finiteness of the absolute moment $\mathbb{E}|X|^r$ is related to a certain behaviour of the *two-sided tail* $\mathbb{P}\big(|X| > x\big)$ of X as $x \to \infty$. Therefore, Theorem 7.3 gives information on the tail behaviour of infinitely divisible distributions. In Section 9 we will return to this and give more detailed results, also for the *left tail* $\mathbb{P}(X < -x)$ and *right tail* $\mathbb{P}(X > x)$ of X as $x \to \infty$. But first it is useful to pay attention to the *support* of an infinitely divisible distribution.

8. Support

Let F be a distribution function on $\mathbb{R}$. As agreed in Section A.2, by the *support* $S(F)$ of F we understand the set of *points of increase* of F; equivalently, $S(F)$ is the smallest *closed* subset S of $\mathbb{R}$ with $m_F(S) = 1$.

According to Proposition A.2.1 the support of the convolution of two distribution functions is equal to the closure of the direct sum of the supports:

$$(8.1)\qquad S(F \star G) = \overline{S(F) \oplus S(G)}.$$

For functions more general than distribution functions, such as Lévy functions, the support is defined similarly.

Let F be infinitely divisible with n-th order factor F_n, say, so $F = F_n^{\star n}$. Then from (8.1) it follows that

$$(8.2)\qquad S(F) = \overline{S(F_n)^{\oplus n}} \qquad [\, n \in \mathbb{N}\,].$$

Also, if F is non-degenerate, then by Proposition I.2.3 $S(F)$ is *unbounded.* We can say more, and start with considering the case where F is *compound-Poisson.* By (3.2) F can then be represented as

$$(8.3)\qquad F(x) = \mathrm{e}^{-\lambda}\, 1_{\mathbb{R}_+}(x) + \sum_{n=1}^{\infty} \Big(\frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}\Big)\, G^{\star n}(x) \qquad [\, x \in \mathbb{R}\,],$$

where $\lambda > 0$ and G is a distribution function that is continuous at zero. From this it is immediately seen that

$$S(F) \supset \{0\} \cup \bigcup_{n=1}^{\infty} S(G^{\star n});$$

we need not have equality here, but we have, if we replace the union over n in the right-hand side by its closure. This is easily verified by showing that a non-zero element outside the closure cannot belong to $S(F)$; cf. the second part of the proof of Proposition A.2.1. Moreover, using (8.1) we see that in the resulting equality we may replace $\bigcup_{n=1}^{\infty} S(G^{\star n})$ by $\mathrm{sg}\,\big\{S(G)\big\}$, the additive semigroup generated by $S(G)$. We conclude that

$$(8.4)\qquad S(F) = \{0\} \cup \overline{\mathrm{sg}\,\big\{S(G)\big\}}.$$

Since the convolution power $F^{\star t}$ of F satisfies (8.3) with λ replaced by λt, it follows that its support does not depend on t:

$$(8.5)\qquad S(F^{\star t}) = S(F) \qquad [\, t > 0\,].$$

Taking $t = 2$ and applying (8.1) $\big($or $t = \frac{1}{2}$ and (8.2)$\big)$, one also proves the following property of compound-Poisson distribution functions F:

(8.6) $\qquad S(F)$ is *closed under addition.*

We turn to the general case, and will relate the support of an infinitely divisible distribution function F to its Lévy canonical triple (a, σ^2, M). Clearly, if F has a normal component, then $S(F) = \mathbb{R}$, so we further assume that $\sigma^2 = 0$. First, suppose that the Lévy function M of F is *bounded.* Then we can apply Theorem 4.18: F has a (finite) *shift parameter* γ given by

$$\gamma = a - \int_{\mathbb{R}\setminus\{0\}} \frac{x}{1+x^2}\, \mathrm{d}M(x), \tag{8.7}$$

and is, up to a shift over γ, of the compound-Poisson form (8.3) with G satisfying $S(G) \setminus \{0\} = S(M)$. So we can use (8.4) to obtain the following preliminary result.

Proposition 8.1. *Let F be an infinitely divisible distribution function with canonical triple $(a, 0, M)$. If M is bounded, then*

$$S(F) = \gamma + \Big(\{0\} \cup \overline{sg\,\{S(M)\}}\Big) = \{\gamma\} \cup \Big(\gamma + \overline{sg\,\{S(M)\}}\Big), \tag{8.8}$$

where γ is the shift parameter of F.

We proceed with considering the case where the Lévy function M of F may be *unbounded* but vanishes everywhere on the negative or on the positive half-line. Hence $M(0-) = 0$ and $M(0+) < 0$ (possibly $-\infty$), or $M(0-) > 0$ (possibly ∞) and $M(0+) = 0$. We start with the case where $M(0-) = 0$, and show in the following proposition that then for $S(F)$ there are three essentially different possibilities. Here it is convenient to make use of a result on $\mathbb{R}_+$. Note that only *bounded* M can satisfy the condition in (i) below. The first condition in (ii) automatically holds for *unbounded* M and the second one for *bounded* M. The condition in (iii) can only be satisfied by *unbounded* M.

Proposition 8.2. *Let F be an infinitely divisible distribution function with canonical triple $(a, 0, M)$ satisfying $M(0-) = 0$.*

(i) *If $M(x) = M(0+)$ for some $x > 0$, then $S(F)$ is given by (8.8) with $\ell_F = \gamma$ as an isolated (possibly its only) point.*

(ii) *If $M(x) > M(0+)$ for all $x > 0$ and $\int_{(0,1]} x\, dM(x) < \infty$, then $S(F)$ is given by $S(F) = [\,\gamma, \infty)$ with γ the shift parameter of F.*

(iii) *If $\int_{(0,1]} x\, dM(x) = \infty$ (but, of course, still $\int_{(0,1]} x^2\, dM(x) < \infty$), then $S(F)$ is given by $S(F) = \mathbb{R}$.*

PROOF. Part (i) follows from Proposition 8.1; note that $\gamma \in S(F)$ but $\gamma + x \notin S(F)$ for all $x \in (0, x_0)$ with $x_0 := \sup\{x > 0 : M(x) = M(0+)\}$. To prove (ii) we let $\int_{(0,1]} x \, dM(x) < \infty$. Then by Theorem 4.13 we have $\ell_F = \gamma > -\infty$. Now, apply Theorem III.8.2 to the infinitely divisible distribution function $H := F(\cdot + \gamma)$ with $\ell_H = 0$, and note that by Corollary 4.14 the ($\mathbb{R}_+$-) canonical function K of H satisfies $S(K) = S(M)$, and $\ell_K = 0$ iff $M(x) > M(0+)$ for all $x > 0$. Thus we have proved (ii) and, in fact, (i) once more, without using Proposition 8.1. Turning to part (iii), we let M satisfy the condition stated there. Then necessarily $\ell_F = -\infty$. Now, we can write M as the sum of two canonical functions M_1 and M_2 which for $x \in (0, 1)$ satisfy

$$M_1(x) = M(1) - \int_{(x,1]} y \, dM(y), \quad M_2(x) = -\int_{(x,1]} (1-y) \, dM(y);$$

note that necessarily $M_2(x) = 0$ for $x \geq 1$. Hence by Proposition 4.5 (iv) we have $F = F_1 \star F_2$, where F_1 is the infinitely divisible distribution function with canonical triple $(a, 0, M_1)$ and F_2 that with triple $(0, 0, M_2)$. Since M_1 satisfies

$$\begin{cases} M_1(0+) = M(1) - \int_{(0,1]} y \, dM(y) = -\infty, \\ \int_{(0,1]} x \, dM_1(x) = \int_{(0,1]} x^2 \, dM(x) < \infty, \end{cases}$$

from part (ii) it follows that $\ell_{F_1} > -\infty$ and $S(F_1) = [\ell_{F_1}, \infty)$. Therefore, $\ell_{F_2} = -\infty$, because $\ell_F = -\infty$. Since by (8.1) the direct sum of $S(F_1)$ and $S(F_2)$ is contained in $S(F)$, we conclude that $S(F) = \mathbb{R}$. ◻

In the same way, or by applying the preceding result not to $F = F_X$ but to F_{-X}, one can prove the following proposition.

Proposition 8.3. *Let F be an infinitely divisible distribution function with canonical triple $(a, 0, M)$ satisfying $M(0+) = 0$.*

(i) *If $M(x) = M(0-)$ for some $x < 0$, then $S(F)$ is given by (8.8) with $r_F = \gamma$ as an isolated (possibly its only) point.*

(ii) *If $M(x) < M(0-)$ for all $x < 0$ and $\int_{(-1,0)} x \, dM(x) > -\infty$, then $S(F)$ is given by $S(F) = (-\infty, \gamma]$ with γ the shift parameter of F.*

(iii) *If $\int_{(-1,0)} x \, dM(x) = -\infty$ (but, of course, still $\int_{(-1,0)} x^2 \, dM(x) < \infty$), then $S(F)$ is given by $S(F) = \mathbb{R}$.*

We finally have to consider an infinitely divisible F with a Lévy function M that gives positive mass to both the negative and the positive half-line, i.e., for which $M(0-) > 0$ and $M(0+) < 0$. Then we split up M as $M = M_1 + M_2$ with $M_1(0-) = 0$ and $M_2(0+) = 0$, and proceed as in the proof of Proposition 8.2: $S(F) = \overline{S(F_1) \oplus S(F_2)}$ where F_1 corresponds to M_1 and F_2 to M_2. Since F_1 and F_2 are non-degenerate, both $S(F_1)$ and $S(F_2)$ are unbounded sets. Now, combine the three possibilities for $S(F_1)$ as given in Proposition 8.2 with those for $S(F_2)$ in Proposition 8.3. Then it easily follows that in eight of the resulting nine cases we have $S(F) = \mathbb{R}$. In the remaining case it is easiest to apply Proposition 8.1; we then obtain part (i) of the following theorem, which further summarizes our findings. The corollary immediately follows from Theorem 4.20.

Theorem 8.4. *Let F be an infinitely divisible distribution function with canonical triple (a, σ^2, M). If $\sigma^2 > 0$, then $S(F) = \mathbb{R}$. Let further $\sigma^2 = 0$ and let γ be the shift parameter of F as given by (8.7) (if the integral there exists).*

(i) *If $M(x) = M(0-)$ for some $x < 0$ and $M(x) = M(0+)$ for some $x > 0$, then*

$$S(F) = \gamma + \Big(\{0\} \cup \overline{sg\,\{S(M)\}}\Big).$$

(ii) *If $M(0-) = 0$, $M(x) > M(0+)$ for $x > 0$ and $\int_{(0,1]} x\, dM(x) < \infty$, then*

$$S(F) = [\,\gamma, \infty).$$

(iii) *If $M(0+) = 0$, $M(x) < M(0-)$ for $x < 0$ and $\int_{(-1,0)} x\, dM(x) > -\infty$, then*

$$S(F) = (-\infty, \gamma\,].$$

(iv) *In all other cases (also where γ is undefined or infinite):*

$$S(F) = \mathbb{R}.$$

Corollary 8.5. *An infinitely divisible distribution function F that is continuous, is supported by a half-line or by all of $\mathbb{R}$.*

Clearly, in general $S(F)$ does not have the property of being *closed under addition*. In the cases (ii) and (iii) of the theorem there is a simple necessary and sufficient condition for F having this property: $\gamma \geq 0$ and $\gamma \leq 0$, respectively. In case (i) the condition $\gamma = 0$ is only sufficient; also when $\gamma \neq 0$, $S(F)$ may be all of $\mathbb{R}$ as in case (iv). Since by Proposition 4.5 (iii) $F^{\star t}$ with $t > 0$ has canonical triple $(ta, t\sigma^2, tM)$, in the four cases of the theorem the support of $F^{\star t}$ is given by

$$S(F^{\star t}) = t\gamma + \Big(\{0\} \cup \overline{\mathrm{sg}\,\{S(M)\}}\Big), \;\; [\,t\gamma, \infty), \;\; (-\infty, t\gamma\,], \;\; \mathbb{R}, \tag{8.9}$$

respectively. Hence in the first three cases we have $S(F^{\star t}) = S(F)$ for all t if $\gamma = 0$.

From part (i) of Theorem 8.4 it follows that an infinitely divisible F for which $\sigma^2 = 0$, $S(M) \subset \mathbb{Z}$ and $\gamma \in \mathbb{Z}$, corresponds to a *discrete* distribution on $\mathbb{Z}$. Since the support $S(p)$ of such a distribution $p = (p_k)_{k\in\mathbb{Z}}$ reduces to

$$S(p) = \{k \in \mathbb{Z} : p_k > 0\},$$

the support of p and the set of *zeroes* of p are complementary sets in $\mathbb{Z}$. For a probability density f of an *absolutely continuous* distribution on $\mathbb{R}$ the situation is not that simple. The support of the corresponding distribution function F can then be written as

$$S(F) = \overline{\{x \in \mathbb{R} : f(x) > 0\}},$$

if one chooses f such that $f(x) = 0$ for all $x \notin S(F)$, which can always be done. Nevertheless, one might ask for possible *zeroes* of f in $S(F)$, even if f is supposed to be *continuous*. Now, let F be infinitely divisible; by Corollary 8.5 $S(F)$ then is either a half-line or all of $\mathbb{R}$. In the first case the $\mathbb{R}_+$-result of Theorem III.8.4 immediately yields the following non-zero property of f; of course, there is an obvious analogue when the support is of the form $(-\infty, \gamma\,]$.

Theorem 8.6. *Let F be an infinitely divisible distribution function on the half-line $[\,\gamma, \infty)$ with a density f that is continuous on (γ, ∞). Then*

$$f(x) > 0 \;\text{ for all } x > \gamma.$$

When $S(F) = \mathbb{R}$, the question above is much harder. In the following theorem we give a sufficient condition for F to have a continuous density f without zeroes. In view of this condition we note that there are infinitely divisible distribution functions F such that $F^{\star t}$ is absolutely continuous only for t larger than some $t_0 > 0$; see Notes.

Theorem 8.7. *Let F be an infinitely divisible distribution function, not supported by a half-line, and suppose that, for some $t \in (0,1)$, $F^{\star t}$ is absolutely continuous with a bounded, continuous density. Then F is absolutely continuous and has a bounded, continuous density f for which*

$$f(x) > 0 \quad \textit{for all } x \in \mathbb{R}.$$

PROOF. Let f_t be the bounded, continuous density of $F^{\star t}$. Since $t \in (0,1)$, we can write $F = F^{\star t} \star F^{\star(1-t)}$, so F is absolutely continuous with density f given by

$$f(x) = \int_{\mathbb{R}} f_t(x-y)\, \mathrm{d}F^{\star(1-t)}(y) \qquad [\, x \in \mathbb{R}\,].$$

Clearly, f is bounded. Moreover, by the dominated convergence theorem f is continuous, and by Corollary 8.5 we have $S(F) = \mathbb{R}$. Now, take $x \in \mathbb{R}$, and note that by the continuity of f_t there exist $y_0 \in \mathbb{R}$, $\varepsilon > 0$ and $c > 0$ such that $f_t(x-y) \geq c$ for all $y \in (y_0 - \varepsilon, y_0 + \varepsilon)$. Then it follows that

$$f(x) \geq c \int_{(y_0-\varepsilon, y_0+\varepsilon)} \mathrm{d}F^{\star(1-t)}(y) > 0,$$

where we used the fact that $S(F^{\star(1-t)}) = \mathbb{R}$; the final case of (8.9) applies, with t replaced by $1-t$. ◻

The condition on $F^{\star t}$ makes this theorem hard to use for deciding that a given continuous density having a zero is not infinitely divisible. As is well known, $F^{\star t}$ has a bounded, continuous density if its characteristic function ϕ^t is absolutely integrable. This condition seems to be slightly more practical, but is in most cases far too strong.

Corollary 8.8. *Let F be an infinitely divisible distribution function, not supported by a half-line, and suppose that its characteristic function ϕ satisfies $\int_{\mathbb{R}} |\phi(u)|^t \, du < \infty$ for some $t \in (0,1)$. Then F is absolutely continuous and has a bounded, continuous density f without zeroes.*

For *symmetric* distributions there is a milder integrability condition if one is only interested in the value of f at zero.

Proposition 8.9. *Let F be a symmetric infinitely divisible distribution function with characteristic function ϕ. If $\int_{\mathbb{R}} |\phi(u)| \, du < \infty$, then F is absolutely continuous and has a bounded, continuous density f with $f(0) > 0$.*

PROOF. It is well known that, under the condition on ϕ, F is absolutely continuous with bounded, continuous density f on $\mathbb{R}$ given by

$$(8.10)\qquad f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-iux}\,\phi(u)\,\mathrm{d}u, \quad \text{so } f(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \phi(u)\,\mathrm{d}u.$$

Since F is symmetric, ϕ is real. Hence $f(0)$ cannot be zero unless ϕ takes negative values, i.e., unless ϕ has a zero, which is prohibited by Proposition 2.4. ◻

In this proposition, however, the condition on ϕ can be replaced by a condition on the density f itself which can be easily verified in many cases.

Proposition 8.10. *Let F be a symmetric infinitely divisible distribution function which is absolutely continuous with density f, and suppose that f is continuous at zero and of bounded variation over every finite interval. Then $f(0) > 0$. In fact, if ϕ is the characteristic function of f, then again $\int_{\mathbb{R}} \big|\phi(u)\big|\,du < \infty$.*

PROOF. It can be shown (see Notes) that if f is continuous at x and of bounded variation over every finite interval, then

$$(8.11)\qquad f(x) = \frac{1}{2\pi}\lim_{t\to\infty}\int_{-t}^{t} \mathrm{e}^{-iux}\,\phi(u)\,\mathrm{d}u.$$

Now, take $x = 0$; since ϕ is positive on $\mathbb{R}$, we can apply the monotone convergence theorem to conclude that $f(0)$ is given by the second part of (8.10) again. Hence $\int_{\mathbb{R}} \big|\phi(u)\big|\,\mathrm{d}u < \infty$ and $f(0) > 0$. ◻

In Section 11 we give an example that illustrates the use of this result.

9. Tail behaviour

In this section we study the *tail behaviour* of (the distribution function of) an infinitely divisible random variable X, and obtain in this way necessary conditions for infinite divisibility, which in concrete situations are often useful; see Section 11 for examples. We shall look at three *global* tail functions of X for $x \geq 0$: the *right tail* $\mathbb{P}(X > x)$, the *left tail* $\mathbb{P}(X < -x)$, and their sum, the *two-sided tail* of X:

$$\mathbb{P}\big(|X| > x\big) = \mathbb{P}(X > x) + \mathbb{P}(X < -x) \qquad [\,x \geq 0\,].$$

We suppose X to be non-degenerate, and we look at the tails in a rather crude way; we restrict ourselves to the asymptotic behaviour of the *logarithms* of the three tail functions. For the two-sided tail this means that we look at $-\log \mathbb{P}(|X| > x)$ for $x \to \infty$; this can always be done because by Proposition I.2.3 X is unbounded, so that $\mathbb{P}(|X| > x) > 0$ for all $x \geq 0$. When considering the right and left tails of X, we should know in advance that $r_X = \infty$ and $\ell_X = -\infty$, respectively. Necessary and sufficient conditions for this are given by Theorem 4.13.

We start with investigating the tails of *compound-Poisson* distributions. So, let $X \stackrel{d}{=} S_N$, where $(S_n)_{n \in \mathbb{Z}_+}$ is an sii-process generated by Y, say (so $S_1 \stackrel{d}{=} Y$), with $\mathbb{P}(Y = 0) = 0$, and N is Poisson distributed and independent of (S_n). Then the two-sided tail of X can be written as

$$\mathbb{P}(|X| > x) = \sum_{n=1}^{\infty} \mathbb{P}(N = n)\, \mathbb{P}(|S_n| > x) \qquad [\, x \geq 0\,].$$

Hence, on the one hand we have

$$\mathbb{P}(|X| > x) \geq \mathbb{P}(N = 1)\, \mathbb{P}(|Y| > x) \qquad [\, x \geq 0\,]; \tag{9.1}$$

this means that an infinitely divisible distribution can have an arbitrarily thick tail. On the other hand, using the fact that $(\alpha + \beta)^k \leq 2^k (\alpha^k + \beta^k)$ for $\alpha, \beta \geq 0$ and $k \in \mathbb{N}$, for $a > 0$ and $k \in \mathbb{N}$ we can estimate as follows:

$$\begin{aligned} \mathbb{P}(|X| > ka) &\geq \mathbb{P}(N = k)\, \mathbb{P}(|S_k| > ka) \geq \\ &\geq \mathbb{P}(N = k) \Big(\big\{\mathbb{P}(Y > a)\big\}^k + \big\{\mathbb{P}(Y < -a)\big\}^k \Big) \geq \\ &\geq \mathbb{P}(N = k) \left(\tfrac{1}{2}\right)^k \big\{\mathbb{P}(|Y| > a)\big\}^k. \end{aligned}$$

Since by Lemma II.9.1 $-\log \mathbb{P}(N = k) \sim k \log k$ as $k \to \infty$, it follows that

$$\limsup_{k \to \infty} \frac{-\log \mathbb{P}(|X| > ka)}{k \log k} \leq 1,$$

for all $a > 0$ such that $\mathbb{P}(|Y| > a) > 0$, so for all $a > 0$ with $a < m_Y$ where $m_Y := \max\{-\ell_Y, r_Y\}$. Now, take such an a and for $x > 0$ let $k_x \in \mathbb{N}$ be such that $(k_x - 1)\, a < x \leq k_x a$. Since $\mathbb{P}(|X| > x) \geq \mathbb{P}(|X| > k_x a)$ and $(k_x \log k_x)/(x \log x) \to 1/a$ as $x \to \infty$, by first letting $x \to \infty$ and then letting a tend to m_Y we conclude that

$$\limsup_{x \to \infty} \frac{-\log \mathbb{P}(|X| > x)}{x \log x} \leq \frac{1}{m_Y}, \tag{9.2}$$

where $1/m_Y := 0$ if $m_Y = \infty$. Let $m_Y < \infty$. Then we can also estimate above, as follows. Since $|Y| \leq m_Y$ a.s., we have $|S_n| \leq n\, m_Y$ a.s., and hence for $k \in \mathbb{N}$

$$\mathbb{P}\big(|X| > k\, m_Y\big) = \sum_{n=k+1}^{\infty} \mathbb{P}(N = n)\, \mathbb{P}\big(|S_n| > k\, m_Y\big) \leq \mathbb{P}(N > k).$$

Since by Lemma II.9.1 $-\log\, \mathbb{P}(N > k) \sim k\, \log k$ as $k \to \infty$, it follows that

$$\liminf_{k\to\infty} \frac{-\log\, \mathbb{P}\big(|X| > k\, m_Y\big)}{k\, \log k} \geq 1.$$

For $x > 0$ we now let $k_x \in \mathbb{Z}_+$ be such that $k_x m_Y \leq x < (k_x + 1)\, m_Y$. Since then $\mathbb{P}\big(|X| > x\big) \leq \mathbb{P}\big(|X| > k_x m_Y\big)$, we conclude that

$$\text{(9.3)} \qquad \liminf_{x\to\infty} \frac{-\log\, \mathbb{P}\big(|X| > x\big)}{x\, \log x} \geq \frac{1}{m_Y}.$$

Finally, by combining (9.2) and (9.3) we obtain a limiting result for the tail of X, in which we may replace m_Y by $\max\{-\ell_M, r_M\}$, where M is the Lévy canonical function of X; this immediately follows from the relation between $G := F_Y$ and M as given by Theorem 4.18. This theorem also shows that we have now obtained the two-sided tail behaviour of all infinitely divisible distributions with M *bounded* and *without normal component*. In fact, these distributions correspond to those of $\gamma + X$ with $\gamma \in \mathbb{R}$ and X compound-Poisson, and because for $x \geq 0$

$$\text{(9.4)} \qquad \mathbb{P}\big(|X| > x + |\gamma|\big) \leq \mathbb{P}\big(|\gamma + X| > x\big) \leq \mathbb{P}\big(|X| > x - |\gamma|\big),$$

the limiting result for the tail of X also holds for that of $\gamma + X$. We summarize.

Proposition 9.1. *Let X be a non-degenerate infinitely divisible random variable with canonical triple $(a, 0, M)$ such that M is bounded. Then the two-sided tail of X satisfies*

$$\text{(9.5)} \qquad \lim_{x\to\infty} \frac{-\log\, \mathbb{P}\big(|X| > x\big)}{x\, \log x} = \frac{1}{\max\{-\ell_M, r_M\}},$$

which should be read as 0 if the maximum is infinite.

In fact, as we shall see, formula (9.5) holds for every non-normal, non-degenerate infinitely divisible random variable X, but in order to show this some work needs to be done.

By adapting the proof of Proposition 9.1 similar results can be derived for the *right* and *left tails* of X separately. It is interesting to note that these results are actually contained in Proposition 9.1. To show this we use a technique that will be useful later in this section as well. Consider the situation of the proposition. Then, by Theorem 4.18, for some $\gamma \in \mathbb{R}$

$$(9.6) \qquad X \stackrel{d}{=} \gamma + X_1 + X_2, \quad \text{with } X_1 \text{ and } X_2 \text{ independent,}$$

where X_1 and X_2 are infinitely divisible with X_1 *nonnegative* and X_2 *nonpositive*. Suppose that $r_M > 0$ or, equivalently, $M(0+) < 0$; then by Theorem 4.13 we have $r_X = \infty$ and hence X_1 is non-degenerate. Since $X_1 = |X_1|$ and the Lévy function M_1 of X_1 satisfies $M_1(0-) = 0$ and $M_1 = M$ on $(0, \infty)$, we can apply Proposition 9.1 to see that the right tail $\mathbb{P}(X_1 > x)$ of X_1 satisfies (9.5) with the right-hand side of it replaced by $1/r_M$. Now, in view of (9.6) the right tail $\mathbb{P}(X > x)$ of X will behave the same as that of X_1, because shifting over γ will not affect tail behaviour and $\mathbb{P}(X_2 > x) = 0$ for all $x \geq 0$.

In order to make this argument precise we state a lemma relating the right tails of X and X_1. Here for $\varepsilon \in (0,1)$ we say that the right tail of X_2 is ε-*smaller* than the right tail of X_1 if

$$(9.7) \qquad \mathbb{P}(X_2 > \varepsilon x) \leq \mathbb{P}(X_1 > x) \qquad \big[x \geq 0 \text{ sufficiently large}\big].$$

Lemma 9.2. *Let* $X \stackrel{d}{=} X_1 + X_2$ *with* X_1 *and* X_2 *independent. Then:*

(i) $\mathbb{P}(X > x) \leq \mathbb{P}\big(X_1 > (1-\varepsilon)\,x\big) + \mathbb{P}(X_2 > \varepsilon\, x)$ *for* $\varepsilon \in [0,1)$, $x \geq 0$.

(ii) $\mathbb{P}(X > x) \geq \mathbb{P}\big(X_1 > (1+\varepsilon)\,x\big)\,\mathbb{P}(X_2 \geq -\varepsilon\, x)$ *for* $\varepsilon \in [0,1)$, $x \geq 0$.

(iii) *For* $\varepsilon \in (0,1)$ *and* $x \geq 0$ *sufficiently large:*

$$\tfrac{1}{2}\,\mathbb{P}\big(X_1 > (1+\varepsilon)\,x\big) \leq \mathbb{P}(X > x) \leq 2\,\mathbb{P}\big(X_1 > (1-\varepsilon)\,x\big),$$

provided, for the second inequality, that the right tail of X_2 *is* ε-*smaller than that of* X_1.

Proof. One easily verifies that for $x \geq 0$ and $\varepsilon \in [0,1)$:

$$\{X_1 + X_2 > x\} \subset \big\{X_1 > (1-\varepsilon)\,x\big\} \cup \{X_2 > \varepsilon\, x\},$$
$$\{X_1 + X_2 > x\} \supset \big\{X_1 > (1+\varepsilon)\,x\big\} \cap \{X_2 \geq -\varepsilon\, x\}.$$

This immediately yields (i) and (ii). Part (iii) is now a direct consequence, because $\lim_{x\to\infty} \mathbb{P}(X_2 \geq -\varepsilon\, x) = 1$ and $\mathbb{P}(X_1 > x) \leq \mathbb{P}\big(X_1 > (1-\varepsilon)\,x\big)$. ◻

We return to considering the right tail of X satisfying (9.6), where X_1 is nonnegative with $\{-\log \mathbb{P}(X_1 > x)\}/(x \log x) \to 1/r_M$ as $x \to \infty$ and X_2 is nonpositive. When $\gamma = 0$, by Lemma 9.2 we have for all $\varepsilon \in (0,1)$ and $x \geq 0$ sufficiently large:

$$\tfrac{1}{2}\,\mathbb{P}\big(X_1 > (1+\varepsilon)\,x\big) \leq \mathbb{P}(X > x) \leq \mathbb{P}(X_1 > x),$$

and hence

$$\frac{1}{r_M} \leq \liminf_{x\to\infty} \Big/ \limsup_{x\to\infty} \frac{-\log \mathbb{P}(X > x)}{x \log x} \leq (1+\varepsilon)\,\frac{1}{r_M},$$

from which by letting $\varepsilon \downarrow 0$ we see that the right tail of X behaves exactly the same as that of X_1: $\{-\log \mathbb{P}(X > x)\}/(x \log x) \to 1/r_M$ as $x \to \infty$. Using this with X replaced by $X - \gamma$ shows that the same property holds in case $\gamma \neq 0$. Observing that the left tail of X is just the right tail of $-X$ and that by Proposition 4.5 (i) the right extremity of the canonical function of $-X$ is given by $-\ell_M$, we are led to the following result.

Proposition 9.3. *Let X be a non-degenerate infinitely divisible random variable with canonical triple $(a, 0, M)$ such that M is bounded. Then the right and left tails of X satisfy*

$$\text{(9.8)} \qquad \lim_{x\to\infty} \frac{-\log \mathbb{P}(X > x)}{x \log x} = \frac{1}{r_M}, \qquad \lim_{x\to\infty} \frac{-\log \mathbb{P}(X < -x)}{x \log x} = \frac{1}{-\ell_M},$$

where $M(0+) < 0$ is supposed in the first case and $M(0-) > 0$ in the second.

We have now derived Proposition 9.3 from Proposition 9.1. Actually, these propositions are equivalent; we shall give the converse proof later in a more general situation. For a single X formula (9.8) is, of course, more precise than (9.5). Therefore, when generalizing our results so far to random variables X with *unbounded* M, we shall first concentrate on right and left tails.

In doing so we start with giving upperbounds for these tails when the support of M is restricted to the positive half-line (and still there is no normal component). Note that by Theorem 4.13 the left tail is then non-trivial only if $\int_{(0,1]} x \,\mathrm{d}M(x) = \infty$. The upperbounds for the tails of X will be obtained via Chebyshev-type inequalities in which the LSt of X occurs. To be sure of finiteness of this LSt we need the following well-known result; see Lemma A.2.8.

Lemma 9.4. *Let X be a random variable with characteristic function ϕ satisfying*

$$\phi(u) = A(u) \qquad [-\varepsilon < u < \varepsilon, \ \textit{some } \varepsilon > 0],$$

where A is a function that is analytic on the disk $|z| < \rho$ in $\mathbb{C}$ with $\rho \geq \varepsilon$. Then $\phi(z) := \mathbb{E}\, e^{izX}$ is well defined for all $z \in \mathbb{C}$ with $|z| < \rho$, and

$$\phi(z) = A(z) \qquad [z \in \mathbb{C} \ \textit{with } |z| < \rho].$$

Proposition 9.5. *Let X be an infinitely divisible random variable with canonical triple $(0, 0, M)$ such that $M(0-) = 0$.*

(i) *If $r_M \leq 1$, then the right tail of X satisfies*

$$\mathbb{P}(X > x) \leq \exp\Big[-\frac{1}{2\, r_M}\, x \log x\Big] \qquad [x \geq 0 \ \textit{sufficiently large}].$$

(ii) *With $b_M := \int_{(0,\infty)} y^2/(1+y^2)^2\, dM(y)$ the left tail of X satisfies*

$$\mathbb{P}(X < -x) \leq \exp\Big[-\frac{1}{2\, b_M}\, x^2\Big] \qquad [x \geq 0].$$

(iii) *For all $c > 0$ the left tail of X satisfies*

$$\mathbb{P}(X < -x) \leq \exp\big[-c\, x^2\big] \qquad [x \geq 0 \ \textit{sufficiently large}].$$

Proof. First, suppose that $r := r_M < \infty$. Since X has canonical triple $(0, 0, M)$, for the characteristic function ϕ of X we have $\phi = A$ on $\mathbb{R}$ where, at least for $z \in \mathbb{R}$, $A(z)$ is defined by

$$A(z) := \exp\Big[\int_{(0,\infty)} \Big(e^{izy} - 1 - \frac{izy}{1+y^2}\Big)\, dM(y)\Big].$$

From (the proof of) Proposition 7.9, however, it follows that A is well defined on all of $\mathbb{C}$ and can be represented as

$$A(z) = \exp\Big[\sum_{n=1}^{\infty} \kappa_n \frac{(iz)^n}{n!}\Big] \qquad [z \in \mathbb{C}],$$

where the cumulants κ_n satisfy $\kappa_1 \leq \frac{1}{2}\kappa_2$ and $\kappa_n \leq r^{n-2}\kappa_2$ for $n \geq 2$. Since this means that A is an analytic function on $\mathbb{C}$, we can apply Lemma 9.4 to conclude that $\phi(z) := \mathbb{E}\, e^{izX}$ is well defined for all $z \in \mathbb{C}$ and satisfies $\phi(z) = A(z)$ for all z. In particular, $\mathbb{E}\, e^{sX}$ and $\mathbb{E}\, e^{-sX}$ are finite for all $s > 0$, and these quantities are given by $A(-is)$ and $A(is)$, respectively.

We are now ready for estimating the right and left tails of X. In view of (i) we let $r \leq 1$; the inequality $\kappa_n \leq r^{n-2}\kappa_2$ then also holds for $n = 1$. It follows that for $x \geq 0$ and $s > 0$

$$\begin{aligned} \mathbb{P}(X > x) &\leq \mathrm{e}^{-sx}\, \mathbb{E}\, \mathrm{e}^{sX} = \\ &= \exp\Big[-sx + \sum_{n=1}^{\infty} \kappa_n \frac{s^n}{n!}\Big] \leq \exp\Big[-sx + \frac{\kappa_2}{r^2}\, \mathrm{e}^{rs}\Big]. \end{aligned}$$

Now, taking $s = (\log x)/r$, we get

$$\mathbb{P}(X > x) \leq \exp\Big[-\frac{1}{r}\, x \log x + \frac{\kappa_2}{r^2}\, x\Big],$$

which yields the upperbound as given in (i) if we take $x \geq \mathrm{e}^{2\kappa_2/r}$. Turning to the left tail, we drop the condition $r \leq 1$ (but still suppose $r < \infty$), and take $x \geq 0$ and $s > 0$. Using the inequality $\mathrm{e}^{-\alpha} - 1 + \alpha \leq \frac{1}{2}\alpha^2$ for $\alpha = sy/(1+y^2) > 0$ with $y > 0$, one then sees that

$$\begin{aligned} \mathbb{P}(X < -x) &\leq \mathrm{e}^{-sx}\, \mathbb{E}\, \mathrm{e}^{-sX} = \mathrm{e}^{-sx} A(is) = \\ &= \exp\Big[-sx + \int_{(0,\infty)} \Big(\mathrm{e}^{-sy} - 1 + \frac{sy}{1+y^2}\Big)\, \mathrm{d}M(y)\Big] \leq \\ &\leq \exp\big[-sx + \tfrac{1}{2}\, b_M\, s^2\big]. \end{aligned}$$

Now, minimizing over $s > 0$, or taking $s = x/b_M$, yields (ii).
Next, suppose that $r_M = \infty$. In order to prove the inequality in (ii) also in this case, we take $r \in (0, \infty)$, and write M as the sum of two canonical functions M_1 and M_2 with

$$M_1 := \begin{cases} M - M(r) & \text{on } (0, r), \\ 0 & \text{on } [r, \infty), \end{cases} \qquad M_2 := \begin{cases} M(r) & \text{on } (0, r), \\ M & \text{on } [r, \infty). \end{cases}$$

Then from Proposition 4.5 (iv) and Theorem 4.13 it follows that

$$X \stackrel{\mathrm{d}}{=} X_1 + X_2 - a_r, \quad \text{with } X_1 \text{ and } X_2 \text{ independent},$$

where $a_r := \int_{(r,\infty)} y/(1+y^2)\, \mathrm{d}M(y) > 0$, X_1 and X_2 are infinitely divisible with canonical triples $(0, 0, M_1)$ and $(a_r, 0, M_2)$, respectively, and X_2 is *nonnegative*. Moreover, as $r_{M_1} < \infty$, X_1 satisfies the inequality in (ii) with $b_M =: b$ replaced by b_r given by $b_r := \int_{(0,r]} y^2/(1+y^2)^2\, \mathrm{d}M(y)$. Hence for $x \geq a_r$ we can estimate as follows:

$$\mathbb{P}(X < -x) \leq \mathbb{P}\big(X_1 < -(x - a_r)\big) \leq \exp\Big[-\frac{1}{2b_r}\, (x - a_r)^2\Big].$$

Since r was chosen arbitrarily in $(0,\infty)$, we may let r tend to infinity, and conclude that (ii) holds as stated, because $a_r \to 0$ and $b_r \to b$ as $r \to \infty$. Finally, we prove (iii) by using (ii) and applying the same splitting of M at $r \in (0, r_M)$ as above, but now we will choose r small rather than large. With a_r and b_r as defined above it follows that for $x \geq 0$ sufficiently large

$$\mathbb{P}(X < -x) \leq \exp\Big[-\frac{1}{2b_r}\,(x-a_r)^2\Big] \leq \exp\Big[-\frac{1}{4b_r}\,x^2\Big].$$

Since $b_r \to 0$ as $r \downarrow 0$, at a given $c > 0$ one can choose r so small that $1/(4b_r) \geq c$; this immediately yields the inequality in (iii). □

Applying part (iii) of this proposition to $a - X$ yields the following result on right tails.

Corollary 9.6. *Let X be an infinitely divisible random variable with canonical triple $(a, 0, M)$ such that $M(0+) = 0$. Then for all $c > 0$ the right tail of X satisfies*

$$\mathbb{P}(X > x) \leq \exp\big[-c\,x^2\big] \qquad [x \geq 0 \text{ sufficiently large}].$$

This corollary has been stated because we want to combine it with the two other results on right tails we have obtained so far, viz. the first parts of Propositions 9.3 and 9.5, and with the well-known fact that as $x \to \infty$

$$\mathbb{P}(X > x) \sim \frac{\sigma}{x\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2/\sigma^2}, \quad \text{so} \quad -\log \mathbb{P}(X > x) \sim \frac{1}{2\sigma^2}\,x^2, \tag{9.9}$$

if X has a *normal* $(0, \sigma^2)$ distribution. Thus we obtain the following main result.

Theorem 9.7. *Let X be a non-degenerate infinitely divisible random variable with canonical triple (a, σ^2, M). Then the right tail of X has the following properties:*

(i) *If $M(0+) = 0$, $\sigma^2 = 0$ and $\int_{[-1,0)} x\,dM(x) > -\infty$, then X satisfies $\mathbb{P}(X > x) = 0$ for all x sufficiently large.*

(ii) *If $M(0+) = 0$, and $\sigma^2 > 0$ or $\int_{[-1,0)} x\,dM(x) = -\infty$, then X satisfies $\mathbb{P}(X > x) > 0$ for all x and*

$$\lim_{x\to\infty} \frac{-\log \mathbb{P}(X > x)}{x^2} = \frac{1}{2\sigma^2},$$

which should be read as ∞ if $\sigma^2 = 0$.

(iii) *If* $M(0+) < 0$, *then* X *satisfies* $\mathbb{P}(X > x) > 0$ *for all* x *and*

$$\lim_{x\to\infty} \frac{-\log \mathbb{P}(X > x)}{x \log x} = \frac{1}{r_M},$$

which should be read as 0 if the right extremity r_M *of* M *is* ∞.

Similarly, the left tail of X *has the following properties:*

(i) *If* $M(0-) = 0$, $\sigma^2 = 0$ *and* $\int_{(0,1]} x \, dM(x) < \infty$, *then* X *satisfies* $\mathbb{P}(X < -x) = 0$ *for all* x *sufficiently large.*

(ii) *If* $M(0-) = 0$, *and* $\sigma^2 > 0$ *or* $\int_{(0,1]} x \, dM(x) = \infty$, *then* X *satisfies* $\mathbb{P}(X < -x) > 0$ *for all* x *and*

$$\lim_{x\to\infty} \frac{-\log \mathbb{P}(X < -x)}{x^2} = \frac{1}{2\sigma^2},$$

which should be read as ∞ *if* $\sigma^2 = 0$.

(iii) *If* $M(0-) > 0$, *then* X *satisfies* $\mathbb{P}(X < -x) > 0$ *for all* x *and*

$$\lim_{x\to\infty} \frac{-\log \mathbb{P}(X < -x)}{x \log x} = \frac{1}{-\ell_M},$$

which should be read as 0 if the left extremity ℓ_M *of* M *is* $-\infty$.

PROOF. First, note that the assertions on the left tail immediately follow from those on the right tail by replacing X by $-X$ and using Proposition 4.5 (i). Thus we concentrate on proving the properties (i), (ii) and (iii) of the right tail.

From Theorem 4.13 we see that the condition stated in (i) is equivalent to the condition that $r_X < \infty$; so (i) follows, and under each of the conditions stated in parts (ii) and (iii) we have $\mathbb{P}(X > x) > 0$ for all x. In proving these parts we distinguish four cases: I–IV, the first two of which concern part (ii).

I. Let $M(0+) = 0$, $\sigma^2 = 0$ and $\int_{[-1,0)} x \, \mathrm{d}M(x) = -\infty$. Then by Corollary 9.6 for all $c > 0$ we have

$$\frac{-\log \mathbb{P}(X > x)}{x^2} \geq c \qquad [\, x \geq 0 \text{ sufficiently large} \,],$$

so this function of x tends to ∞ as $x \to \infty$, as desired.

II. Let $M(0+) = 0$ and $\sigma^2 > 0$. Then we split off the normal component: $X \stackrel{\mathrm{d}}{=} X_1 + X_2$, where X_1 and X_2 are independent, X_1 has a normal $(0, \sigma^2)$

distribution and X_2 has canonical triple $(a, 0, M)$. Let $\delta > 0$ and $c > 0$. Then by (9.9) and Corollary 9.6 we have for $x \geq 0$ sufficiently large

$$\mathbb{P}(X_1 > x) \geq \exp\Big[-\Big(\frac{1}{2\sigma^2} + \delta\Big)\, x^2\Big], \quad \mathbb{P}(X_2 > x) \leq \mathrm{e}^{-c\,x^2},$$

and hence for every $\varepsilon \in (0, 1)$

$$\mathbb{P}(X_2 > \varepsilon x) \,/\, \mathbb{P}(X_1 > x) \leq \exp\Big[-\Big(c\,\varepsilon^2 - \frac{1}{2\sigma^2} - \delta\Big)\, x^2\Big].$$

Now, choose c so large that $c\,\varepsilon^2 > 1/(2\sigma^2) + \delta$; then it follows that the right tail of X_2 is ε-smaller than the right tail of X_1. Hence by Lemma 9.2 for $x \geq 0$ sufficiently large

$$\tfrac{1}{2}\, \mathbb{P}\big(X_1 > (1+\varepsilon)\, x\big) \leq \mathbb{P}(X > x) \leq 2\, \mathbb{P}\big(X_1 > (1-\varepsilon)\, x\big),$$

which by (9.9) implies that

$$\frac{1}{2\sigma^2}\,(1-\varepsilon)^2 \leq \liminf_{x\to\infty} \,/\, \limsup_{x\to\infty} \frac{-\log \mathbb{P}(X > x)}{x^2} \leq \frac{1}{2\sigma^2}\,(1+\varepsilon)^2.$$

By letting $\varepsilon \downarrow 0$ we see that the limiting result of (ii) holds for X, as desired.

III. Let $M(0+) < 0$, $\sigma^2 = 0$ and $M(0-) = 0$. We split up M at a point $r \in (0, r_M \wedge 1)$, as in the proof of Proposition 9.5. So we have $X \overset{\mathrm{d}}{=} X_1 + X_2$, where X_1 and X_2 are independent, X_1 satisfies the conditions of Proposition 9.3 and X_2 those of Proposition 9.5 (i). Let $\varepsilon \in (0, 1)$ and $\delta > 0$. Then it follows that for $x \geq 0$ sufficiently large

$$\mathbb{P}(X_1 > x) \geq \exp\Big[-\Big(\frac{1}{\varepsilon\, r_M} + \delta\Big)\, \varepsilon x\, \log \varepsilon x\Big],$$

and hence

$$\mathbb{P}(X_2 > \varepsilon x) \,/\, \mathbb{P}(X_1 > x) \leq \exp\Big[-\Big(\frac{1}{2r} - \frac{1}{\varepsilon\, r_M} - \delta\Big)\, \varepsilon x\, \log \varepsilon x\Big].$$

From this we see that r can be taken close to 0 such that the right tail of X_2 is ε-smaller than that of X_1. Now we can proceed as in case II and apply Lemma 9.2 to conclude that the right tails of X and X_1 behave the same, i.e., X satisfies the limiting result of (iii), as desired.

IV. Let $M(0+) < 0$, and $\sigma^2 > 0$ or $M(0-) > 0$. We split up M at zero, and thus have $X \overset{\mathrm{d}}{=} X_1 + X_2$, where X_1 and X_2 are independent, X_1 satisfies the conditions of case III and X_2 those of case (i) or (ii). Let $\delta > 0$. Then it follows that for $x \geq 0$ sufficiently large

$$\mathbb{P}(X_1 > x) \geq \exp\Big[-\Big(\frac{1}{r_M} + \delta\Big)\, x\, \log x\Big], \quad \mathbb{P}(X_2 > x) \leq \mathrm{e}^{-c\,x^2},$$

for some $c > 0$, and hence for every $\varepsilon \in (0,1)$

$$\mathbb{P}(X_2 > \varepsilon x) \,/\, \mathbb{P}(X_1 > x) \le \exp\Big[-c\,\varepsilon^2 x^2 + \Big(\frac{1}{r_M} + \delta\Big)\, x \log x\Big].$$

Since this upperbound is ≤ 1 for x sufficiently large, we conclude that the right tail of X_2 is ε-smaller than that of X_1. Applying Lemma 9.2 again shows that X, like X_1, satisfies the limiting result of (iii). □

Finally, as announced we show that by using the properties of right and left tails as given by Theorem 9.7 we can also easily deal with *two-sided tails*; it turns out that the limiting result of Proposition 9.1 holds for *all* infinitely divisible distributions except the normal ones. Recall that

$$\mathbb{P}\big(|X| > x\big) = \mathbb{P}(X > x) + \mathbb{P}(X < -x) \qquad [\,x \ge 0\,].$$

Theorem 9.8. *Let X be an infinitely divisible random variable that is not normal or degenerate. Then the two-sided tail of X satisfies*

$$\lim_{x\to\infty} \frac{-\log \mathbb{P}\big(|X| > x\big)}{x \log x} = \frac{1}{\max\{-\ell_M, r_M\}}, \tag{9.10}$$

which should be read as 0 if the maximum is infinite. Here ℓ_M and r_M are the left and right extremities of the Lévy function M of X.

Proof. Since X is not normal, we have $M(0-) > 0$ or $M(0+) < 0$. Now, for reasons of symmetry we may restrict ourselves to proving the theorem in case M satisfies $M(0+) < 0$ and $r_M \ge -\ell_M$. Setting $\alpha := 1/r_M$, by Theorem 9.7 for the right tail of X we then have

$$\lim_{x\to\infty} \frac{-\log \mathbb{P}(X > x)}{x \log x} = \alpha, \tag{9.11}$$

and we have to show that this limiting result also holds with X replaced by $|X|$. When $\alpha = 0$, this is immediately clear from the fact that then $\mathbb{P}\big(|X| > x\big) \ge \mathbb{P}(X > x)$; so, further we suppose that $\alpha > 0$.
First, let $M(0-) = 0$ or, equivalently, $\ell_M = 0$. Then Theorem 9.7 implies that for some $c > 0$ the left tail of X satisfies

$$\mathbb{P}(X < -x) \le \mathrm{e}^{-c\,x^2} \qquad [\,x \ge 0 \text{ sufficiently large}\,]. \tag{9.12}$$

Now, combining (9.11) and (9.12) we see that for $\delta > 0$

$$\mathbb{P}(X < -x) \,/\, \mathbb{P}(X > x) \le \exp\big[-c\,x^2 + (\alpha + \delta)\, x \log x\big],$$

which does not exceed one for x sufficiently large; hence for these x

$$\mathbb{P}(X > x) \leq \mathbb{P}\big(|X| > x\big) \leq 2\,\mathbb{P}(X > x).$$

We conclude that the two-sided tail behaves exactly the same as the right tail.

Next, let $M(0-) > 0$ or, equivalently, $\ell_M < 0$. Then setting $\beta := -1/\ell_M$, by Theorem 9.7 for the left tail of X we have

$$\lim_{x\to\infty} \frac{-\log \mathbb{P}(X < -x)}{x \log x} = \beta. \tag{9.13}$$

Note that $0 < \alpha \leq \beta$. When $\alpha < \beta$, we combine (9.11) and (9.13) to see that for $\delta > 0$ with $\delta < \frac{1}{2}(\beta - \alpha)$

$$\mathbb{P}(X < -x) \,/\, \mathbb{P}(X > x) \leq \exp\Big[-\big\{(\beta-\delta) - (\alpha+\delta)\big\}\, x \log x\Big],$$

which does not exceed one for x sufficiently large; as above we conclude that the two-sided tail behaves like the right tail. When $\alpha = \beta$, we look at $R(x) := \max\big\{\mathbb{P}(X > x), \mathbb{P}(X < -x)\big\}$ and observe that as $x \to \infty$

$$\frac{-\log R(x)}{x \log x} = \min\left\{\frac{-\log \mathbb{P}(X > x)}{x \log x}, \frac{-\log \mathbb{P}(X < -x)}{x \log x}\right\} \longrightarrow \alpha,$$

because of (9.11) and (9.13). Since obviously

$$R(x) \leq \mathbb{P}\big(|X| > x\big) \leq 2\,R(x) \qquad [\,x \geq 0\,],$$

we conclude that, also when $\alpha = \beta$, the two-sided tail of X behaves as desired. □

This theorem, together with (9.4) and (9.9), yields the following *characterization of the normal distribution.*

Corollary 9.9. *A non-degenerate infinitely divisible random variable X has a normal distribution iff it satisfies*

$$\limsup_{x\to\infty} \frac{-\log \mathbb{P}\big(|X| > x\big)}{x \log x} = \infty. \tag{9.14}$$

Theorem 9.8 is remarkable in that it not only puts bounds on the tails of infinitely divisible distributions, but also shows that the tails are not too irregular, since the limit in (9.10) exists. Corollary 9.9 stresses the surprising fact that the normal distribution is separated from all other infinitely divisible distributions by a deviating (two-sided) tail behaviour.

When the Lévy function M has bounded support, Theorem 9.8 gives the exact rate with which $-\log \mathbb{P}(|X| > x)$ tends to infinity. In the other case we only know that

$$(9.15)\qquad \lim_{x\to\infty} \frac{-\log \mathbb{P}(|X| > x)}{x \log x} = 0.$$

In general not more than this can be said; the convergence in (9.15) can be arbitrarily slow. For the *compound-exponential* distributions, which by Theorem 5.2 (ii) satisfy (9.15), it can be shown that $-\log \mathbb{P}(|X| > x) = O(x)$ as $x \to \infty$; it is not known, however, whether $\{-\log \mathbb{P}(|X| > x)\}/x$ has a finite limit as $x \to \infty$.

10. Log-convexity

There are only few simple sufficient conditions for a *probability density* on $\mathbb{R}$ to be infinitely divisible. In Section III.10 we have seen that a density on $\mathbb{R}_+$ is infinitely divisible if it is completely monotone or, more generally, log-convex. Recall that a function f on $(0,\infty)$ is said to be *completely monotone* if it has alternating derivatives:

$$(-1)^n f^{(n)}(x) \geq 0 \qquad [n \in \mathbb{Z}_+;\ x > 0],$$

and that the nonvanishing *log-convex* functions consist of the positive functions on $(0,\infty)$ for which $\log f$ is convex. Since by (A.3.12) log-convexity implies convexity, probability densities cannot be log-convex on *all* of $\mathbb{R}$. We therefore consider densities that are log-convex *on both sides of zero*, and ask whether they are infinitely divisible.

In determining this, in the present section we restrict ourselves to distributions that are *symmetric*. Thus we consider densities f on $\mathbb{R}$ of the form

$$(10.1)\qquad f(x) = \tfrac{1}{2}\, g(|x|) \qquad [x \in \mathbb{R}],$$

where g is a probability density on $\mathbb{R}_+$; cf. (4.12). For the characteristic function ϕ of f we have, with π the Lt of g:

$$(10.2)\qquad \phi(u) = \int_0^\infty (\cos ux)\, g(x)\,\mathrm{d}x = \mathrm{Re}\,\pi(-iu).$$

First, one might wonder whether infinite divisibility of g implies that of f, but the situation is not that simple; in Section 11 we will give an example of an infinitely divisible g such that ϕ has real zeroes. On the other hand, we do have infinite divisibility of f if g is restricted to the class of *completely monotone* densities.

In order to show this, we let g be completely monotone on $(0,\infty)$, and use the fact (Proposition A.3.11) that g can then be represented as a *mixture of exponential densities*:

$$g(x) = \int_{(0,\infty)} \lambda\, \mathrm{e}^{-\lambda x}\, \mathrm{d}H(\lambda), \quad \text{so} \quad \pi(s) = \int_{(0,\infty)} \frac{\lambda}{\lambda + s}\, \mathrm{d}H(\lambda),$$

for some distribution function H on $(0,\infty)$. For f given by (10.1) it follows that

$$f(x) = \int_{(0,\infty)} \tfrac{1}{2}\, \lambda\, \mathrm{e}^{-\lambda|x|}\, \mathrm{d}H(\lambda) \qquad [x \in \mathbb{R}], \tag{10.3}$$

so f is a *mixture of Laplace densities.* Now observe that the characteristic function ϕ of f can be written as

$$\phi(u) = \int_{(0,\infty)} \frac{\lambda^2}{\lambda^2 + u^2}\, \mathrm{d}H(\lambda) = \pi_1(u^2), \tag{10.4}$$

where π_1 is defined by

$$\pi_1(s) = \int_{(0,\infty)} \frac{\lambda^2}{\lambda^2 + s}\, \mathrm{d}H(\lambda) = \int_{(0,\infty)} \frac{\lambda}{\lambda + s}\, \mathrm{d}H(\sqrt{\lambda}).$$

From the second expression for π_1 we see that π_1, like π, is the pLSt of a distribution with a completely monotone density. Hence π_1 is an infinitely divisible pLSt, and Proposition 6.1 can be applied to conclude that ϕ is *infinitely divisible.* As complete monotonicity of g is equivalent to the complete monotonicity of f restricted to $(0,\infty)$, we can summarize as follows.

Theorem 10.1. *A symmetric distribution which has a density that is completely monotone on* $(0,\infty)$*, is infinitely divisible. Equivalently, a mixture of Laplace densities as in (10.3) is infinitely divisible.*

In Chapter VI we shall see that symmetry is not necessary; a density f with the property that both $x \mapsto f(x)$ and $x \mapsto f(-x)$ are completely monotone on $(0,\infty)$, is infinitely divisible.

Of course, there is a discrete counterpart to Theorem 10.1 for symmetric distributions $(p_k)_{k\in\mathbb{Z}}$ on $\mathbb{Z}$. The appropriate analogue of (10.1) for such distributions is given by

$$p_k = \frac{1}{2-q_0}\, q_{|k|} \qquad [\, k \in \mathbb{Z}\,], \tag{10.5}$$

where $(q_k)_{k\in\mathbb{Z}_+}$ is a distribution on $\mathbb{Z}_+$. The fact that we want to take $q_0 > 0$ here, causes some problems when 'discretizing' the proof of Theorem 10.1. The analogue of (10.1) given by $p_k = \frac{1}{2}\,(q_k + q_{-k})$ (with $q_k = 0$ for $k < 0$), looks more attractive, but only yields the infinite divisibility of (p_k) such that $(p_0, 2p_1, 2p_2, \ldots)$ is completely monotone. Recall that a sequence $(q_k)_{k\in\mathbb{Z}_+}$ is said to be *completely monotone* if

$$(-1)^n \Delta^n q_k \geq 0 \qquad [\, n, k \in \mathbb{Z}_+\,],$$

where $\Delta^0 q_k := q_k$, $\Delta q_k := q_{k+1} - q_k$ and $\Delta^n := \Delta \circ \Delta^{n-1}$ for $n \in \mathbb{N}$.

Theorem 10.2. *A symmetric distribution on $\mathbb{Z}$ which is completely monotone on $\mathbb{Z}_+$, is infinitely divisible.*

PROOF. Let $(p_k)_{k\in\mathbb{Z}}$ be a distribution as indicated in the theorem. Then it has the form (10.5) with $(q_k)_{k\in\mathbb{Z}_+}$ a completely monotone probability distribution on $\mathbb{Z}_+$; take $q_k = \big(2/\{1+p_0\}\big)\, p_k$ for $k \in \mathbb{Z}_+$. Using the fact (Proposition A.4.10) that (q_k) can be represented as a *mixture of geometric distributions*:

$$q_k = \int_{[0,1)} (1-p)\, p^k \,\mathrm{d}H(p) \qquad [\, k \in \mathbb{Z}_+\,],$$

for some distribution function H on $[\,0,1)$, one easily shows that the characteristic function ϕ of (p_k) can be written as

$$\phi(u) = \frac{1}{2-q_0} \int_{[0,1)} (1-p) \Big(\frac{1}{1-p\,\mathrm{e}^{-iu}} + \frac{1}{1-p\,\mathrm{e}^{iu}} - 1\Big)\, \mathrm{d}H(p) =$$

$$= \int_{[0,1)} \frac{1-r(p)}{1-r(p)\,\cos u}\, \mathrm{d}G(p),$$

where the functions r and G are defined on $[\,0,1)$ by

$$r(p) = \frac{2p}{1+p^2}, \quad G(p) = \frac{1}{2-q_0} \int_{[0,p]} (1+s)\, \mathrm{d}H(s).$$

Now, $r(p) \in [0,1)$ for all p, and $q_0 = 1 - \mu_H$ with μ_H the first moment of H, so G is a distribution function on $[0,1)$. It follows that

$$\phi(u) = P(\cos u),$$

where P is the pgf of a mixture of geometric distributions, hence of a distribution that is completely monotone and therefore infinitely divisible; cf. Theorem II.10.4. Since $u \mapsto \cos u$ is a characteristic function, we can apply Proposition 3.1 and conclude that ϕ is an infinitely divisible characteristic function. ◻◻

We briefly return to the following *question* from the beginning of this section:

(10.6) $\qquad g$ log-convex on $(0,\infty) \overset{?}{\Longrightarrow} f$ in (10.1) infinitely divisible.

Unfortunately, it is not known whether this implication holds. It is not difficult to prove that ϕ in (10.2) is positive if g is *convex*, and hence if g is log-convex. This means that examples of convex functions g, for which f is not infinitely divisible, will be hard to find—if any exist. On the other hand, it is hard to get a handle on proving the infinite divisibility of f if g is log-convex. Many log-convex functions are also completely monotone, and log-convex functions that are not completely monotone are hard to handle. Though the set of log-convex densities is convex, there seems to be no easy set of 'extreme points' from which these densities can be built. Numerical evidence seems to indicate that the answer to (10.6) is affirmative.

It is remarkable that for *characteristic functions* a question similar to (10.6) has a positive answer; real (and hence even) characteristic functions that are log-convex on $(0,\infty)$ are infinitely divisible. In order to show this, together with a discrete counterpart, we use the following well-known criterion due to Pólya and a periodic variant of it.

Lemma 10.3. *Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be an even continuous function satisfying $\phi(0) = 1$.*

(i) *If ϕ is nonincreasing and convex on $(0,\infty)$, then ϕ is the characteristic function of a symmetric distribution on $\mathbb{R}$.*

(ii) *If ϕ is nonincreasing and convex on $(0,\pi)$ and 2π-periodic, then ϕ is the characteristic function of a symmetric distribution on $\mathbb{Z}$.*

Theorem 10.4. *Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be an even continuous function satisfying $\phi(0) = 1$.*

(i) *If ϕ is nonincreasing and log-convex on $(0, \infty)$, then ϕ is the characteristic function of a symmetric infinitely divisible distribution on $\mathbb{R}$.*

(ii) *If ϕ is nonincreasing and log-convex on $(0, \pi)$ and 2π-periodic, then ϕ is the characteristic function of a symmetric infinitely divisible distribution on $\mathbb{Z}$.*

PROOF. Let ϕ be nonincreasing and log-convex on $(0, \infty)$. Then so is ϕ^t for any $t > 0$. Since by (A.3.12) log-convexity implies convexity, from part (i) of the lemma it follows that ϕ^t is a characteristic function for all $t > 0$. Hence ϕ is infinitely divisible by Proposition 2.5. Part (ii) is similarly obtained from the second part of the lemma. ◻

As a special case we obtain the infinite divisibility of any real characteristic function that is *completely monotone* on $(0, \infty)$. By Bernstein's theorem these functions are given by the functions ϕ of the form

$$(10.7) \qquad \phi(u) = \int_{\mathbb{R}_+} \mathrm{e}^{-\lambda|u|}\, \mathrm{d}H(\lambda) = \widehat{H}\big(|u|\big) \qquad [\,u \in \mathbb{R}\,],$$

where H is a distribution function on $\mathbb{R}_+$. It follows that if $H(0) = 0$, then the distribution corresponding to ϕ is absolutely continuous with density f given by

$$(10.8) \qquad f(x) = \frac{1}{\pi} \int_{(0,\infty)} \frac{\lambda}{\lambda^2 + x^2}\, \mathrm{d}H(\lambda) \qquad [\,x \in \mathbb{R}\,],$$

so f is a *mixture of Cauchy densities.* Thus Theorem 10.4 (i) leads to the following result.

Theorem 10.5. *If π is a pLSt, then the function ϕ defined by*

$$(10.9) \qquad \phi(u) = \pi\big(|u|\big) \qquad [\,u \in \mathbb{R}\,],$$

is an infinitely divisible characteristic function. In particular, any mixture of Cauchy densities as in (10.8) is infinitely divisible.

If π is a pLSt, then so is $s \mapsto \pi(s^\gamma)$ for every $\gamma \in (0, 1]$; this follows by combining (III.3.8) and Example III.4.9, or from Proposition A.3.7 (vi) and Bernstein's theorem. Thus Theorem 10.5 yields the following supplement of Proposition 6.1.

Corollary 10.6. *If π is a pLSt, then $u \mapsto \pi\big(|u|^\gamma\big)$ is an infinitely divisible characteristic function for every $\gamma \in (0,1\,]$.*

Infinitely divisible discrete distributions, such as occur in part (ii) of Theorem 10.4, can be obtained from those of part (i). In showing this we use the following result, which gives a sufficient condition for *Poisson's summation formula* to hold; cf. (A.2.15) and Corollary A.2.6.

Lemma 10.7. *Let ϕ be a nonnegative integrable characteristic function that is nonincreasing on $(0,\infty)$, and let f be the corresponding continuous density. Then ϕ and f are related by*

$$\sum_{k\in\mathbb{Z}} \phi(u+2k\pi) = \sum_{k\in\mathbb{Z}} f(k)\, e^{iuk} \qquad [\,u \in \mathbb{R}\,]. \tag{10.10}$$

An important consequence of this identity is the fact that its left-hand side, when normalized, can be viewed as the characteristic function of a distribution $(p_k)_{k\in\mathbb{Z}}$ on $\mathbb{Z}$ which is proportional to $\big(f(k)\big)$. We combine this observation with Theorem 10.4.

Theorem 10.8. *Let ϕ be a positive integrable characteristic function that is log-convex on $(0,\infty)$, and let f be the corresponding continuous density. Then:*

(i) *The following function ψ is well defined and positive:*

$$\psi(u) := \sum_{k\in\mathbb{Z}} \phi(u+2k\pi) \Big/ \sum_{\ell\in\mathbb{Z}} \phi(2\ell\pi) \qquad [\,u \in \mathbb{R}\,];$$

it is a 2π-periodic characteristic function that is nonincreasing and log-convex on $(0,\pi)$.

(ii) *The following sequence $(p_k)_{k\in\mathbb{Z}}$ is well defined and nonnegative:*

$$p_k := f(k) \Big/ \sum_{\ell\in\mathbb{Z}} f(\ell) \qquad [\,k \in \mathbb{Z}\,];$$

it is an infinitely divisible distribution on $\mathbb{Z}$ with characteristic function ψ as given in (i).

Proof. As ϕ is bounded and log-convex on $(0,\infty)$, it is nonincreasing on $(0,\infty)$. Therefore, the conditions of Lemma 10.7 are satisfied and hence formula (10.10) holds. In view of this and of Theorem 10.4 (ii) both parts of the theorem are proved as soon as we have showed that ψ is nonincreasing and log-convex on $(0,\pi)$.

To do so we first note that because of the convexity of ϕ:

$$\phi(s) - \phi(s+h) \geq \phi(t) - \phi(t+h) \qquad [0 < s < t,\, h > 0],$$

and write

$$\chi(u) := \sum_{k \in \mathbb{Z}} \phi(u + 2k\pi) = \sum_{n=0}^{\infty} \big\{\phi(2n\pi + u) + \phi(2(n+1)\pi - u)\big\}.$$

Now take $0 < u < v < \pi$; since $2n\pi + u < 2(n+1)\pi - v$, it follows that the difference $\chi(u) - \chi(v)$ is nonnegative; it can be written as

$$\sum_{n=0}^{\infty} \Big[\big\{\phi(2n\pi + u) - \phi(2n\pi + v)\big\} + \\ - \big\{\phi(2(n+1)\pi - v) - \phi(2(n+1)\pi - u)\big\}\Big].$$

We conclude that χ, and hence ψ, is nonincreasing on $(0, \pi)$.
Next, note that ϕ, being even, is not only log-convex on $(0, \infty)$, but also on $(-\infty, 0)$. As the interval $(0, \pi) + 2k\pi$ with $k \in \mathbb{Z}$ is completely contained in $(0, \infty)$ or in $(-\infty, 0)$, it follows that the function $u \mapsto \phi(u + 2k\pi)$ with $k \in \mathbb{Z}$ is log-convex on $(0, \pi)$. Hence ψ, as the sum of these functions (up to a constant), is also log-convex on $(0, \pi)$; cf. Proposition A.3.10. ◻◻

In Section 11 we will give an example that illustrates this theorem. Note that, because of their convexity and evenness, the characteristic functions ϕ and ψ, if not identically one, cannot have a derivative at zero and hence cannot have a finite first moment.

Finally, we briefly look at log-concavity; a positive function f on $\mathbb{R}$ is *log-concave* iff $\log f$ is concave. Log-concavity of a *probability density* does not imply infinite divisibility; see Section 11 for an example. Nevertheless, several well-known infinitely divisible densities on $\mathbb{R}$ are log-concave, for instance the *normal* densities and the *Laplace* densities. It would be interesting to have simple sufficient conditions in terms of the canonical triple (a, σ^2, M) for an infinitely divisible density f to be log-concave; cf. Theorem III.10.6.

The *standard normal* distribution not only has a log-concave density, but also a log-concave *characteristic function*. The *standard Cauchy* distribution has a log-concave characteristic function as well. In general, however, log-concavity of a characteristic function does not imply infinite divisibility; see Section 11 for an example.

11. Examples

In Chapter III we have seen several examples of infinitely divisible distributions on $\mathbb{R}_+$; they can sometimes be used to illustrate the results of the present chapter. But here, of course, we are more interested in having illustrative examples of infinitely divisible distributions that are not concentrated on a half-line. The only concrete examples of this type we have seen so far, are the *sym-gamma* distributions (including *Laplace*) with characteristic functions ϕ of the form

$$(11.1)\qquad \phi(u) = \Big(\frac{\lambda^2}{\lambda^2 + u^2}\Big)^r \qquad [\,\lambda > 0,\, r > 0\,]$$

and the *symmetric stable* distributions (including *normal* and *Cauchy*) with characteristic functions ϕ of the form

$$(11.2)\qquad \phi(u) = \exp\big[-\lambda\,|u|^\gamma\big] \qquad [\,\lambda > 0,\, 0 < \gamma \le 2\,];$$

the sym-gamma distribution with shape parameter $r \le 1$ is compound-exponential. We now give several other examples of distributions that on account of the results of this chapter are or are *not* infinitely divisible. In doing so we roughly follow the order of the previous sections.

We start with considering two related and interesting distributions, the *Gumbel* distribution and the *logistic* distribution. Their infinite divisibility is not easily seen without knowing that they can be obtained in a very special way.

Example 11.1. Let Y have a standard exponential distribution. Consider X such that

$$X \stackrel{d}{=} -\log Y.$$

Then X has an absolutely continuous distribution with distribution function F and density f given by

$$F(x) = \exp\big[-e^{-x}\big], \quad f(x) = \exp\big[-(x + e^{-x})\big] \qquad [\,x \in \mathbb{R}\,];$$

note that f is *log-concave*. This distribution is called the *Gumbel* distribution; it is an *extreme-value* distribution because, as one easily verifies by using Helly's theorem,

$$\max\{Y_1, \ldots, Y_n\} - \log n \xrightarrow{d} X \qquad [\,n \to \infty\,],$$

where $Y_1, Y_2, \ldots$ are independent with $Y_i \stackrel{\mathrm{d}}{=} Y$ for all i. On the other hand, it is well known that

$$\max\{Y_1, \ldots, Y_n\} \stackrel{\mathrm{d}}{=} Y_1 + \tfrac{1}{2} Y_2 + \cdots + \tfrac{1}{n} Y_n \qquad [\, n \in \mathbb{N} \,].$$

Since the exponential distribution is infinitely divisible, it follows from Propositions 2.1 and 2.3 that X is *infinitely divisible*, too. In Example 11.10 we will determine the Lévy and Kolmogorov canonical triples of X. □

Example 11.2. Let Y_1 and Y_2 be independent, both having the Gumbel distribution of Example 11.1. Consider X such that

$$X \stackrel{\mathrm{d}}{=} Y_1 - Y_2.$$

Then X has an absolutely continuous distribution, and both its distribution function F and a density f can be obtained by convolution:

$$F(x) = \frac{1}{1 + \mathrm{e}^{-x}}, \quad f(x) = \tfrac{1}{4} \frac{1}{\cosh^2 \frac{1}{2} x} = \tfrac{1}{2} \frac{1}{\cosh x + 1} \quad [\, x \in \mathbb{R} \,];$$

note that f is *log-concave*. This distribution is called the *logistic* distribution. It is *infinitely divisible*; this immediately follows from Proposition 2.1 and the infinite divisibility of the Gumbel distribution. In Example 11.11 the Lévy canonical triple of X will be given. □

Next we consider a few positive functions ϕ on $\mathbb{R}$; we show them to be characteristic functions and determine whether they are infinitely divisible.

Example 11.3. For $\gamma \in (0, 2\,]$ consider the function ϕ on $\mathbb{R}$ defined by

$$\phi(u) = \frac{1}{1 + |u|^\gamma} \qquad [\, u \in \mathbb{R} \,].$$

Since $\phi(u) = \pi\big(|u|^\gamma\big)$ with π the pLSt of the standard exponential distribution, from Proposition 6.1 it follows that ϕ is an *infinitely divisible characteristic function.* Use of Example 4.9 shows that ϕ is even *compound-exponential.* We further take $\gamma = 1$. Then the underlying distribution function F_0 in (5.1) is Cauchy, so $F_0^{\star t}(x) = F_0(x/t)$ for $t > 0$, and hence the distribution corresponding to ϕ is absolutely continuous with density f given by

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{t}{t^2 + x^2}\, \mathrm{e}^{-t}\, \mathrm{d}t \qquad [\, x \in \mathbb{R} \,].$$

From Theorem 5.2 it follows that the canonical triple is $(0, 0, M)$ where M has density m very similar to f:

$$m(x) = \frac{1}{\pi} \int_0^\infty \frac{1}{t^2 + x^2} \, \mathrm{e}^{-t} \, \mathrm{d}t \qquad [\, x \neq 0 \,].$$

See also Theorem 10.5; ϕ is of Pólya type. □

Example 11.4. Consider the function ϕ on $\mathbb{R}$ defined by

$$\phi(u) = \frac{1 + |u|}{1 + 2|u|} \qquad [\, u \in \mathbb{R} \,].$$

Then ϕ is an *infinitely divisible characteristic function.* This can be shown in several ways. One can observe that ϕ can be written as

$$\phi(u) = \frac{1}{2 - \phi_1(u)}, \quad \text{with } \phi_1(u) := \frac{1}{1 + |u|};$$

since ϕ_1 is a characteristic function (cf. Example 11.3), it follows that ϕ is of the *compound-geometric* form (3.6). Alternatively, one can use Theorem 10.4; ϕ is an even continuous function that is nonincreasing and log-convex on $(0, \infty)$. Or apply Theorem 10.5:

$$\phi(u) = \tfrac{1}{2} + \tfrac{1}{2} \, \frac{1}{1 + 2|u|} = \pi\big(|u|\big), \quad \text{with } \pi(s) := \tfrac{1}{2} + \tfrac{1}{2} \, \frac{1}{1 + 2s},$$

and note that π is a pLSt; it corresponds to a mixture of the degenerate distribution at zero and an exponential distribution. This example also shows that an infinitely divisible distribution with infinite second moment can have a characteristic function ϕ satisfying $|\phi| > \delta$ for some $\delta > 0$; cf. Proposition 2.11. □

Example 11.5. Consider the function ϕ on $\mathbb{R}$ defined by

$$\phi(u) = \tfrac{1}{2} + \tfrac{1}{2} \, \mathrm{e}^{-u^2} \qquad [\, u \in \mathbb{R} \,].$$

Then ϕ is a *characteristic function*; the corresponding distribution is a mixture of the degenerate distribution at zero and a normal distribution. Thus $\phi(u) = \widetilde{F}(u)$ for some distribution function F on $\mathbb{R}$. From Lemma 9.4 it is seen that $\phi(z) := \widetilde{F}(z)$ is well defined for all $z \in \mathbb{C}$ with

$$\phi(z) = \tfrac{1}{2} + \tfrac{1}{2} \, \mathrm{e}^{-z^2} \qquad [\, z \in \mathbb{C} \,].$$

Since $\phi(z) = 0$ for $z \in \mathbb{C}$ such that $z^2 = -\pi i$, Theorem 2.12 (iii) implies that ϕ is *not* infinitely divisible. This also immediately follows from Corollary 9.9. This example shows that infinite divisibility may be lost by shifting mass to zero; cf. Example 11.4. It also illustrates (6.1); the class of infinitely divisible distributions is *not* closed under mixing. ◫

Example 11.6. Let T be a $(0, \infty)$-valued random variable with distribution function G and pLSt π. Consider the function ϕ on $\mathbb{R}$ defined by

$$\phi(u) = \pi(\tfrac{1}{4}u^2) = \int_{(0,\infty)} e^{-\frac{1}{4}tu^2} \, dG(t) \qquad [u \in \mathbb{R}].$$

Then ϕ is a *characteristic function*; the corresponding distribution is a variance mixture of normal distributions and has density f given by

$$f(x) = \frac{1}{\sqrt{\pi}} \int_{(0,\infty)} e^{-x^2/t} \frac{1}{\sqrt{t}} \, dG(t) \qquad [x \in \mathbb{R}].$$

In general, ϕ will *not* be infinitely divisible; see Example 11.5, and compare with Theorem 10.5. From Proposition 3.6 or directly from Proposition 6.1, however, it follows that if π is infinitely divisible, then so is ϕ. Now, put $V := 1/T$, and let H be the distribution function of V. Then f can be rewritten as

$$f(x) = \frac{1}{\sqrt{\pi}} \int_{(0,\infty)} e^{-vx^2} \sqrt{v} \, dH(v) \qquad [x \in \mathbb{R}],$$

which by taking V standard *gamma* (r) distributed transforms into the density of the *student* (r) distribution:

$$f(x) = \frac{1}{B(r, \frac{1}{2})} \Big(\frac{1}{1+x^2}\Big)^{r+\frac{1}{2}} \qquad [x \in \mathbb{R}].$$

Thus, if we would know that $1/V$ is infinitely divisible for gamma distributed V, then we would have proved the infinite divisibility of the student distribution. We return to this in Chapter VI. ◫

Example 11.7. Consider the function ϕ on $\mathbb{R}$ defined by

$$\phi(u) = 1 + u^2 - |u| \sqrt{2 + u^2} \qquad [u \in \mathbb{R}].$$

As $\phi(u) = \pi(u^2)$ with π the infinitely divisible pLSt from Example III.11.9, it follows from Proposition 6.1 that ϕ is an *infinitely divisible characteristic function.* ◫

Example 11.8. Consider the function ϕ on $\mathbb{R}$ defined by

$$\phi(u) = \frac{1}{\cosh u} = \frac{2}{e^u + e^{-u}} \qquad [\, u \in \mathbb{R} \,].$$

One is tempted to use Theorem 10.5 and write $\phi(u) = \pi_1\big(|u|\big)$ with $\pi_1(s) := 1/\cosh s$, but π_1 is *not* a pLSt, as is shown in Example III.11.12. There it is also noted that π with $\pi(s) := 1/\cosh\sqrt{s}$ is indeed an infinitely divisible pLSt; since $\phi(u) = \pi(u^2)$, it follows from Proposition 6.1 that ϕ is an *infinitely divisible characteristic function.* For an alternative proof we refer to Example V.9.18, where also a corresponding density is given. ◻◻

Example 11.9. Consider the function ϕ on $\mathbb{R}$ defined by

$$\phi(u) = \frac{1 - e^{-|u|}}{|u|} \qquad \big[\, u \neq 0;\ \phi(0) := 1 \,\big].$$

Since $\phi(u) = \pi\big(|u|\big)$ with π the pLSt of the uniform distribution on $(0,1)$, it follows from Theorem 10.5 that ϕ is an *infinitely divisible characteristic function.* By (10.8) the corresponding distribution has a density f that is a *mixture of Cauchy densities*:

$$f(x) = \int_0^1 \frac{1}{\pi} \frac{\lambda}{\lambda^2 + x^2}\, d\lambda = \frac{1}{2\pi} \log\Big(1 + \frac{1}{x^2}\Big) \qquad [\, x \neq 0 \,].$$

The infinite divisibility of f, and hence of ϕ, can now also be concluded from Theorem 10.1. In fact, for $x > 0$ we can write

$$-\pi\, f'(x) = \frac{1}{x} - \frac{x}{x^2+1} = \int_0^\infty e^{-\lambda x} \{1 - \cos \lambda\}\, d\lambda,$$

so $-f'$ is *completely monotone* on $(0,\infty)$, and hence so is f because $f \geq 0$. It follows that f can also be represented as a *mixture of Laplace densities* in the following way:

$$f(x) = \int_0^\infty \tfrac{1}{2}\lambda\, e^{-\lambda|x|}\, h(\lambda)\, d\lambda \qquad [\, x \neq 0 \,],$$

where the density h on $(0,\infty)$ is given by $h(\lambda) = (2/\pi)\,(1 - \cos\lambda)/\lambda^2$. ◻◻

We proceed with three examples on canonical representations, and first return to the *Gumbel* and *logistic* distributions.

Example 11.10. Let X have the *Gumbel* distribution with density f given by

$$f(x) = \exp\left[-(x + \mathrm{e}^{-x})\right] \qquad [\, x \in \mathbb{R}\,];$$

in Example 11.1 we have seen that X is infinitely divisible. For the characteristic function ϕ of X we can write

$$\phi(u) = \int_{\mathbb{R}} \mathrm{e}^{iux} \exp\left[-\mathrm{e}^{-x}\right] \mathrm{e}^{-x}\, \mathrm{d}x = \int_0^\infty y^{-iu}\, \mathrm{e}^{-y}\, \mathrm{d}y = \Gamma(1 - iu).$$

Now, using well-known expressions for the gamma function and for Euler's constant γ (see Section A.5), we can rewrite $\log\, \phi(u)$ as

$$\begin{aligned} \log\, \phi(u) &= -\int_0^\infty \left(iu + \frac{1 - \mathrm{e}^{iux}}{1 - \mathrm{e}^{-x}} \right) \frac{1}{x}\, \mathrm{e}^{-x}\, \mathrm{d}x = \\ &= iu\gamma + \int_0^\infty (\mathrm{e}^{iux} - 1 - iux)\, \frac{\mathrm{e}^{-x}}{x\,(1 - \mathrm{e}^{-x})}\, \mathrm{d}x. \end{aligned}$$

Thus we have found for ϕ the Kolmogorov representation (7.15); in the canonical triple (μ, κ, H) the mean μ is given by $\mu = \gamma$, the variance κ by

$$\kappa = \int_0^\infty \frac{x\, \mathrm{e}^{-x}}{1 - \mathrm{e}^{-x}}\, \mathrm{d}x = \int_0^1 \frac{-\log\, y}{1 - y}\, \mathrm{d}y = \frac{\pi^2}{6},$$

and the distribution function H is absolutely continuous with density h on $(0, \infty)$ given by

$$h(x) = \frac{6}{\pi^2} \frac{x\, \mathrm{e}^{-x}}{1 - \mathrm{e}^{-x}} \qquad [\, x > 0\,].$$

From (7.14) it follows that the Lévy triple is $(a, 0, M)$ with a determined by (7.7) with $\kappa_1 = \mu = \gamma$ and M absolutely continuous with density m on $(0, \infty)$ given by

$$m(x) = \frac{\mathrm{e}^{-x}}{x\,(1 - \mathrm{e}^{-x})} \qquad [\, x > 0\,].$$

Note that $M(0-) = 0$ and $\int_{(0,1]} x\, \mathrm{d}M(x) = \infty$, so by Theorem 9.7 (ii) the left tail of X is non-trivial, but thin. Actually, it is extremely thin; it is given by $-\log\, \mathbb{P}(X < -x) = \mathrm{e}^x$ for all x. The right tail of X is easily seen to satisfy $-\log\, \mathbb{P}(X > x) \sim x$ as $x \to \infty$; cf. Theorem 9.7 (iii). ◻◻

Example 11.11. Let X have the *logistic* distribution with density f given by

$$f(x) = \tfrac{1}{4}\,\frac{1}{\cosh^2 \frac{1}{2}x} = \tfrac{1}{2}\,\frac{1}{\cosh x + 1} \qquad [x \in \mathbb{R}];$$

in Example 11.2 we have seen that X is infinitely divisible. Since X can be obtained as $X \stackrel{d}{=} Y_1 - Y_2$ with Y_1 and Y_2 independent and both having a Gumbel distribution, we can use the results of Example 11.10. Thus for the characteristic function ϕ of X we find

$$\phi(u) = \Gamma(1 - iu)\,\Gamma(1 + iu) = \frac{\pi u}{\sinh \pi u};$$

cf. Section A.5. Moreover, because of Proposition 4.5 the Lévy triple of X is $(0, 0, M)$ with M absolutely continuous with density m given by

$$m(x) = \frac{\mathrm{e}^{-|x|}}{|x|\,(1 - \mathrm{e}^{-|x|})} \qquad [x \neq 0].$$

Hence from Theorem 4.11 it follows that ϕ can be represented as

$$\phi(u) = \exp\Big[2\int_0^\infty (\cos\, ux - 1)\,\frac{\mathrm{e}^{-x}}{x\,(1 - \mathrm{e}^{-x})}\,\mathrm{d}x\Big].$$

Finally, we note that the two-sided tail of X satisfies $-\log \mathbb{P}\big(|X| > x\big) \sim x$ as $x \to \infty$; cf. Theorem 9.8. □

Example 11.12. Let X be a random variable with a symmetric, *infinitely divisible* distribution and with $\mathbb{V}\mathrm{ar}\, X = 1$, and let Z be standard-normal. Then we have the following *moment inequality*:

$$\mathbb{E}\,|X|^r \le \mathbb{E}\,|Z|^r \qquad [0 < r \le 2].$$

To prove this we use the Kolmogorov representation for the characteristic function ϕ_X of X; in our case it reduces to

$$\phi_X(u) = \exp\Big[\int_{\mathbb{R}} (\cos ux - 1)\,\frac{1}{x^2}\,\mathrm{d}H(x)\Big],$$

where H is a (symmetric) distribution function. Since $\cos t \ge 1 - \frac{1}{2}t^2$ for all $t \in \mathbb{R}$, it follows that

$$\phi_X(u) \ge \phi_Z(u) \text{ for all } u \in \mathbb{R}.$$

Now, take $r \in (0,2]$, and for $\lambda > 0$ let S_λ be a random variable that is symmetric stable (λ) with exponent r, so $\phi_{S_\lambda}(u) = \exp\left[-\lambda\,|u|^r\right]$; see Example 4.9. Then by Parseval's identity for characteristic functions

$$\mathbb{E}\,\phi_{S_\lambda}(X) = \mathbb{E}\,\phi_X(S_\lambda) \geq \mathbb{E}\,\phi_Z(S_\lambda) = \mathbb{E}\,\phi_{S_\lambda}(Z).$$

Since by dominated convergence $\mathbb{E}\,|X|^r = \lim_{\lambda\downarrow 0}\{1 - \mathbb{E}\,\phi_{S_\lambda}(X)\}/\lambda$, and similarly for $\mathbb{E}\,|Z|^r$, the desired inequality follows. ◫

We next present some further examples and counter-examples in the context of *log-convexity* and *log-concavity* for densities and characteristic functions; see Section 10.

Example 11.13. Let X have an absolutely continuous distribution with density f given by

$$f(x) = c\,\mathrm{e}^{-|x|}\,\mathrm{e}^{-x^2} \qquad [\,x \in \mathbb{R}\,],$$

where $c > 0$ is a norming constant. Clearly, f is *log-concave.* Nevertheless, X is *not* infinitely divisible. This can be seen by noting that X cannot be normal, and considering the two-sided tail of X; for $x \geq 0$ we have

$$\mathbb{P}\big(|X| > x\big) \leq 2c \int_x^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y \leq \frac{2c}{x}\int_x^\infty y\,\mathrm{e}^{-y^2}\,\mathrm{d}y = \frac{c}{x}\,\mathrm{e}^{-x^2},$$

so X has too thin a tail to satisfy the limiting relation of Theorem 9.8. In fact, combining the estimation above with (9.9) one easily shows that $-\log \mathbb{P}\big(|X| > x\big) \sim x^2$ as $x \to \infty$, so now one can use Corollary 9.9 to conclude that X is not infinitely divisible. This example also shows that densities f of the form

$$f(x) = c\,\mathrm{e}^{-|x|}\,f_0(x) \qquad [\,x \in \mathbb{R}\,],$$

with f_0 an infinitely divisible density (not concentrated on a half-line), need *not* be infinitely divisible; cf. (6.3). ◫

Example 11.14. Consider the following mixture of normal characteristic functions:

$$\phi(u) = \tfrac{1}{2}\,\mathrm{e}^{-u^2} + \tfrac{1}{2}\,\mathrm{e}^{-2u^2}.$$

Then, differentiating $\log \phi$ twice, one easily shows that ϕ is *log-concave.* Nevertheless, ϕ is *not* infinitely divisible. This can be seen as in Example 11.13 by using Corollary 9.9; ϕ cannot be normal, and the corresponding density f is given by

$$f(x) = \frac{1}{4\sqrt{\pi}}\, e^{-\frac{1}{4}x^2} + \frac{1}{4\sqrt{2\pi}}\, e^{-\frac{1}{8}x^2} \qquad [\, x \in \mathbb{R}\,],$$

so the two-sided tail of X satisfies $-\log \mathbb{P}\big(|X| > x\big) \sim \frac{1}{8}x^2$ as $x \to \infty$. □

Example 11.15. For $r > 0$ consider the *double-gamma* density f_r given by

$$f_r(x) = \frac{1}{2\Gamma(r)}\, |x|^{r-1}\, e^{-|x|} \qquad [\, x \in \mathbb{R};\ x \neq 0 \text{ when } r < 1\,];$$

it can be written as in (10.1): $f_r(x) = \frac{1}{2}\, g_r\big(|x|\big)$ for all x, where g_r is the standard gamma$\,(r)$ density on $\mathbb{R}_+$. It is well known that g_r is infinitely divisible for every r. Now, for $r \leq 1$ also f_r is *infinitely divisible*; this follows from Theorem 10.1 because then f_r (or g_r) is completely monotone on $(0, \infty)$. Thus, taking e.g. $r = \frac{1}{2}$, we see that the following density is infinitely divisible:

$$f_{\frac{1}{2}}(x) = \frac{1}{2\sqrt{\pi}}\, \frac{1}{\sqrt{|x|}}\, e^{-|x|} \qquad [\, x \neq 0\,].$$

For $r > 1$, however, f_r is continuous at zero with $f_r(0) = 0$, so by Proposition 8.10 f_r is *not* infinitely divisible. The last result can also be obtained by using Theorem 2.12 (i); because of (10.2) the characteristic function ϕ_r of f_r is given by

$$\phi_r(u) = \operatorname{Re}\Big(\frac{1}{1 - iu}\Big)^r = \frac{\cos\,(r \arctan u)}{(1 + u^2)^{r/2}},$$

which has real zeroes iff $r > 1$. Taking for instance $r = 2$, we get simpler expressions:

$$f_2(x) = \tfrac{1}{2}\, |x|\, e^{-|x|}, \quad \phi_2(u) = \frac{1 - u^2}{(1 + u^2)^2}.$$

Returning to the case where $r \leq 1$, we note that, more generally, for every $\alpha \in (0, 1\,]$ and $r \in (0, 1/\alpha\,]$ the density f given by

$$f(x) = \frac{\alpha}{2\Gamma(r)}\, |x|^{\alpha r - 1} \exp\big[-|x|^\alpha\big],$$

is *infinitely divisible*; see Example III.11.3. □

Example 11.16. For $r > 1$ consider the density f given by

$$f(x) = \tfrac{1}{2}(r-1)\,\frac{1}{\big(1+|x|\big)^r} \qquad [x \in \mathbb{R}].$$

Since f is completely monotone on $(0,\infty)$ (cf. Example III.11.5), it follows from Theorem 10.1 that f is *infinitely divisible.* This distribution is sometimes called the *double-Pareto* distribution. ◻◻

Example 11.17. Consider the probability distribution $(p_k)_{k\in\mathbb{Z}}$ on $\mathbb{Z}$ given by

$$p_k = c\,\frac{1}{\big(1+|k|\big)^2} \qquad \big[\,k\in\mathbb{Z};\; c := 1/(\tfrac{1}{3}\pi^2-1)\,\big].$$

Since $(p_k)_{k\in\mathbb{Z}_+}$ is completely monotone (cf. Example II.11.4), it follows from Theorem 10.2 that (p_k) is *infinitely divisible.* ◻◻

Example 11.18. For $\lambda > 0$ consider the sequence $(p_k)_{k\in\mathbb{Z}}$ defined by

$$p_k = c_\lambda\,\frac{1}{\lambda^2+k^2} \qquad \big[\,k\in\mathbb{Z};\; c_\lambda := \lambda/(\pi\coth\lambda\pi)\,\big].$$

Then (p_k) is a *probability* distribution on $\mathbb{Z}$ which is *infinitely divisible.* In order to show this we observe that (p_k) is proportional to $\big(f(k)\big)$, where f is the Cauchy (λ) density; recall that f and its characteristic function ϕ are given by

$$f(x) = \frac{1}{\pi}\,\frac{\lambda}{\lambda^2+x^2}, \qquad \phi(u) = \mathrm{e}^{-\lambda|u|}.$$

Note that ϕ is of the type considered in Theorem 10.4 (i); ϕ is *log-convex* on $(0,\infty)$. Since ϕ is also nonincreasing on $(0,\infty)$, we have Poisson's summation formula as given in Lemma 10.7, which implies that

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \phi(2k\pi) = \frac{\mathrm{e}^{\lambda\pi}+\mathrm{e}^{-\lambda\pi}}{\mathrm{e}^{\lambda\pi}-\mathrm{e}^{-\lambda\pi}} = \coth\lambda\pi.$$

It now follows that the sequence (p_k) above can be viewed as a probability distribution on $\mathbb{Z}$. Moreover, we can apply Theorem 10.8 to conclude that (p_k) is infinitely divisible with a characteristic function that is 2π-periodic, and nonincreasing and log-convex on $(0,\pi)$, so of the type as considered in Theorem 10.4 (ii). ◻◻

The final example shows that, as opposed to the discrete component (cf. Proposition I.2.2), the continuous-singular and absolutely continuous components of an infinitely divisible distribution need not be infinitely divisible.

Example 11.19. Let F be a distribution function that is compound-Poisson, so there exist $\lambda > 0$ and a distribution function G such that for $x \in \mathbb{R}$

$$F(x) = \mathrm{e}^{-\lambda}\, 1_{[0,\infty)}(x) + \lambda\, \mathrm{e}^{-\lambda}\, G(x) + \sum_{n=2}^{\infty} \Bigl(\frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}\Bigr)\, G^{\star n}(x).$$

First, take for G a continuous-singular distribution function on $(0,1)$ such that $G^{\star 2}$ is absolutely continuous; this can be done (see Notes). Then the continuous-singular component of F is given by G, which is *not* infinitely divisible because it has its support in $(0,1)$; cf. Proposition I.2.3.
Next, take for G an absolutely continuous distribution function such that $\widetilde{G}(u_0) = 0$ for some $u_0 \in \mathbb{R}$; the uniform distribution on $(0,1)$, for instance, has this property with $u_0 = 2\pi$. Then, as one easily verifies, the absolutely continuous component of F has characteristic function ϕ_{ac} given by

$$\phi_{\mathrm{ac}}(u) = \frac{1}{\mathrm{e}^{\lambda} - 1}\, \bigl\{\exp\bigl[\lambda\, \widetilde{G}(u)\bigr] - 1\bigr\},$$

which is *not* infinitely divisible as $\phi_{\mathrm{ac}}(u_0) = 0$; cf. Proposition 2.4. ◻

12. Notes

The basic properties of infinitely divisible distributions and characteristic functions can be found in many textbooks. We name Tucker (1967), where the definition of the logarithm of a nonvanishing characteristic function is very carefully treated; Lukacs (1970); Feller (1971), with emphasis on distributions on the half-line; Loève (1977, 1978). More recently, brief treatments of infinite divisibility are given in two books on Lévy processes: Bertoin (1996) and Sato (1999).

Theorem 2.12, on zeroes of infinitely divisible characteristic functions, can be found in Lukacs (1970). Infinitely divisible compound distributions are treated in the context of *subordination* by Feller (1971); not all of the results in Section 3 can be found there, though.

The canonical representations of Sections 4 and 7 have been known for a long time. The first results can be found in de Finetti (1929); Kolmogorov

(1932), where Theorem 7.7 is given; Lévy (1934), who gave the most general result; Khintchine (1937b), who gave a different proof of Lévy's formula. Derivations of these formulas are given in several standard books on probability theory; apart from the references mentioned above, e.g., in Breiman (1968), and Laha and Rohatgi (1979). A very classical reference is Gnedenko and Kolmogorov (1968), where also Propositions 4.2 and 4.3 are proved; an interesting treatment is presented in Stroock (1993). A derivation based on Choquet theory (extreme points in a convex set) is given by Johansen (1966). A representation for $\mathbb{R}$-valued infinitely divisible random variables similar to that in Theorem III.3.9 in the $\mathbb{R}_+$-case occurs in Csőrgö et al. (1988); see also Csőrgö (1989). The formula for ℓ_F in Theorem 4.13 was first given by Tucker (1961); estimates can be found in Baxter and Shapiro (1960), and in Esseen (1965). Theorem 4.20 is due to Blum and Rosenblatt (1959); the proof we give, and especially the inequality for random walks used in it, is due to Huff (1974); a proof using sample functions of Lévy processes is given by Millar (1995). For early results see Hartman and Wintner (1942); compare, however, Orey (1968). Related work has been done by Tucker (1962, 1964); the first reference contains a proof of Theorem 4.23, which is also proved by Fisz and Varadarajan (1963). There seems to be no simple necessary and sufficient condition on the Lévy function for an infinitely divisible distribution to be absolutely continuous; cf. Tucker (1965). Not all of the detailed results in Section 4 are easily available in the literature.

Compound-exponential distributions are considered by Steutel (1970) in the context of mixtures, by van Harn (1978) in connection with p-functions, and by Klebanov et al. (1984) as a class of 'geometrically infinitely divisible' distributions.

Relations between the moments of infinitely divisible distributions and the corresponding canonical functions were first derived by Shapiro (1956), and in more detail by Ramachandran (1969) and by Wolfe (1971b). Since the relations between moments of the Lévy function and cumulants are not as simple as in the half-line case, it does not seem possible to define 'fractional cumulants' here. The cumulant inequality of Proposition 7.8 is proved in Gupta et al. (1994), who use it for testing.

Supports of infinitely divisible distributions have been considered by Rubin (1967), Tucker (1975) and Brown (1976), and more recently by Tortrat (1988); part of the results of Section 8 can be found in these references, under varying conditions. Sharpe (1969a) has given the first result on zeroes of densities on $\mathbb{R}$, but under rather stronger conditions than our Theorem 8.7. Sharpe (1995) contains very nice general results, but these are not suited for deciding on infinite divisibility of individual densities. Hudson and Tucker (1975b) prove that an infinitely divisible density is positive a.e. on its support, which may sound trivial, but is not; for more precise results we refer to Byczkowski et al. (1996). Formula (8.11), used in the proof of Proposition 8.10, is taken from Kawata (1972).

Tails of infinitely divisible distributions have been studied by many authors, in varying degrees of precision. Among the many authors are Zolotarev (1965), Kruglov (1970), Ruegg (1970, 1971), Horn (1972) and Steutel (1974). The results in Theorems 9.7 and 9.8 are basically due to Sato (1973). Riedel (1975) characterizes the normal distribution in terms of one-sided tails; see also Rossberg et al. (1985), and Sato and Steutel (1998). Csőrgö and Mason (1991) prove slightly more detailed results using a stochastic representation of infinitely divisible random variables.

Densities that are completely monotone on both sides of zero, are considered in Steutel (1970). The infinite divisibility of real characteristic functions that are log-convex on both sides of zero, was noted by Horn (1970). This class of functions, which is closed under mixing, multiplication and pointwise limits, is studied in Keilson and Steutel (1972); they also give Theorem 10.8. A different proof of Theorem 10.5 occurs in Kelker (1971).

The infinite divisibility of the student distribution mentioned in Example 11.6 has been examined by many authors. We name Kelker (1971); Grosswald (1976), who gave the first complete proof by using properties of Bessel functions; Ismail (1977), who also uses Bessel functions; Epstein (1977), who gives a long 'elementary' proof. See also the Notes of Chapter VI, where we return to the infinite divisibility of $1/V$ with V gamma (r). The work on Bessel functions and infinite divisibility was continued by Barndorff-Nielsen and Halgreen (1977), Kent (1978), Ismail and Kelker (1979), Ismail and May (1979), and Ismail and Miller (1982). The moment inequality in Example 11.12 is due to Klaassen (1981). Examples by Rubin (1967) of continuous-singular distribution functions F on $(0, 1)$ such that $F^{\star 2}$ is absolutely continuous, gave rise to Example 11.19.

Chapter V

SELF-DECOMPOSABILITY AND STABILITY

1. Introduction

Self-decomposable and stable distributions, as introduced in Section I.5, derive their importance from being limit distributions in the central limit problem. In this chapter, however, self-decomposability and stability will not be studied in this context; we will show that the self-decomposable and stable distributions form interesting subclasses of the class of *infinitely divisible* distributions, having attractive properties such as unimodality. Again, results for distributions on $\mathbb{Z}_+$ and on $\mathbb{R}_+$ are much easier obtained than for distributions on $\mathbb{R}$. In this introductory section we collect some general observations, whereas in the subsequent sections the $\mathbb{Z}_+$-, $\mathbb{R}_+$- and $\mathbb{R}$-case are treated separately.

We first recall the definition of self-decomposability. A random variable X is said to be *self-decomposable* if for every $\alpha \in (0,1)$ it can be written (in distribution) as

(1.1) $$X \stackrel{\mathrm{d}}{=} \alpha X + X_\alpha,$$

where in the right-hand side the random variables X and X_α are independent. Clearly, self-decomposability is a property of the *distribution* of X; therefore, a distribution function F and a characteristic function ϕ are called *self-decomposable* if a corresponding random variable has this property, and (1.1) can be rewritten in terms of F and ϕ in an obvious way. Doing so one easily verifies that (1.1) with some fixed α implies the same relation with α replaced by α^n for any $n \in \mathbb{N}$. Hence for self-decomposability of X we need only require (1.1) for all α in some left neighbourhood of one, and it will be no surprise that several results on self-decomposability will be obtained by letting $\alpha \uparrow 1$. If a self-decomposable random variable X has a finite left extremity ℓ_X, then for every α the component X_α also has ℓ_{X_α} finite with $\ell_X = \alpha\,\ell_X + \ell_{X_\alpha}$, so

$$(1.2) \qquad \ell_{X_\alpha} = (1-\alpha)\,\ell_X.$$

It follows that if X is $\mathbb{R}_+$-valued, then its components X_α are $\mathbb{R}_+$-valued as well; therefore, in this case we will use probability Laplace-Stieltjes transforms (pLSt's) to rewrite (1.1) rather than characteristic functions. In the $\mathbb{R}_+$-case we will also often suppose that $\ell_X = 0$; this is not an essential restriction because, as is easily verified, for every $a \in \mathbb{R}$:

$$(1.3) \qquad X \text{ self-decomposable} \iff a + X \text{ self-decomposable}.$$

Any *degenerate* distribution is self-decomposable, but mostly we tacitly exclude the trivial case of a distribution degenerate at zero, so we then assume that $\mathbb{P}(X=0) < 1$. It will then be clear that no $\mathbb{Z}_+$-valued random variable X with $\ell_X = 0$ can satisfy (1.1) for all α; in fact, we will show that any non-degenerate self-decomposable distribution is *absolutely continuous*. Nevertheless, by replacing the ordinary product αX in (1.1) by the $\mathbb{Z}_+$-valued 'product' $\alpha \odot X$ as introduced in Section A.4, we can define a meaningful concept of *self-decomposability* for $\mathbb{Z}_+$-valued random variables X with properties, like unimodality, similar to those in the $\mathbb{R}_+$-case. Moreover, by generalizing the multiplication $\odot$ one is led to classes of *infinitely divisible* distributions, both on $\mathbb{Z}_+$ and on $\mathbb{R}_+$, that are self-decomposable with respect to *composition semigroups* of transforms such as occur in branching processes.

We next turn to the stable distributions. A random variable X is said to be *weakly stable* if for every $n \in \mathbb{N}$ it can be written (in distribution) as

$$(1.4) \qquad X \overset{\mathrm{d}}{=} c_n\,(X_1 + \cdots + X_n) + d_n,$$

where $c_n > 0$, $d_n \in \mathbb{R}$, and $X_1, \ldots, X_n$ are independent with $X_i \overset{\mathrm{d}}{=} X$ for all i. Note that $c_1 = 1$ and $d_1 = 0$. The random variable X is called *strictly stable* if it is weakly stable with $d_n = 0$ in (1.4) for all n. Again, a distribution function F and a characteristic function ϕ are called *weakly/strictly stable* if a corresponding random variable has this property, and (1.4) can be rewritten in terms of F and ϕ in an obvious way. For given $m, n \in \mathbb{N}$ one easily verifies that (1.4) holds as stated if it holds with n replaced by m

and by nm; hence for weak stability of X we need only require (1.4) for large n. Any *degenerate* distribution is strictly stable with $c_n = 1/n$ for all n. A weakly stable X that has a *symmetric* distribution, i.e., for which $X \stackrel{d}{=} -X$, is easily shown to be strictly stable. On the other hand, most weakly stable random variables can be made strictly stable by adding a suitable constant. To see this, we first note that, similar to (1.3), for every $a \in \mathbb{R}$:

$$(1.5) \qquad X \text{ weakly stable } \iff a + X \text{ weakly stable.}$$

In fact, if X satisfies (1.4), then so does $a + X$ with the same c_n and with d_n replaced by $d_n' := d_n - a\,(nc_n - 1)$. When X is weakly stable with $c_n = 1/n$ and $d_n \neq 0$ for some $n \geq 2$, it follows that $a + X$ will not be strictly stable for any a. In case $c_n \neq 1/n$ for all $n \geq 2$, however, one is tempted to take $a = d_n/(nc_n - 1)$; then $d_n' = 0$. Of course, we can only do so if $d_n/(nc_n - 1)$ is independent of $n \geq 2$. Now, this is indeed the case; look at $c_2 c_n\,(X_1 + \cdots + X_{2n})$, and split up the sum into two sums of n X's and into n sums of two X's, then it easily follows that

$$c_2 c_n\,(X_1 + \cdots + X_{2n}) \stackrel{d}{=} \begin{cases} X - (d_2 + 2c_2 d_n), \\ X - (d_n + nc_n d_2), \end{cases}$$

so $d_2 + 2c_2 d_n = d_n + nc_n d_2$, and hence $d_n/(nc_n - 1) = d_2/(2c_2 - 1)$. If a weakly stable random variable X has a finite left extremity ℓ_X, then c_n and d_n in (1.4) satisfy

$$(1.6) \qquad \ell_X = nc_n \ell_X + d_n, \quad \text{so} \quad d_n = (1 - nc_n)\,\ell_X.$$

Hence $X - \ell_X$, with left extremity zero, is strictly stable and, in case $c_n = 1/n$ for all n, even X itself is strictly stable, as is $a + X$ for any $a \in \mathbb{R}$. Thus in the $\mathbb{R}_+$-case we can and will restrict ourselves to strict stability. In view of the preceding discussion we make the following convention:

From now on 'stability' means 'strict stability'.

In the $\mathbb{Z}_+$-case *stability* is defined by (1.4) with $d_n = 0$ and with ordinary multiplication replaced by the discrete multiplication $\odot$.

Our main goal is to exhibit both the self-decomposable and stable distributions as special infinitely divisible distributions by showing the special form of their *canonical representations* and to prove the interesting property of *unimodality*. We first do so in Sections 2 and 3 for *distributions on* $\mathbb{R}_+$, because this is a classical case in which explicit results can be obtained in a straightforward way, by use of properties of *completely monotone* functions. Then, in Sections 4 and 5 we use the discrete multiplication $\odot$ and treat, analogous to the $\mathbb{R}_+$-case, self-decomposable and stable *distributions on* $\mathbb{Z}_+$, using properties of *absolutely monotone* functions. In Sections 6 and 7 we treat the much more complicated (classical) case of *distributions on* $\mathbb{R}$; as we wish to proceed as in the $\mathbb{R}_+$-case, we have to accept a small restriction. Section 8 contains a generalization in terms of composition semigroups of transforms, both for distributions on $\mathbb{Z}_+$ and on $\mathbb{R}_+$. The final two sections, 9 and 10, contain examples and notes.

2. Self-decomposability on the nonnegative reals

Let X be an $\mathbb{R}_+$-valued random variable. Then, as noted in Section 1, we can use pLSt's to rewrite (1.1); with π the pLSt of X it follows that X is self-decomposable iff for all $\alpha \in (0,1)$ there exists a pLSt π_α such that

$$(2.1) \qquad \pi(s) = \pi(\alpha s)\,\pi_\alpha(s).$$

Mostly we will consider LSt's only for positive values of the argument. Since a pLSt is positive on $(0,\infty)$, relation (2.1) can be written as

$$(2.2) \qquad \pi_\alpha(s) = \frac{\pi(s)}{\pi(\alpha s)}.$$

It follows that the distributions of the components X_α of X in (1.1) are uniquely determined by X. The function π_α defined by (2.2) will be called the π_α*-function* of π, also when π is not yet known to be a pLSt (but has no zeroes). Using Bernstein's theorem (Theorem A.3.6) one immediately obtains the following *criterion*.

Proposition 2.1. *A pLSt* π *is self-decomposable iff for all* $\alpha \in (0,1)$ *the* π_α*-function of* π *is completely monotone.*

Recall that a real-valued function ρ on $(0,\infty)$ is said to be *completely monotone* if ρ possesses derivatives of all orders, alternating in sign:

$$(-1)^n \rho^{(n)}(s) \geq 0 \qquad [\,n \in \mathbb{Z}_+;\ s > 0\,].$$

In the sequel we wil use, without further comment, several properties of completely monotone functions as reviewed in Proposition A.3.7. Using Proposition 2.1 one easily shows that the class of self-decomposable distributions on $\mathbb{R}_+$ is *closed under scale transformation*, *under convolution* and *under weak convergence*. In the next two propositions we make these elementary properties explicit.

Proposition 2.2.

(i) *If X is a self-decomposable $\mathbb{R}_+$-valued random variable, then so is aX for every $a \in \mathbb{R}_+$. Equivalently, if π is a self-decomposable pLSt, then so is $\pi^{(a)}$ with $\pi^{(a)}(s) := \pi(as)$ for every $a \in \mathbb{R}_+$.*

(ii) *If X and Y are independent self-decomposable $\mathbb{R}_+$-valued random variables, then $X+Y$ is self-decomposable. Equivalently, if π_1 and π_2 are self-decomposable pLSt's, then their pointwise product $\pi_1\pi_2$ is a self-decomposable pLSt.*

Proposition 2.3. *If a sequence $(X^{(m)})$ of self-decomposable $\mathbb{R}_+$-valued random variables converges in distribution to X, then X is self-decomposable. Equivalently, if a sequence $(\pi^{(m)})$ of self-decomposable pLSt's converges (pointwise) to a pLSt π, then π is self-decomposable.*

The use of Proposition 2.1 is also illustrated by the following simple example.

Example 2.4. For $r > 0$, $\lambda > 0$, let X have the *gamma* (r, λ) distribution, so its density f and pLSt π are given by

$$f(x) = \frac{\lambda^r}{\Gamma(r)}\, x^{r-1}\, \mathrm{e}^{-\lambda x}, \quad \pi(s) = \Big(\frac{\lambda}{\lambda+s}\Big)^r.$$

Then for $\alpha \in (0,1)$ the π_α-function of π can be written as

$$\pi_\alpha(s) = \Big(\frac{\lambda+\alpha s}{\lambda+s}\Big)^r = \Big\{\alpha + (1-\alpha)\,\frac{\lambda}{\lambda+s}\Big\}^r.$$

Now, the function $s \mapsto \alpha + (1-\alpha)\,\lambda/(\lambda+s)$ is an infinitely divisible pLSt; this was shown in Example III.11.8. Hence π_α is completely monotone by Proposition III.2.3. We conclude that X is *self-decomposable*. ◻◻

We next want to derive a canonical representation for a self-decomposable pLSt π. To this end we first recall some basic facts from Section III.4.

Infinite divisibility of an $\mathbb{R}_+$-valued random variable X with pLSt π means that the functions π^t with $t > 0$ are all completely monotone. This condition is, however, equivalent to the complete monotonicity of only the *ρ-function* of π defined by

$$(2.3) \qquad \rho(s) := -\frac{\mathrm{d}}{\mathrm{d}s} \log \pi(s) = -\frac{\pi'(s)}{\pi(s)}, \quad \text{so } \pi(s) = \exp\Big[-\int_0^s \rho(u)\,\mathrm{d}u\Big].$$

Moreover, any function π of this form with ρ completely monotone on $(0, \infty)$ is the pLSt of an infinitely divisible distribution on $\mathbb{R}_+$. This observation, together with Bernstein's theorem, yields a simple derivation of the *canonical representation* of an infinitely divisible pLSt π:

$$(2.4) \qquad \pi(s) = \exp\Big[-\int_{\mathbb{R}_+} (1 - \mathrm{e}^{-sx}) \frac{1}{x}\,\mathrm{d}K(x)\Big],$$

where the *canonical function* K is an LSt-able function that satisfies

$$(2.5) \qquad \int_{(1,\infty)} \frac{1}{x}\,\mathrm{d}K(x) < \infty, \quad \widehat{K} = \rho, \quad K(0) = \ell_X, \quad \lim_{x\to\infty} K(x) = \mathbb{E}X.$$

Now it turns out that *self-decomposability* can be handled in a similar way. Before showing this we prove that the self-decomposable distributions are *infinitely divisible*; later on we will precisely indicate (in terms of the canonical function K) which infinitely divisible distributions are self-decomposable.

Theorem 2.5. *A self-decomposable distribution on $\mathbb{R}_+$ is infinitely divisible.*

Proof. Let π be a self-decomposable pLSt with factors π_α as in (2.1). In order to show that π is infinitely divisible, we express the ρ-function of π in terms of the π_α:

$$\rho(s) = \frac{1}{\pi(s)} \lim_{\alpha\uparrow 1} \frac{\pi(\alpha s) - \pi(s)}{(1-\alpha)s} = \lim_{\alpha\uparrow 1} \frac{\pi(\alpha s)}{\pi(s)} \, \frac{1 - \pi(s)/\pi(\alpha s)}{(1-\alpha)s},$$

so we get

$$(2.6) \qquad \rho(s) = \lim_{\alpha\uparrow 1} \frac{1}{1-\alpha} \, \frac{1 - \pi_\alpha(s)}{s}.$$

From (A.3.4) it now follows that ρ is the limit of completely monotone functions; hence ρ is completely monotone, and π is infinitely divisible. ◻

Let π be a pLSt. We look for a function, the complete monotonicity of which characterizes the self-decomposability of π. To this end we note that in Section III.4 the ρ-function of π is obtained from the functions π^t with $t > 0$ by

$$\rho(s) = \lim_{t \downarrow 0} -\frac{1}{t} \frac{\mathrm{d}}{\mathrm{d}s} \pi^t(s).$$

Since in the self-decomposability context the π_α-function of π seems to take over the role of π^t, one is tempted to look at $-\pi'_\alpha/(1-\alpha)$ for $\alpha \uparrow 1$:

$$\pi(s)^2 \lim_{\alpha \uparrow 1} -\frac{1}{1-\alpha}\, \pi'_\alpha(s) =$$

$$= \lim_{\alpha \uparrow 1} \frac{1}{1-\alpha} \frac{\pi(s)^2}{\pi(\alpha s)^2} \{\alpha\, \pi(s)\, \pi'(\alpha s) - \pi'(s)\, \pi(\alpha s)\} =$$

$$= \lim_{\alpha \uparrow 1} \Big\{ \pi'(s)\, \frac{\pi(s) - \pi(\alpha s)}{1-\alpha} - \pi(s)\, \frac{\pi'(s) - \pi'(\alpha s)}{1-\alpha} - \pi(s)\, \pi'(s) \Big\} =$$

$$= \pi'(s)\, s\, \pi'(s) - \pi(s)\, s\, \pi''(s) - \pi(s)\, \pi'(s) = \pi(s)^2 \big[s\, \rho(s) \big]'.$$

We now define the ρ_0*-function* of π by

$$\rho_0(s) := \frac{\mathrm{d}}{\mathrm{d}s} \big[s\, \rho(s) \big] \qquad [\, s > 0 \,], \tag{2.7}$$

with ρ as in (2.3), and allow π here to be any positive nonincreasing convex function on $(0, \infty)$ with $\pi(0+) = 1$ and a continuous second derivative. Since by (A.3.5) such a π satisfies $\lim_{s \downarrow 0} s\, \pi'(s) = 0$, so $\lim_{s \downarrow 0} s\, \rho(s) = 0$, we can express ρ in terms of ρ_0 by

$$\rho(s) = \frac{1}{s} \int_0^s \rho_0(u)\, \mathrm{d}u = \int_0^1 \rho_0(vs)\, \mathrm{d}v \qquad [\, s > 0 \,]; \tag{2.8}$$

here the integrals are supposed to exist, which is the case if ρ_0 is nonnegative. We are now ready to prove the following *criterion* for self-decomposability.

Theorem 2.6. *Let π be a positive nonincreasing convex function on $(0, \infty)$ with $\pi(0+) = 1$ and a continuous second derivative. Then π is the pLSt of a self-decomposable distribution on $\mathbb{R}_+$ iff its ρ_0-function is completely monotone.*

$-\pi'_\alpha$ is completely monotone. Since, as we saw above, the ρ_0-function of π can be obtained as

$$\rho_0(s) = \lim_{\alpha \uparrow 1} -\frac{1}{1-\alpha}\,\pi'_\alpha(s), \tag{2.9}$$

we conclude that ρ_0, as a limit of completely monotone functions, is completely monotone.
Conversely, let the ρ_0-function of π be completely monotone. Then ρ_0 is nonnegative, so we have (2.8). It follows that the ρ-function of π is a mixture of completely monotone functions, and hence is itself completely monotone. Therefore, π is a pLSt and, in fact, an infinitely divisible pLSt; cf. Theorem III.4.1. We can now apply Proposition 2.1. Take $\alpha \in (0,1)$ and note that the π_α-function of π is a positive differentiable function on $(0,\infty)$ with $\pi_\alpha(0+) = 1$. So we can compute the ρ-function ρ_α of π_α and find

$$\rho_\alpha(s) = \rho(s) - \alpha\,\rho(\alpha s) = \int_\alpha^1 \rho_0(vs)\,\mathrm{d}v,$$

where we used (2.8); hence ρ_α, as a mixture of completely monotone functions, is completely monotone. As above it follows that π_α is a pLSt and, in fact, an infinitely divisible pLSt. We conclude that the pLSt π is self-decomposable. ◻

Corollary 2.7. *The components X_α in (1.1) of a self-decomposable $\mathbb{R}_+$-valued random variable X are infinitely divisible. Equivalently, the factors π_α in (2.1) of a self-decomposable pLSt π are infinitely divisible.*

Let π be as in Theorem 2.6. We wish to view the ρ_0-function of π as the ρ-function of a positive differentiable function π_0 on $(0,\infty)$ with $\pi_0(0+) = 1$; in view of Theorem III.4.1 complete monotonicity of ρ_0 is then equivalent to π_0 being an infinitely divisible pLSt. Clearly, such a function π_0 is determined by π because $\big(s\,\rho(s)\big)' = \big(-\log \pi_0(s)\big)'$, and hence

$$\pi_0(s) = \exp\big[-s\,\rho(s)\big] \qquad [\,s > 0\,], \tag{2.10}$$

with ρ the ρ-function of π. The function π_0 in (2.10) is called the π_0-*function* of π. Thus we are led to the following variant of Theorem 2.6; note, however, that we may start from more general functions π.

Theorem 2.8. *Let π be a positive differentiable function on $(0, \infty)$ with $\pi(0+) = 1$. Then π is the pLSt of a self-decomposable distribution on $\mathbb{R}_+$ iff its π_0-function is an infinitely divisible pLSt.*

PROOF. Let π be a self-decomposable pLSt, so by Theorem 2.6 its ρ_0-function is completely monotone. As noted above, this implies that π_0 is an infinitely divisible pLSt; note that, indeed, π_0 is positive and differentiable on $(0, \infty)$ and satisfies $\pi_0(0+) = 1$ because of (A.3.5).
Conversely, let π be such that its π_0-function is an infinitely divisible pLSt, so the ρ-function ρ_0 of π_0 is completely monotone. Now, by (2.3) applied to π_0 it is seen that the ρ-function of π can be written as

$$\rho(s) = \frac{1}{s}\,\{-\log\,\pi_0(s)\} = \frac{1}{s}\int_0^s \rho_0(u)\,du, \tag{2.11}$$

so ρ is of the form (2.8). This means that we can proceed as in the second part of the proof of Theorem 2.6 to conclude that π is a self-decomposable pLSt. □

The criterion of Theorem 2.8 can be reformulated so as to obtain the following *representation theorem* for self-decomposable pLSt's.

Theorem 2.9. *A function π on $\mathbb{R}_+$ is the pLSt of a self-decomposable distribution on $\mathbb{R}_+$ iff π has the form*

$$\pi(s) = \exp\Big[\int_0^s \frac{\log\,\pi_0(u)}{u}\,du\Big] \qquad [\,s \ge 0\,], \tag{2.12}$$

with π_0 the pLSt of an infinitely divisible random variable X_0, for which necessarily

$$\mathbb{E}\,\log\,(X_0 + 1) < \infty. \tag{2.13}$$

PROOF. Let π be a self-decomposable pLSt, so by Theorem 2.8 its π_0-function is an infinitely divisible pLSt. Now, inserting in (2.3) the expression for ρ in the first part of (2.11), one sees that π takes the form (2.12). The converse statement immediately follows from Theorem 2.8; the function π in (2.12) is positive and differentiable on $(0, \infty)$ with $\pi(0+) = 1$ and its π_0-function is π_0.
The logarithmic moment condition (2.13) follows from the fact that the integral in (2.12) has to be finite; this concerns a general property of pLSt's, which is proved in Proposition A.3.2. □

In view of (2.12) the π_0-function of a self-decomposable pLSt π will be called the *underlying* (infinitely divisible) pLSt of π. Note that the ρ-function of π_0 is given by the ρ_0-function of π, so by the second part of (2.5) the canonical function K_0 of π_0 satisfies

$$(2.14) \qquad \widehat{K}_0 = \rho_0.$$

It is now easy to get a *canonical representation* for self-decomposable pLSt's similar to that in (2.4) for infinitely divisible pLSt's. To see this set

$$I(a) := 1 - \mathrm{e}^{-a}, \quad J(a) := \int_0^a \frac{I(t)}{t}\,\mathrm{d}t \qquad [\,a \geq 0\,].$$

Then the underlying pLSt π_0 of a self-decomposable pLSt π can be represented, as all infinitely divisible pLSt's, in terms of its canonical function K_0 by

$$(2.15) \qquad \pi_0(s) = \exp\Big[-\int_{\mathbb{R}_+} I(sx)\,\frac{1}{x}\,\mathrm{d}K_0(x)\Big] \qquad [\,s \geq 0\,],$$

where necessarily $\int_{(1,\infty)} (1/x)\,\mathrm{d}K_0(x) < \infty$. Inserting this representation for π_0 in (2.12) and changing the order of integration, we obtain a similar representation for the self-decomposable pLSt π; we only have to replace I by J.

Theorem 2.10 (Canonical representation). *A function π on $\mathbb{R}_+$ is the pLSt of a self-decomposable distribution on $\mathbb{R}_+$ iff π has the form*

$$(2.16) \qquad \pi(s) = \exp\Big[-\int_{\mathbb{R}_+} J(sx)\,\frac{1}{x}\,dK_0(x)\Big] \qquad [\,s \geq 0\,],$$

with K_0 an LSt-able function; the integrand for $x = 0$ is defined by continuity. Here the function K_0 is unique, and necessarily satisfies

$$(2.17) \qquad \int_{(1,\infty)} (\log\, x)\,\frac{1}{x}\,dK_0(x) < \infty.$$

PROOF. We only have to show yet that (2.17) holds if π is of the form (2.16). For this it is sufficient to note that the integral in (2.16) with $s = 1$ is finite and that $J(x) \geq c\,\log\,x$ for $x \geq 1$ and some $c > 0$. □

The function K_0 in Theorem 2.10 will be called the *second canonical function* of π, and of the corresponding distribution function F and of a corresponding random variable X. It is the (first) canonical function of the underlying pLSt π_0, and is most easily determined by using (2.14). The condition on K_0 given by (2.17) is equivalent to that on π_0 in (2.13) because, as has been shown in the proofs above, both conditions are equivalent to the finiteness of $-\log \pi(1)$. Moreover, (2.17) is equivalent to the first condition in (2.5) for the (first) canonical function K of π; this is an immediate consequence of an expression for K in terms of K_0 which will now be derived. Since $\widehat{K} = \rho$, the ρ-function of π, we can use (2.8) to write

$$\begin{aligned}\widehat{K}(s) &= \int_0^1 \widehat{K}_0(vs)\,\mathrm{d}v = \int_{\mathbb{R}_+} \Big(\int_0^1 \mathrm{e}^{-vsy}\,\mathrm{d}v\Big)\,\mathrm{d}K_0(y) = \\ &= K_0(0) + \int_{(0,\infty)} \Big(\int_0^y \mathrm{e}^{-sx}\,\mathrm{d}x\Big)\,\frac{1}{y}\,\mathrm{d}K_0(y) = \\ &= K_0(0) + \int_0^\infty \mathrm{e}^{-sx} \Big(\int_{(x,\infty)} \frac{1}{y}\,\mathrm{d}K_0(y)\Big)\,\mathrm{d}x,\end{aligned}$$

so for K we get

$$K(x) = K_0(0) + \int_0^x \Big(\int_{(t,\infty)} \frac{1}{y}\,\mathrm{d}K_0(y)\Big)\,\mathrm{d}t \qquad [x \ge 0]. \tag{2.18}$$

It follows that $K(0) = K_0(0)$ and by Fubini's theorem that $K(x)$ and $K_0(x)$ have the same limits as $x \to \infty$, and

$$\int_{(1,\infty)} \frac{1}{x}\,\mathrm{d}K(x) = \int_{(1,\infty)} (\log x)\,\frac{1}{x}\,\mathrm{d}K_0(x). \tag{2.19}$$

Hence the equivalence noted above indeed holds, and if X is a random variable with pLSt π, then by the last two parts of (2.5) we have

$$K_0(0) = \ell_X, \quad \lim_{x\to\infty} K_0(x) = \mathbb{E}X. \tag{2.20}$$

The preceding discussion also leads to the following *characterization* of the self-decomposable distributions on $\mathbb{R}_+$ among the infinitely divisible ones; cf. Theorem 2.5. From now on it is convenient to use distribution functions F rather than pLSt's, and to suppose that $\ell_F = 0$; cf. (1.3).

Theorem 2.11. *A distribution function F with $\ell_F = 0$ is self-decomposable iff it is infinitely divisible having an absolutely continuous canonical function K with a nonincreasing density k on $(0,\infty)$. In this case, for k one may take*

$$k(x) = \int_{(x,\infty)} \frac{1}{y}\,\mathrm{d}K_0(y) \qquad [x > 0], \tag{2.21}$$

where K_0 is the second canonical function of F.

PROOF. The direct part of the theorem, including the expression for k in (2.21), immediately follows by using the first part of (2.20) in (2.18). Turning to the converse we let F be infinitely divisible such that K has a nonincreasing density k, which may be taken right-continuous. Since by (2.5) the function $x \mapsto k(x)/x$ is integrable over $(1,\infty)$, we have $\lim_{x\to\infty} k(x) = 0$, so we can write $k(x) = \int_{(x,\infty)} \mathrm{d}(-k)$ for $x > 0$. Comparison with (2.21) now suggests looking at the right-continuous nondecreasing function K_0 with $K_0(x) = 0$ for $x < 0$ and on $(0,\infty)$ defined by

$$K_0(x) = \int_{(0,x]} y\,\mathrm{d}\big(-k(y)\big) = K(x) - x\,k(x) \qquad [\,x > 0\,]. \tag{2.22}$$

Here the second expression for K_0, which easily follows by use of Fubini's theorem, shows that $K_0(x)$ is indeed finite for all $x > 0$ and that $K_0(0) = 0$; note that $K(0) = 0$ because $\ell_F = 0$. Now, use (a generalized version of) (A.3.2) to show that for $s > 0$

$$\int_{\mathbb{R}_+} \mathrm{e}^{-sx}\,\mathrm{d}K_0(x) = s\int_0^\infty \mathrm{e}^{-sx}\,K_0(x)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}s}\big[s\,\widehat{K}(s)\big].$$

So, K_0 is an LSt-able function and, because of (2.5) and (2.7), its LSt $\widehat{K}_0$ is precisely the ρ_0-function of F. From Theorem 2.6 it follows that F is self-decomposable. ◻

Of course, a function K as in Theorem 2.11 is *concave* on $(0,\infty)$. Now, it is well known that, conversely, any such function has a nonincreasing density on $(0,\infty)$. Hence Theorem 2.11 may be reformulated in the following way; since $\ell_F = K(0)$, we need not require $\ell_F = 0$ here.

Corollary 2.12. *A distribution on $\mathbb{R}_+$ is self-decomposable iff it is infinitely divisible having a canonical function that is concave on* $(0,\infty)$.

Before using Theorem 2.11 for proving some attractive properties of self-decomposable distributions on $\mathbb{R}_+$, we give a simple example where both the canonical functions K and K_0 can be computed explicitly, and prove some useful *closure properties*.

Example 2.13. Consider the *gamma* (r,λ) distribution with pLSt π given by

$$\pi(s) = \Big(\frac{\lambda}{\lambda + s}\Big)^r.$$

In Example 2.4 we have seen, by computing the π_α-function of π, that π is *self-decomposable*. Alternatively, one can use Theorems 2.6 or 2.11. The (first) canonical density k was found in Example III.4.8:

$$k(x) = r\,\mathrm{e}^{-\lambda x} \qquad [\,x > 0\,],$$

so, indeed, k is nonincreasing on $(0, \infty)$. Since the ρ-function of π is given by $\rho(s) = r/(\lambda + s)$, for the ρ_0-function of π we find $\rho_0(s) = r\lambda/(\lambda + s)^2$ which, indeed, is completely monotone. From (2.14) it follows that the second canonical function K_0 has a density k_0 given by

$$k_0(x) = r\lambda\, x\,\mathrm{e}^{-\lambda x} \qquad [\,x > 0\,].$$

Moreover, using (2.10) we can compute the underlying infinitely divisible pLSt π_0 of π; we find

$$\pi_0(s) = \exp\big[-r\,s/(\lambda + s)\big] = \exp\big[-r\,\{1 - \lambda/(\lambda + s)\}\big],$$

so π_0 is of the compound-Poisson type. ◻◻

Self-decomposability is preserved under scale transformations, convolutions and taking limits; see Propositions 2.2 and 2.3. We now consider some other operations; they also occur in Section III.6 on closure properties of general infinitely divisible distributions on $\mathbb{R}_+$.

Proposition 2.14. *Let π be a self-decomposable pLSt. Then:*

(i) *For $a > 0$ the a-th power π^a of π is a self-decomposable pLSt.*

(ii) *For $a > 0$ the pLSt $\pi^{(a)}$ with $\pi^{(a)}(s) := \pi(a + s)/\pi(a)$ is self-decomposable.*

(iii) *For $\gamma \in (0, 1\,]$ the function $\pi^{(\gamma)}$ with $\pi^{(\gamma)}(s) := \pi(s^\gamma)$ is a self-decomposable pLSt.*

Proof. Part (i) follows by taking the a-th power in (2.1) and using Proposition III.2.3; π and its factors π_α are infinitely divisible because of Theorem 2.5 and Corollary 2.7. One can also use Theorem 2.6; the ρ_0-function of π^a is a times the ρ_0-function of π. Next, one easily verifies that the ρ_0-function $\rho_0^{(a)}$ of $\pi^{(a)}$ in (ii) can be expressed in terms of the ρ- and ρ_0-function of π by

$$\rho_0^{(a)}(s) = \rho_0(a + s) - a\,\rho'(a + s);$$

since this function is completely monotone, $\pi^{(a)}$ is self-decomposable. Finally, consider the function $\pi^{(\gamma)}$ in (iii), and recall that by (III.3.8) it is a pLSt for every pLSt π (self-decomposable or not); it can be written as $\pi^{(\gamma)} = \pi \circ (-\log \pi_0)$ with π_0 the infinitely divisible (stable) pLSt of Example III.4.9. Now, using (2.1) with α replaced by α^γ, one sees that $\pi^{(\gamma)}$ is self-decomposable. ◻

In view of part (iii) we note that if π and π_0 are self-decomposable, then $\pi \circ (-\log \pi_0)$ need *not* be self-decomposable; we refer to Section 9 for an example. Further, if π is self-decomposable, then the pLSt's $\pi^{(a)}$ with $a > 0$ and $\pi^{(\alpha)}$ with $\alpha \in (0,1)$ defined by

$$\pi^{(a)}(s) := \frac{\pi(a)\,\pi(s)}{\pi(a+s)}, \quad \pi^{(\alpha)}(s) := 1 - \alpha + \alpha\,\pi(s),$$

are *not* self-decomposable; this is because the corresponding distributions have positive mass at zero (cf. Proposition A.3.3 for the first pLSt) and, as will be proved in a moment, a non-degenerate self-decomposable distribution is absolutely continuous. In particular, the factors π_α of the *gamma* pLSt π, which were computed in Example 2.4, are *not* self-decomposable; note that by Example 2.13 the underlying pLSt π_0 of π is *not* self-decomposable either. In fact, self-decomposability of the π_α turns out to be equivalent to that of π_0.

Theorem 2.15. *Let π be a self-decomposable pLSt. Then its factors π_α with $\alpha \in (0,1)$ are self-decomposable iff the underlying infinitely divisible pLSt π_0 of π is self-decomposable. In fact, for $\alpha \in (0,1)$ the π_0-function of π_α equals the π_α-function of π_0: $\pi_{\alpha,0} = \pi_{0,\alpha}$.*

PROOF. Recall that π_α and π_0 are defined by

$$\pi_\alpha(s) = \frac{\pi(s)}{\pi(\alpha s)}, \quad \pi_0(s) = \exp\big[-s\,\rho(s)\big],$$

where ρ is the ρ-function of π. Now, as is easily verified and already noted in the proof of Theorem 2.6, the ρ-function ρ_α of π_α is related to ρ by $\rho_\alpha(s) = \rho(s) - \alpha\,\rho(\alpha s)$, so

$$\pi_{\alpha,0}(s) := \exp\big[-s\,\rho_\alpha(s)\big] = \frac{\pi_0(s)}{\pi_0(\alpha s)} =: \pi_{0,\alpha}(s);$$

this proves the final statement of the theorem. The rest is now easy; use Theorem 2.8 for π_α, and (2.1) and Corollary 2.7 for π_0. ◻

We return to Theorem 2.11; it enables us to make use of several results from Chapter III. First of all, recall that an infinitely divisible distribution function F on $\mathbb{R}_+$ satisfies the following functional equation:

$$\int_{[0,x]} u \,\mathrm{d}F(u) = \int_{[0,x]} F(x-u)\,\mathrm{d}K(u) \qquad [\,x \geq 0\,].$$

In Section III.4 we used this equation to easily show that F is absolutely continuous if its canonical function K is absolutely continuous and satisfies $\int_{(0,\infty)} (1/x)\,\mathrm{d}K(x) = \infty$. Since by Theorem 2.11 the canonical function K of a self-decomposable F with $\ell_F = 0$ has these properties, we conclude that such an F is *absolutely continuous*. Making better use of the functional equation led to Proposition III.4.16 which, in turn, yielded the general result in Theorem III.10.4 on infinitely divisible distributions on $\mathbb{R}_+$ having a nonincreasing canonical density k on $(0,\infty)$. Therefore we can state the following result.

Theorem 2.16. *Let F be a self-decomposable distribution function with $\ell_F = 0$. Then F is absolutely continuous and has a unique density f for which*

$$\text{(2.23)} \qquad x\,f(x) = \int_0^x f(x-u)\,k(u)\,du \qquad [\,x > 0\,],$$

where k is a canonical density that is nonincreasing on $(0,\infty)$. Moreover, f is continuous and positive on $(0,\infty)$, f is bounded on $(0,\infty)$ if $k(0+) > 1$, and

(i) *If f is nonincreasing near 0 with $f(0+) = \infty$, then $k(0+) \leq 1$.*
(ii) *If $f(0+)$ exists in $(0,\infty)$, then $k(0+) = 1$.*
(iii) *If f is nondecreasing near 0 with $f(0+) = 0$, then $k(0+) \geq 1$.*

This theorem can be used to show that a self-decomposable distribution on $\mathbb{R}_+$ is *unimodal*; see Section A.2 for definition and properties. In case the left extremity is zero, for the corresponding density f that is continuous on $(0,\infty)$, this means that there are only the following two possibilities: (1) f is nonincreasing on $(0,\infty)$; (2) f is nondecreasing and not constant on $(0,x_1]$ and nonincreasing on $[\,x_1,\infty)$, for some $x_1 > 0$. The next theorem also characterizes these possibilities in terms of a corresponding canonical density k that is nonincreasing on $(0,\infty)$.

Theorem 2.17. *A self-decomposable distribution on $\mathbb{R}_+$ is unimodal. If its left extremity is zero, then the corresponding continuous density f is nonincreasing iff the nonincreasing canonical density k satisfies $k(0+) \leq 1$.*

PROOF. Let F be a self-decomposable distribution function on $\mathbb{R}_+$. We suppose that $\ell_F = 0$; this is not an essential restriction. Let f and k be the densities as indicated in the theorem; they are related as in the functional equation (2.23), f is positive and k can be taken *right-continuous*, of course. First we note that if f is nonincreasing on $(0, \infty)$, then from parts (i) and (ii) of Theorem 2.16 it immediately follows that $k(0+) \leq 1$. Next we show that for the rest of the proof we may restrict ourselves to functions k with some further nice properties.

Suppose that k can be obtained as the monotone (pointwise) limit of a sequence $(k_n)_{n \in \mathbb{N}}$ of nonincreasing canonical densities. Then by the monotone convergence theorem we have for $s > 0$

$$\lim_{n \to \infty} \int_0^\infty (1 - \mathrm{e}^{-sx}) \, \frac{1}{x} \, k_n(x) \, \mathrm{d}x = \int_0^\infty (1 - \mathrm{e}^{-sx}) \, \frac{1}{x} \, k(x) \, \mathrm{d}x,$$

and hence by (2.4) $\lim_{n \to \infty} \widehat{F}_n(s) = \widehat{F}(s)$ for $s > 0$, where F_n is the infinitely divisible distribution function with canonical density k_n. From the continuity theorem for pLSt's it follows that (F_n) converges weakly to F. Since unimodality is preserved under weak convergence, we conclude that F is unimodal as soon as all F_n are unimodal. Similarly, F is concave on $(0, \infty)$ as soon as all F_n are concave on $(0, \infty)$, and hence F has a nonincreasing density on $(0, \infty)$ if every F_n has such a density. Also, note that if the approximating sequence (k_n) is *nondecreasing*, then $k(0+) \leq 1$ implies that $k_n(0+) \leq 1$ for all n.

These observations will now be used in four successive steps for showing that k may be taken special, as indicated below.

1. Define $k_n(x) := \min\{k(1/n), k(x)\}$ for $n \in \mathbb{N}$ and $x > 0$. Then each k_n is a nonincreasing canonical density which, moreover, is bounded. Since $k_n \uparrow k$ as $n \to \infty$, we may further assume that k is like k_n, i.e., that k is *bounded.*

2. Define $k_n(x) := n \int_x^{x+1/n} k(t) \, \mathrm{d}t$ for $n \in \mathbb{N}$ and $x > 0$. Then each k_n is a bounded nonincreasing canonical density which, moreover, is continuous. Since by the right-continuity of k we have $k_n \uparrow k$ as $n \to \infty$, we may further assume that k is bounded and *continuous.*

3. Define k_n as in the preceding step. Then each k_n is a bounded nonincreasing continuous canonical density which, moreover, is differentiable with derivative given by $k_n'(x) = n\{k(x+1/n) - k(x)\}$ for $x > 0$. Since $k_n \uparrow k$ as $n \to \infty$, we may further assume that k is like k_n, i.e., that k is bounded with a *bounded continuous derivative* on $(0, \infty)$.
4. Define $k_n(x) := k(x) + \mathrm{e}^{-nx}$ for $n \in \mathbb{N}$ and $x > 0$. Then each k_n is a bounded canonical density with a bounded continuous derivative which, moreover, is negative. Since $k_n \downarrow k$ as $n \to \infty$, for proving just the unimodality of f we may further assume that k is bounded with a bounded continuous *negative* derivative on $(0, \infty)$.
We finally prove the remaining assertions of the theorem for F having a canonical density k as after step 3. Then we can differentiate the functional equation (2.23) to see that f has a continuous derivative on $(0, \infty)$ satisfying

$$x\, f'(x) = \{k(0+) - 1\}\, f(x) + \int_0^x f(x-u)\, k'(u)\, \mathrm{d}u \qquad [\,x > 0\,]. \tag{2.24}$$

Since $k' \leq 0$, it follows that in case $k(0+) \leq 1$ we have $f'(x) \leq 0$ for all x, i.e., f is nonincreasing on $(0, \infty)$. Thus the final statement of the theorem has been proved.
So now assume that $k(0+) > 1$. In order to show that F is unimodal also in this case, we suppose that k has the additional property of having a *negative* derivative; cf. step 4 above. First, note that by (2.23) there exists $\varepsilon > 0$ such that $x\, f(x) \geq F(x)$ for $x \in (0, \varepsilon)$. Since k' is bounded, so $k'(u) \geq -c$ for all $u > 0$ and some $c > 0$, from (2.24) it follows that for $x \in (0, \varepsilon)$

$$x\, f'(x) \geq \{k(0+) - 1\}\, f(x) - c\, F(x) \geq \{k(0+) - 1 - cx\}\, f(x).$$

We conclude that $f'(x) > 0$ for all $x > 0$ sufficiently small. Therefore, we can define

$$x_1 := \sup\{x > 0 : f' \geq 0 \text{ on } (0, x]\},$$

which necessarily is finite. By the continuity of f' it follows that $f'(x_1) = 0$. Moreover, we can differentiate once more in (2.24) to show that for $x > 0$

$$x\, f''(x) = \{k(0+) - 2\} f'(x) + f(0+)\, k'(x) + \int_0^x f'(x-u)\, k'(u)\, \mathrm{d}u, \tag{2.25}$$

and see that f'' is continuous on $(0, \infty)$ with $f''(x_1) < 0$, because $k' < 0$. It follows that f is *strictly* decreasing on some right neighbourhood of x_1.

Now, we will show that f has this property on *all* of the interval (x_1, ∞). For this it is sufficient to take $x_2 > x_1$ such that f is *nonincreasing* on (x_1, x_2), and to prove that $f'(x_2) < 0$. To do so we set $a := x_2 - x_1$, apply (2.24) with $x = x_2$, and replace $k(0+) - 1$ by $\{k(a) - 1\} - \int_0^a k'(u)\,du$. Since $k' < 0$ and $f(x_2) \le f(x_2 - u)$ for all $u \in (0, a)$, with strict inequality for u close to a, we then see that

$$x_2\, f'(x_2) < \{k(a) - 1\}\, f(x_2) + \int_a^{x_2} f(x_2 - u)\, k'(u)\,du. \tag{2.26}$$

If $k(a) \le 1$, then obviously $f'(x_2) < 0$. So we now assume $k(a) > 1$. Put $f(z) := 0$ for $z \le 0$, and note that $f(x_2) \le f(x_1)$ and $f(x_2 - u) \ge f(x_1 - u)$ for all $u \in (a, x_2)$. Then the upperbound in (2.26) can be estimated further to obtain

$$x_2\, f'(x_2) < \{k(a) - 1\}\, f(x_1) + \int_a^{x_2} f(x_1 - u)\, k'(u)\,du.$$

Next, the procedure above resulting in (2.26) is reversed; since we have $f(x_1) \ge f(x_1 - u)$ for all $u \in (0, a)$, it follows that

$$x_2\, f'(x_2) < \{k(0+) - 1\}\, f(x_1) + \int_0^{x_2} f(x_1 - u)\, k'(u)\,du.$$

As in the integral here the upperbound may be replaced by x_1, we can apply (2.24) with $x = x_1$ to conclude that

$$x_2\, f'(x_2) < x_1\, f'(x_1). \tag{2.27}$$

But $f'(x_1) = 0$, so $f'(x_2) < 0$ also when $k(a) > 1$. ∎

The results (i), (ii) and (iii) in Theorem 2.16 can be improved somewhat. Since we now know that f is unimodal, f is *monotone* near zero, so the limit $f(0+)$ exists in $[0, \infty]$. Using also the second part of the preceding theorem then leads to the following result; here $f(0+) > 0$ includes the possibility that $f(0+) = \infty$, and similarly $k(0+) > 1$ includes $k(0+) = \infty$.

Corollary 2.18. *Let f be the continuous density of a self-decomposable distribution with left extremity zero, and let k be a corresponding nonincreasing canonical density. Then $f(0+)$ exists in $[0, \infty]$, f is nonincreasing if $f(0+) > 0$, and*

(i) $f(0+) = \infty \;\Rightarrow\; k(0+) \le 1$,
(ii) $0 < f(0+) < \infty \;\Rightarrow\; k(0+) = 1$,
(iii) $f(0+) = 0 \;\Rightarrow\; k(0+) > 1$,
(iv) $k(0+) < 1 \;\Rightarrow\; f(0+) = \infty$,
(v) $k(0+) = 1 \;\Rightarrow\; f(0+) > 0$,
(vi) $k(0+) > 1 \;\Rightarrow\; f(0+) = 0$.

In the case where $k(0+) = 1$, the limit $f(0+)$ need not be finite. In fact, using the canonical representation (2.4) for the Lt π of f and the fact (see Proposition A.3.4) that $f(0+) = \lim_{s\to\infty} s\,\pi(s)$, one can show that then

$$(2.28)\qquad f(0+) < \infty \iff \int_0^1 \frac{1-k(x)}{x}\,dx < \infty.$$

The proof of Theorem 2.17 gives rise to the following special result; it will be needed in Section 6 to prove the unimodality of a general self-decomposable distribution on $\mathbb{R}$.

Proposition 2.19. *Let f be the continuous density of a self-decomposable distribution with left extremity zero, and suppose that the canonical density k of f is bounded with $k(0+) > 1$ and has a bounded continuous negative derivative on $(0,\infty)$. Then $f(0+) = 0$, f is unimodal with*

$$x_1 := \sup\{x > 0 : f' \geq 0 \text{ on } (0,x]\}$$

as a positive mode, and f is positive and log-concave on $(0, x_1)$; in fact, f has a second derivative that is continuous on $(0,\infty)$ with

$$(2.29)\qquad \{f'(x)\}^2 > f(x)\,f''(x) \qquad [0 < x \leq x_1].$$

PROOF. From the proof of Theorem 2.17 and from Corollary 2.18 it follows that the density f has a continuous second derivative and further satisfies $f(0+) = 0$, $f(x) > 0$ for all $x > 0$, and $f'(x) > 0$ for all $x > 0$ sufficiently small; moreover, x_1 is a positive mode of f, and f satisfies the equations (2.24) and (2.25). In view of this and of (2.29) we consider the following function d:

$$d(x) := -x\,\{f(x)\}^2 \big[\log f(x)\big]'' = x\,\{f'(x)\}^2 - x\,f(x)\,f''(x);$$

we have to show that it is positive on $(0, x_1]$. To this end we rewrite d as

$$(2.30)\qquad d(x) = f(x)\,f'(x) + \int_0^x \{f'(x)\,f(x-u) - f(x)\,f'(x-u)\}\,k'(u)\,du,$$

and take $x \in (0, x_1]$. Since $-f(x)\,f'(x-u)\,k'(u) \geq 0$ and $f(x-u) \leq f(x)$ for $u \in (0,x)$, we can estimate in the following way:

$$\begin{aligned} d(x) &\geq f(x)\,f'(x)\,\Big(1 + \int_0^x k'(u)\,du\Big) = \\ &= f(x)\,f'(x)\,\Big(1 - \big\{k(0+) - k(x)\big\}\Big). \end{aligned}$$

It follows that $d(x) > 0$ for all $x > 0$ sufficiently small. Now, suppose that $d(x) \le 0$ for some $x \in (0, x_1]$. Then, since d is continuous, there exists $x_0 \le x_1$ such that

$$d(x_0) = 0, \quad d(x) > 0 \ \text{ for } x \in (0, x_0).$$

From this we see that $f'/f = (\log f)'$ is (strictly) decreasing on $(0, x_0)$, so

$$f'(x_0)\, f(x_0 - u) - f(x_0)\, f'(x_0 - u) < 0 \ \text{ for } \ u \in (0, x_0).$$

As $f'(x_0) \ge 0$, by (2.30) this would imply that $d(x_0) > 0$. So we have obtained a contradiction; we conclude that d is positive on all of $(0, x_1]$. ◻◻

We conclude this section with showing a phenomenon that leads us to the special self-decomposable distributions of the next section. Return to Theorem 2.9; let $\mathcal{L}_1$ be the set of nonnegative functions on $\mathbb{R}_+$ of the form $-\log \pi$ where π is the pLSt of an infinitely divisible distribution with finite logarithmic moment, and let $\mathcal{L}_2$ be the set of functions of the form $-\log \pi$ where π is a self-decomposable pLSt. Then $\mathcal{L}_1$ and $\mathcal{L}_2$ are *semi-linear spaces* in the following sense:

$$\begin{cases} h \in \mathcal{L}_i, \ \lambda > 0 \implies \lambda h \in \mathcal{L}_i, \\ h_1, h_2 \in \mathcal{L}_i \implies h_1 + h_2 \in \mathcal{L}_i; \end{cases}$$

see Proposition 2.14 (i) for the case $i = 2$. Now, Theorem 2.9 says that the following mapping T is 1–1 from $\mathcal{L}_1$ *onto* $\mathcal{L}_2$:

$$(2.31) \qquad (Th)(s) := \int_0^s \frac{h(u)}{u}\, du \qquad [\, h \in \mathcal{L}_1;\ s \ge 0\,].$$

Moreover, T is *semi-linear* in the sense that

$$\begin{cases} T(\lambda h) = \lambda\,(Th) \ \text{ for } h \in \mathcal{L}_1 \text{ and } \lambda > 0, \\ T(h_1 + h_2) = Th_1 + Th_2 \ \text{ for } h_1, h_2 \in \mathcal{L}_1. \end{cases}$$

Let us look for the possible *eigenvalues* and *eigenfunctions* of T, i.e., the constants $\tau > 0$ and functions $h \in \mathcal{L}_1$ for which

$$(2.32) \qquad Th = \tau h.$$

By differentiation it follows that such a τ and h satisfy $\tau\, h'(s)/h(s) = 1/s$, so for some $\lambda > 0$ we have

$$(2.33) \qquad h(s) = \lambda\, s^{1/\tau} \qquad [\, s \ge 0\,].$$

Now, any such function h obviously satisfies (2.32). But by Theorem III.4.1 it is indeed in $\mathcal{L}_1$ iff h' is completely monotone, which is equivalent to saying that $\tau \geq 1$. We conclude that any $\tau \geq 1$ is an eigenvalue of T and that the corresponding eigenfunctions h are given by (2.33) with $\lambda > 0$. For the pLSt π with $-\log \pi = h$ this means that

$$(2.34) \qquad \pi(s) = \exp\left[-\lambda\, s^{1/\tau}\right].$$

We already met this infinitely divisible pLSt π in Example III.4.9; note, however, that π is even *self-decomposable* because also $h \in \mathcal{L}_2$. In the next section these special self-decomposable pLSt's will be recognized as being *stable* with *exponent* $1/\tau$.

3. Stability on the nonnegative reals

Let X be an $\mathbb{R}_+$-valued random variable. Then, according to Section 1, X is called (strictly) *stable* if for every $n \in \mathbb{N}$ there exists $c_n > 0$ such that

$$(3.1) \qquad X \stackrel{\mathrm{d}}{=} c_n\,(X_1 + \cdots + X_n),$$

where $X_1, \ldots, X_n$ are independent with $X_i \stackrel{\mathrm{d}}{=} X$ for all i. In terms of the pLSt π of X equation (3.1) reads as follows:

$$(3.2) \qquad \pi(s) = \bigl\{\pi(c_n s)\bigr\}^n.$$

From (3.1) or (3.2) it is immediately clear that the stable distributions are in the class of our interest: *A stable distribution on $\mathbb{R}_+$ is infinitely divisible.* The stable distributions turn out to be even *self-decomposable*.

Before being able to show this we need to do some preliminary work. Let π be a stable pLSt. Then from (3.2) it follows that for $m, n \in \mathbb{N}$

$$\pi(c_{mn} s) = \bigl\{\pi(c_n c_{mn} s)\bigr\}^n = \bigl\{\pi(c_m c_n c_{mn} s)\bigr\}^{mn} = \pi(c_m c_n s),$$

and hence, since π is strictly decreasing,

$$(3.3) \qquad c_{mn} = c_m c_n \qquad [m, n \in \mathbb{N}].$$

Now, both (3.2) and (3.3) can be generalized as follows; there exists a continuous function $c : (0,\infty) \to (0,\infty)$ such that

$$(3.4) \qquad \begin{cases} \pi(s) = \bigl\{\pi\bigl(c(x)s\bigr)\bigr\}^x & \text{for } x > 0, \\ c(xy) = c(x)\, c(y) & \text{for } x > 0 \text{ and } y > 0. \end{cases}$$

We will show this together with the following consequence.

Theorem 3.1. *A pLSt π is stable iff there exists $\gamma > 0$ such that*

$$\pi(s) = \{\pi(x^{-1/\gamma}s)\}^x \qquad [x > 0]. \tag{3.5}$$

PROOF. Let π be stable; then we have (3.2) where (c_n) is a sequence of positive constants satisfying (3.3). On account of the latter relation the following function c on $(0, \infty) \cap \mathbb{Q}$ is well defined:

$$c(x) := c_n/c_k \ \text{ if } \ x = n/k \ \text{ with } n, k \in \mathbb{N}.$$

Then $c(n) = c_n$ for $n \in \mathbb{N}$, and one easily verifies that (3.4) holds for rational x and y. In order to show that c can be continuously extended such that (3.4) holds for real x and y, we let $x > 0$ and take a sequence (x_n) in $\mathbb{Q}$ such that $\lim_{n\to\infty} x_n = x$; then we have

$$\lim_{n\to\infty} \pi\big(c(x_n)s\big) = \{\pi(s)\}^{1/x}. \tag{3.6}$$

By the continuity of π it follows that the sequence $\big(c(x_n)\big)$ must be bounded and can have at most one, non-zero, limit point; this means that $\big(c(x_n)\big)$ has a finite non-zero limit. Using (3.6) once more, one sees that the value of this limit does not depend on the approximating sequence (x_n). Therefore, the function c on $(0, \infty) \cap \mathbb{Q}$ can be extended to all of $(0, \infty)$ in the following way:

$$c(x) := \lim_{n\to\infty} c(x_n) \ \text{ if } \ x = \lim_{n\to\infty} x_n \ \text{ with } x_n \in \mathbb{Q} \ \text{ for all } n.$$

Relation (3.4) now easily follows. Hence we have (3.6) for (x_n) in $(0, \infty)$, and we can proceed as above to show that c is continuous on $(0, \infty)$.
Now, it is well known that a continuous function c with the multiplicative property of (3.4) has the form $c(x) = x^r$ for some $r \in \mathbb{R}$. By substituting this in the first part of (3.4) and letting $x \to \infty$ we see that necessarily $r < 0$. Thus we can write $r = -1/\gamma$ for some $\gamma > 0$; this results in (3.5).
Conversely, if π satisfies (3.5) for some $\gamma > 0$, then it also satisfies (3.2) for all n with c_n given by $c_n = n^{-1/\gamma}$, so π is stable. ◻

For a stable pLSt π the positive constant γ for which (3.5) holds, is called the *exponent* (of stability) of π (or of a corresponding random variable X). From Theorem 3.1 and its proof it will be clear that, for given $\gamma > 0$, the stable distributions with exponent γ can be characterized as follows.

Corollary 3.2. *For $\gamma > 0$ an $\mathbb{R}_+$-valued random variable X is stable with exponent γ iff for every $n \in \mathbb{N}$ it can be written as*

$$X \stackrel{d}{=} n^{-1/\gamma}\,(X_1 + \cdots + X_n), \tag{3.7}$$

where $X_1, \ldots, X_n$ are independent with $X_i \stackrel{d}{=} X$ for all i.

Theorem 3.1 is basic in the sense that it can be used to easily obtain both a useful characterization theorem and a canonical representation for the stable distributions on $\mathbb{R}_+$. Let the random variable X be stable with exponent γ. Then for $x > 0$ and $y > 0$ its pLSt π satisfies

$$\pi\big((x+y)^{1/\gamma}s\big) = \big\{\pi(s)\big\}^{x+y} = \pi(x^{1/\gamma}s)\,\pi(y^{1/\gamma}s),$$

and hence, if X' denotes a random variable independent of X with $X' \stackrel{d}{=} X$,

$$(x+y)^{1/\gamma}X \stackrel{d}{=} x^{1/\gamma}X + y^{1/\gamma}X' \qquad [\,x > 0,\ y > 0\,]. \tag{3.8}$$

Dividing both sides by $(x+y)^{1/\gamma}$ leads to the direct part of the following *characterization* result.

Theorem 3.3. *For $\gamma > 0$ an $\mathbb{R}_+$-valued random variable X is stable with exponent γ iff, with X' as above, X can be written as*

$$X \stackrel{d}{=} \alpha X + \beta X' \tag{3.9}$$

for all $\alpha, \beta \in (0,1)$ with $\alpha^\gamma + \beta^\gamma = 1$.

PROOF. We are left with showing the converse part of the theorem. Let X then satisfy (3.9) for all α and β as indicated. We will use induction to show that (3.7) holds for all n. The equality with $n = 2$ follows from taking $\alpha = \beta = (\frac{1}{2})^{1/\gamma}$. Next, let $n \geq 2$, and take $\alpha = \big\{n/(n+1)\big\}^{1/\gamma}$ and $\beta = \big\{1/(n+1)\big\}^{1/\gamma}$. Then it follows that

$$X \stackrel{d}{=} (n+1)^{-1/\gamma}\,\big\{n^{1/\gamma}X + X'\big\},$$

from which it will be clear that also the induction step can be made. From Corollary 3.2 we conclude that X is stable with exponent γ. ◻

An immediate consequence of this theorem is the following important result; just use the definition of self-decomposability.

Theorem 3.4. *A stable distribution on $\mathbb{R}_+$ is self-decomposable.*

We proceed with showing how Theorem 3.1 gives rise to a *canonical representation* for the stable distributions on $\mathbb{R}_+$. Let $\gamma > 0$, and let π be a stable pLSt with exponent γ, so π satisfies (3.5). Taking first $s = 1$ in this relation and then $x = s^{-\gamma}$ with $s > 0$, we see that π also satisfies

$$\log \pi(1) = s^{-\gamma} \log \pi(s), \quad \text{so} \quad \pi(s) = \exp\big[-\lambda\, s^{\gamma}\big],$$

where we put $\lambda := -\log \pi(1)$, which is positive. Now, any such function π with $\lambda > 0$ obviously satisfies (3.5). But it is indeed the pLSt of an infinitely divisible distribution on $\mathbb{R}_+$ iff γ is restricted to $(0, 1\,]$; this immediately follows from Theorem III.4.1 and the fact that the ρ-function of π is given by

$$(3.10) \qquad \rho(s) := -\frac{\mathrm{d}}{\mathrm{d}s} \log \pi(s) = \lambda\gamma\, s^{\gamma-1} \qquad [\, s > 0\,].$$

Thus we have proved the following result.

Theorem 3.5 (Canonical representation). *For $\gamma > 0$ a function π on $\mathbb{R}_+$ is the pLSt of a stable distribution on $\mathbb{R}_+$ with exponent γ iff $\gamma \le 1$ and π has the form*

$$(3.11) \qquad \pi(s) = \exp\big[-\lambda\, s^{\gamma}\big] \qquad [\, s \ge 0\,],$$

where $\lambda > 0$.

So on $\mathbb{R}_+$, there are no stable distributions with exponent $\gamma > 1$, and those with exponent $\gamma = 1$ are given by the *degenerate* distributions. The stable distributions with exponent $\gamma \le 1$ coincide with the infinitely divisible distributions on $\mathbb{R}_+$ whose canonical functions K have LSt's ρ of the form (3.10) with $\lambda > 0$; see the second part of (2.5). Inversion yields the following characterization of the *non-degenerate stable* distributions on $\mathbb{R}_+$ among the *infinitely divisible* ones.

Theorem 3.6. *For $\gamma \in (0, 1)$ a distribution on $\mathbb{R}_+$ is stable with exponent γ iff it is infinitely divisible having an absolutely continuous canonical function K with a density k of the form*

$$(3.12) \qquad k(x) = c\, x^{-\gamma} \qquad [\, x > 0\,]$$

with $c > 0$. Here $c = \lambda\gamma / \Gamma(1-\gamma)$ if the distribution has pLSt (3.11).

Similarly, the non-degenerate stable distributions on $\mathbb{R}_+$ can be characterized among the *self-decomposable* ones; compute the ρ_0-function (2.7) of π in (3.11) and invert it to obtain K_0 with $\widehat{K}_0 = \rho_0$, or make use of Theorem 3.6 and the relation (2.22) between K and K_0.

Theorem 3.7. *For $\gamma \in (0,1)$ a distribution on $\mathbb{R}_+$ is stable with exponent γ iff it is self-decomposable having an absolutely continuous second canonical function K_0 with a density k_0 of the form*

$$(3.13) \qquad k_0(x) = c\, x^{-\gamma} \qquad [\, x > 0\,]$$

with $c > 0$. Here $c = \lambda\gamma^2/\Gamma(1-\gamma)$ if the distribution has pLSt (3.11).

From Theorems 3.6 and 3.7 it follows that the first and second canonical densities k and k_0 of a stable pLSt π with exponent $\gamma \in (0,1)$ are related by

$$(3.14) \qquad k(x) = \frac{1}{\gamma}\, k_0(x) \qquad [\, x > 0\,].$$

Since $\widehat{K}(s) = \big(-\log\, \pi(s)\big)'$ and $\widehat{K}_0(s) = \big(-\log\, \pi_0(s)\big)'$ with π_0 the underlying infinitely divisible pLSt of π, relation (3.14) reflects the observation (for $\gamma < 1$) at the end of the preceding section. It can be formulated as a characterization of the stable pLSt's as in the theorem below; recall that $\mathcal{L}_1$ and $\mathcal{L}_2$ are the semi-linear spaces of functions of the form $-\log\, \pi$ with π an infinitely divisible pLSt with finite logarithmic moment and with π a self-decomposable pLSt, respectively.

Theorem 3.8. *For $\gamma \in (0,1\,]$ a pLSt π is stable with exponent γ iff the function $h = -\log\, \pi$ is an eigenfunction at the eigenvalue $1/\gamma$ of the semi-linear bijective mapping $T : \mathcal{L}_1 \to \mathcal{L}_2$ given by*

$$(Th)(s) := \int_0^s \frac{h(u)}{u}\, du \qquad [\, h \in \mathcal{L}_1;\ s \ge 0\,].$$

The mapping T has no other eigenvalues than $1/\gamma$ with $\gamma \in (0,1\,]$.

Finally, we pay some attention to the non-degenerate stable *distributions* on $\mathbb{R}_+$ themselves, rather than to their transforms or canonical functions. Let the $\mathbb{R}_+$-valued random variable X be stable with exponent $\gamma \in (0,1)$. Then from (1.6) and (3.7) it follows that

$$(3.15) \qquad \ell_X = 0, \quad \mathbb{E}X = \infty;$$

to show this one can also use (2.5) and (2.20). In view of Proposition 3.8, however, X does have a finite logarithmic moment:

$$\mathbb{E} \log (X+1) < \infty. \tag{3.16}$$

Combining Theorems III.7.4 and 3.6 shows that for $r > 0$ we even have

$$\mathbb{E}X^r < \infty \iff r < \gamma; \tag{3.17}$$

see Section 9 for an explicit expression for $\mathbb{E}X^r$. Since X is self-decomposable, we can apply Theorems 2.16 and 3.6 to conclude that X has an absolutely continuous distribution with a density f that is continuous, positive and bounded on $(0, \infty)$, and for some $c > 0$ satisfies

$$x\, f(x) = c \int_0^x f(x-u)\, u^{-\gamma}\, du \qquad [\, x > 0\,]. \tag{3.18}$$

Where there are simple explicit expressions for the pLSt and both the canonical densities of X, the density f itself seems to be generally intractable. Only in case $\gamma = \frac{1}{2}$ an explicit expression for f can be given; see Section 9. Last but not least we mention the following consequence of Theorem 2.17 and its corollary; note that by Theorem 3.6 the canonical density k of X satisfies $k(0+) = \infty$.

Theorem 3.9. *A non-degenerate stable distribution on* $\mathbb{R}_+$ *is absolutely continuous and has a density* f *that is unimodal and continuous with* $f(0+) = 0$ *and hence not monotone on* $(0, \infty)$.

4. Self-decomposability on the nonnegative integers

Let X be a $\mathbb{Z}_+$-valued random variable. In order to define a meaningful concept of self-decomposability for X we replace the ordinary product αX in (1.1), with $\alpha \in (0,1)$, by the $\mathbb{Z}_+$-valued *α-fraction* $\alpha \odot X$ of X as introduced in Section A.4:

$$\alpha \odot X := Z_1 + \cdots + Z_X,$$

where $Z_1, Z_2, \ldots$ are independent Bernoulli$\,(\alpha)$ variables with values in $\{0,1\}$ and independent of X. We will only need $\alpha \odot X$ *in distribution*; its probability generating function (pgf) is given by

$$P_{\alpha \odot X}(z) = P_X(1 - \alpha + \alpha z).$$

We refer to Section 8 for more background behind this simplest choice of a $\mathbb{Z}_+$-valued fraction and for *all* possible choices.

Thus we are led to the following definition: A $\mathbb{Z}_+$-valued random variable X is said to be *discrete self-decomposable* if for every $\alpha \in (0,1)$ it can be written (in distribution) as

$$X \stackrel{d}{=} \alpha \odot X + X_\alpha, \tag{4.1}$$

where in the right-hand side the random variables $\alpha \odot X$ and X_α are independent. The corresponding distribution $(p_k)_{k\in\mathbb{Z}_+}$ and pgf P are also called *discrete self-decomposable.* Because $\mathbb{P}(\alpha \odot X = 0) > 0$, the components X_α of X are $\mathbb{Z}_+$-valued as well. Therefore, we can use pgf's to rewrite equation (4.1); P is discrete self-decomposable iff for every $\alpha \in (0,1)$ there exists a pgf P_α such that

$$P(z) = P(1-\alpha + \alpha z)\, P_\alpha(z). \tag{4.2}$$

We determine to what extent the general observations of Section 1 remain true in the $\mathbb{Z}_+$-case. One easily verifies that (4.2) with some fixed α implies the same relation with α replaced by α^n for any $n \in \mathbb{N}$. Hence for discrete self-decomposability of P we need only require (4.2) for all α in some left neighbourhood of one. Contrary to their $\mathbb{R}_+$-valued counterparts, however, discrete self-decomposable random variables necessarily have left extremities equal to zero.

Proposition 4.1. *A discrete self-decomposable random variable X has the property that $\mathbb{P}(X = 0) > 0$.*

Proof. Let X be discrete self-decomposable with pgf P. Then letting $\alpha \uparrow 1$ in (4.2) we see that $\lim_{\alpha\uparrow 1} P_\alpha(z) = 1$ for all $z \in (0,1]$. This limiting relation must then also hold for $z = 0$ because for $\alpha \in (0,1)$ and $z \in (0,1)$ we can estimate as follows:

$$\begin{aligned} 1 - P_\alpha(0) &= \{1 - P_\alpha(z)\} + \{P_\alpha(z) - P_\alpha(0)\} \le \\ &\le \{1 - P_\alpha(z)\} + z/(1-z). \end{aligned}$$

It follows that $P_\alpha(0) > 0$ for all α sufficiently close to one. Applying (4.2) for such an α shows that $P(0) > 0$. □

Obviously, also the components X_α in (4.1) of a discrete self-decomposable X take the value zero with positive probability, so (1.2) trivially holds in the $\mathbb{Z}_+$-case. Proposition 4.1 also shows, however, that the discrete analogue of (1.3) does *not* hold, and that *degenerate* distributions on $\mathbb{Z}_+$, which are self-decomposable in the classical sense, can*not* be discrete self-decomposable. Since non-degenerate self-decomposable distributions on $\mathbb{R}_+$ are absolutely continuous, it follows that no distribution on $\mathbb{Z}_+$ can be both classically and discrete self-decomposable. Therefore, it is not confusing when discrete self-decomposability is just called *self-decomposability*; this we will mostly do.

The self-decomposable distributions on $\mathbb{Z}_+$ turn out to have many properties analogous to those of their counterparts on $\mathbb{R}_+$ in Section 2. This is not surprising if one sets $\overline{P}(s) := P(1-s)$ and $\overline{P}_\alpha(s) := P_\alpha(1-s)$, and notes that (4.2) can then be written as

$$\overline{P}(s) = \overline{P}(\alpha s)\,\overline{P}_\alpha(s),$$

which is very similar to equation (2.1) for a self-decomposable pLSt π. Nevertheless, we give the precise statements and proofs, because the notation is rather different and the proofs are sometimes essentially simpler; moreover, as argued before, we want to make the $\mathbb{Z}_+$-case readable independent of the $\mathbb{R}_+$-case.

The use of completely monotone functions in Section 2 is replaced, in the present section, by that of absolutely monotone functions. Recall that a real-valued function R on $[0,1)$ is said to be *absolutely monotone* if R possesses nonnegative derivatives of all orders:

$$R^{(n)}(z) \geq 0 \qquad [\, n \in \mathbb{Z}_+;\ 0 \leq z < 1\,].$$

Proposition A.4.4 contains several properties of absolutely monotone functions; they will be used without further comment. Moreover, by Theorem A.4.3 an absolutely monotone function can be represented on $[0,1)$ as a power series with nonnegative coefficients. Therefore, rewriting (4.2) as

$$P_\alpha(z) = \frac{P(z)}{P(1-\alpha+\alpha z)} \tag{4.3}$$

and calling this function the P_α*-function* of P, we immediately obtain the following *criterion* for self-decomposability.

Proposition 4.2. *A pgf P is self-decomposable iff for every $\alpha \in (0,1)$ the P_α-function of P is absolutely monotone.*

Using this proposition one easily shows that the class of self-decomposable distributions on $\mathbb{Z}_+$ is *closed under convolution* and *under weak convergence*; see the two propositions below. Closure under scale transformation does not hold: If X is a self-decomposable $\mathbb{Z}_+$-valued random variable, then aX is *not* self-decomposable for any $a \in \mathbb{Z}_+$ with $a \geq 2$; this immediately follows from the result to be given in Corollary 4.14. In Proposition 4.15 we will show, however, that if X is self-decomposable, then so is $\alpha \odot X$ for every $\alpha \in (0,1)$.

Proposition 4.3. *If X and Y are independent self-decomposable $\mathbb{Z}_+$-valued random variables, then $X+Y$ is self-decomposable. Equivalently, if P_1 and P_2 are self-decomposable pgf's, then their pointwise product P_1P_2 is a self-decomposable pgf.*

Proposition 4.4. *If a sequence $(X^{(m)})$ of self-decomposable $\mathbb{Z}_+$-valued random variables converges in distribution to X, then X is self-decomposable. Equivalently, if a sequence $(P^{(m)})$ of self-decomposable pgf's converges (pointwise) to a pgf P, then P is self-decomposable.*

The criterion of Proposition 4.2 is also used in the next two examples. The first one is somewhat unexpected; in the next section it will become clear, however, that it can be viewed as the discrete analogue of the (self-decomposable) degenerate distribution on $\mathbb{R}_+$. The second one holds no surprise; it is the discrete analogue of the (self-decomposable) gamma distribution of Example 2.4.

Example 4.5. For $\lambda > 0$, let X have the *Poisson* (λ) distribution, so its distribution $(p_k)_{k\in\mathbb{Z}_+}$ and pgf P are given by

$$p_k = \frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda}, \quad P(z) = \exp\big[-\lambda\,(1-z)\big].$$

Then for $\alpha \in (0,1)$ the P_α-function of P can be written as

$$P_\alpha(z) = \exp\big[-\lambda\,(1-\alpha)(1-z)\big],$$

which is absolutely monotone. Hence X is *self-decomposable*. □

Example 4.6. For $r > 0$, $p \in (0,1)$, let X have the *negative-binomial* (r,p) distribution, so its distribution $(p_k)_{k \in \mathbb{Z}_+}$ and pgf P are given by

$$p_k = \binom{r+k-1}{k} p^k (1-p)^r, \quad P(z) = \Big(\frac{1-p}{1-pz}\Big)^r.$$

Then for $\alpha \in (0,1)$ the P_α-function of P can be written as

$$P_\alpha(z) = \Big(\frac{1-p(1-\alpha+\alpha z)}{1-pz}\Big)^r = \Big\{\alpha + (1-\alpha)\,\frac{1-p}{1-pz}\Big\}^r.$$

Now, from Example II.11.15 it is seen that P_α is the r-th power of an infinitely divisible pgf, so by Proposition II.2.3 P_α is absolutely monotone. We conclude that X is *self-decomposable*. □□

Before deriving a canonical representation for self-decomposable pgf's we recall some basic facts from Section II.4. As agreed in Section II.1, *infinite divisibility* of a $\mathbb{Z}_+$-valued random variable X means *discrete* infinite divisibility, so X then satisfies $\mathbb{P}(X=0) > 0$. An X with this property and with pgf P is infinitely divisible iff the functions P^t with $t > 0$ are all absolutely monotone. This condition is, however, equivalent to the absolute monotonicity of only the *R-function* of P defined by

$$(4.4) \qquad R(z) := \frac{\mathrm{d}}{\mathrm{d}z} \log P(z) = \frac{P'(z)}{P(z)}, \quad \text{so } P(z) = \exp\Big[-\int_z^1 R(x)\,\mathrm{d}x\Big].$$

Moreover, any function P of this form with R absolutely monotone on $[0,1)$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$. This observation easily leads to the *canonical representation* of an infinitely divisible pgf P:

$$(4.5) \qquad P(z) = \exp\Big[-\sum_{k=0}^{\infty} \frac{r_k}{k+1}\,(1-z^{k+1})\Big],$$

where the *canonical sequence* $(r_k)_{k\in\mathbb{Z}_+}$ consists of nonnegative numbers for which

$$(4.6) \qquad \sum_{k=0}^{\infty} \frac{r_k}{k+1} < \infty, \quad \sum_{k=0}^{\infty} r_k z^k = R(z), \quad \sum_{k=0}^{\infty} r_k = \mathbb{E}X.$$

Now, for *self-decomposable* distributions on $\mathbb{Z}_+$ similar results can be obtained. We first show that they are *infinitely divisible.*

Theorem 4.7. *A self-decomposable distribution on $\mathbb{Z}_+$ is infinitely divisible.*

PROOF. Let P be a self-decomposable pgf with factors P_α as in (4.2). Since by Proposition 4.1 we have $P(0) > 0$, we are ready as soon as we can show that the R-function of P is absolutely monotone. Now, this function can be written in terms of the P_α:

$$R(z) = \frac{1}{P(z)} \lim_{\alpha \uparrow 1} \frac{P(1-\alpha+\alpha z) - P(z)}{(1-\alpha)(1-z)} =$$

$$= \lim_{\alpha \uparrow 1} \frac{P(1-\alpha+\alpha z)}{P(z)} \, \frac{1 - P(z)/P(1-\alpha+\alpha z)}{(1-\alpha)(1-z)},$$

so we get

$$R(z) = \lim_{\alpha \uparrow 1} \frac{1}{1-\alpha} \frac{1 - P_\alpha(z)}{1-z}. \tag{4.7}$$

From (A.4.6) it now follows that R is the limit of absolutely monotone functions; hence R is absolutely monotone, and P is infinitely divisible. ◻◻

Let P be a pgf. We look for a function, the absolute monotonicity of which characterizes the self-decomposability of P. To this end we note that the R-function of P can be obtained from the functions P^t with $t > 0$ by

$$R(z) = \lim_{t \downarrow 0} \frac{1}{t} \frac{\mathrm{d}}{\mathrm{d}z} P^t(z).$$

Since in the self-decomposability context the P_α-function of P seems to take over the role of P^t, one is tempted to look at $P_\alpha'/(1-\alpha)$ for $\alpha \uparrow 1$:

$$\lim_{\alpha \uparrow 1} \frac{1}{1-\alpha} P_\alpha'(z) =$$

$$= \lim_{\alpha \uparrow 1} \frac{P'(z)\, P(1-\alpha+\alpha z) - \alpha\, P(z)\, P'(1-\alpha+\alpha z)}{(1-\alpha)\, P(1-\alpha+\alpha z)^2} =$$

$$= \frac{(1-z)\, P'(z)^2 - (1-z)\, P(z)\, P''(z) + P(z)\, P'(z)}{P(z)^2} =$$

$$= -\big[(1-z)\, R(z)\big]'.$$

The resulting function will be called the R_0-*function* of P, so

$$R_0(z) := -\frac{\mathrm{d}}{\mathrm{d}z}\big[(1-z)\, R(z)\big] \qquad [\,0 \le z < 1\,], \tag{4.8}$$

with R as in (4.4); here we allow P to be any positive nondecreasing convex function on $[\,0,1)$ with $P(1-) = 1$ and a continuous second derivative.

Since by (A.4.7) such a P has the property that $\lim_{z\uparrow 1}(1-z)\,P'(z)=0$, and hence $\lim_{z\uparrow 1}(1-z)\,R(z)=0$, we can express R in terms of R_0 by

$$(4.9)\qquad R(z)=\frac{1}{1-z}\int_z^1 R_0(x)\,\mathrm{d}x=\int_0^1 R_0(1-y+yz)\,\mathrm{d}y;$$

here the integrals are supposed to exist, which is the case if R_0 is nonnegative. We are now ready to prove the following *criterion* for self-decomposability.

Theorem 4.8. *Let P be a positive nondecreasing convex function on $[0,1)$ with $P(1-)=1$ and a continuous second derivative. Then P is the pgf of a self-decomposable distribution on $\mathbb{Z}_+$ iff its R_0-function is absolutely monotone.*

PROOF. Let P be a self-decomposable pgf with factors P_α, $\alpha\in(0,1)$; then P'_α is absolutely monotone. Since, as we saw above, the R_0-function of P can be obtained as

$$(4.10)\qquad R_0(z)=\lim_{\alpha\uparrow 1}\frac{1}{1-\alpha}\,P'_\alpha(z),$$

we conclude that R_0, as a limit of absolutely monotone functions, is absolutely monotone.

Conversely, let the R_0-function of P be absolutely monotone. Then R_0 is nonnegative, so we have (4.9). It follows that the R-function of P is a mixture of absolutely monotone functions, and hence is itself absolutely monotone. Therefore, P is a pgf and, in fact, an infinitely divisible pgf; cf. Theorem II.4.3. We can now apply Proposition 4.2. Take $\alpha\in(0,1)$ and note that the P_α-function of P is a positive differentiable function on $[0,1)$ with $P_\alpha(1-)=1$. So we can compute the R-function R_α of P_α and find

$$R_\alpha(z)=R(z)-\alpha\,R(1-\alpha+\alpha z)=\int_\alpha^1 R_0(1-y+yz)\,\mathrm{d}y,$$

where we used (4.9); hence R_α, as a mixture of absolutely monotone functions, is absolutely monotone. As above it follows that P_α is a pgf and, in fact, an infinitely divisible pgf. We conclude that the pgf P is self-decomposable. ◻

Corollary 4.9. *The components X_α in (4.1) of a self-decomposable $\mathbb{Z}_+$-valued random variable X are infinitely divisible. Equivalently, the factors P_α in (4.2) of a self-decomposable pgf P are infinitely divisible.*

Let P be as in Theorem 4.8. We wish to view the R_0-function of P as the R-function of a positive differentiable function P_0 on $[0,1)$ with $P_0(1-) = 1$; in view of Theorem II.4.3 absolute monotonicity of R_0 is then equivalent to P_0 being an infinitely divisible pgf. Clearly, such a function P_0 is determined by P because $\big(-(1-z)\,R(z)\big)' = \big(\log P_0(z)\big)'$, and hence

$$P_0(z) = \exp\big[-(1-z)\,R(z)\big] \qquad [\,0 \le z < 1\,], \tag{4.11}$$

with R the R-function of P. The function P_0 in (4.11) is called the P_0-*function* of P. Thus we are led to the following variant of Theorem 4.8; note, however, that we may start from more general functions P.

Theorem 4.10. *Let P be a positive differentiable function on $[0,1)$ with $P(1-) = 1$. Then P is the pgf of a self-decomposable distribution on $\mathbb{Z}_+$ iff its P_0-function is an infinitely divisible pgf.*

Proof. Let P be a self-decomposable pgf, so by Theorem 4.8 its R_0-function is absolutely monotone. As noted above, this implies that P_0 is an infinitely divisible pgf; observe that, indeed, P_0 is positive and differentiable on $[0,1)$ and satisfies $P_0(1-) = 1$ because of (A.4.7).
Conversely, let P be such that its P_0-function is an infinitely divisible pgf, so the R-function R_0 of P_0 is absolutely monotone. Now, by (4.4) applied to P_0 it is seen that the R-function of P can be written as

$$R(z) = \frac{1}{1-z}\,\big\{-\log P_0(z)\big\} = \frac{1}{1-z}\int_z^1 R_0(x)\,\mathrm{d}x, \tag{4.12}$$

so R is of the form (4.9). This means that we can proceed as in the second part of the proof of Theorem 4.8 to conclude that P is a self-decomposable pgf. ◻

The criterion of Theorem 4.10 can be reformulated so as to obtain the following *representation theorem* for self-decomposable pgf's.

Theorem 4.11. *A function P on $[0,1]$ is the pgf of a self-decomposable distribution on $\mathbb{Z}_+$ iff P has the form*

$$P(z) = \exp\Big[\int_z^1 \frac{\log P_0(x)}{1-x}\,dx\Big] \qquad [\,0 \le z \le 1\,] \tag{4.13}$$

with P_0 the pgf of an infinitely divisible random variable X_0, for which necessarily

$$\mathbb{E}\,\log(X_0+1) < \infty. \tag{4.14}$$

PROOF. Let P be a self-decomposable pgf, so by Theorem 4.10 its P_0-function is an infinitely divisible pgf. Now, inserting in (4.4) the expression for R in the first part of (4.12), one sees that P takes the form (4.13). The converse statement immediately follows from Theorem 4.10; the function P in (4.13) is positive and differentiable on $[0,1)$ with $P(1-) = 1$ and its P_0-function is P_0.
The logarithmic moment condition (4.14) follows from the fact that the integral in (4.13) has to be finite; this concerns a general property of pgf's, which is proved in Proposition A.4.2. ◻◻

In view of (4.13) the P_0-function of a self-decomposable pgf P will be called the *underlying* (infinitely divisible) pgf of P. Note that the R-function of P_0 is given by the R_0-function of P, so by the middle part of (4.6) the canonical sequence $(r_{0,k})_{k\in\mathbb{Z}_+}$ of P_0 satisfies

$$(4.15)\qquad \sum_{k=0}^{\infty} r_{0,k} z^k = R_0(z).$$

It is now easy to get a *canonical representation* for self-decomposable pgf's similar to that in (4.5) for infinitely divisible pgf's. To see this we define for $k \in \mathbb{Z}_+$ and $z \in [0,1]$:

$$I_k(z) := 1 - z^{k+1}, \quad J_k(z) := \int_z^1 \frac{I_k(x)}{1-x}\,\mathrm{d}x = \sum_{j=0}^{k} \frac{I_j(z)}{j+1}.$$

Then the underlying pgf P_0 of a self-decomposable pgf P can be represented, as all infinitely divisible pgf's, in terms of its canonical sequence $(r_{0,k})_{k\in\mathbb{Z}_+}$ by

$$(4.16)\qquad P_0(z) = \exp\Big[-\sum_{k=0}^{\infty} \frac{r_{0,k}}{k+1} I_k(z)\Big] \qquad [0 \le z \le 1],$$

where necessarily $\sum_{k=0}^{\infty} r_{0,k}/(k+1) < \infty$. Inserting this representation for P_0 in (4.13) and changing integration and summation, we obtain a similar representation for the self-decomposable pgf P; we only have to replace I by J.

Theorem 4.12 (Canonical representation). *A function P on $[0,1]$ is the pgf of a self-decomposable distribution on $\mathbb{Z}_+$ iff P has the form*

$$(4.17)\qquad P(z) = \exp\Big[-\sum_{k=0}^{\infty} \frac{r_{0,k}}{k+1} J_k(z)\Big] \qquad [0 \le z \le 1],$$

where the quantities $r_{0,k}$ with $k \in \mathbb{Z}_+$ are nonnegative. Here the sequence $(r_{0,k})_{k\in\mathbb{Z}_+}$ is unique, and necessarily satisfies

$$(4.18)\qquad \sum_{k=0}^{\infty} \{\log(k+1)\}\, \frac{r_{0,k}}{k+1} < \infty.$$

PROOF. We only have to show yet that (4.18) holds if P is of the form (4.17). For this it is sufficient to note that the sum in (4.17) with $z = 0$ is finite and that $J_k(0) = \sum_{j=0}^{k} 1/(j+1) \sim \log(k+1)$ as $k \to \infty$. □

The sequence $(r_{0,k})_{k\in\mathbb{Z}_+}$ in Theorem 4.12 will be called the *second canonical sequence* of P (and of the corresponding distribution and of a corresponding random variable). It is the canonical sequence of the underlying pgf P_0; it is most easily determined by using (4.15). The condition on $(r_{0,k})$ given by (4.18) is equivalent to that on P_0 in (4.14) because, as has been shown in the proofs above, both conditions are equivalent to finiteness of $-\log P(0)$. Moreover, (4.18) is equivalent to the first condition in (4.6) for the (first) canonical sequence (r_k) of P; this is an immediate consequence of an expression for (r_k) in terms of $(r_{0,k})$ which will now be derived. Since (r_k) has gf R, the R-function of P, we can use (4.9) to write

$$\sum_{k=0}^{\infty} r_k z^k = \frac{1}{1-z}\int_z^1 R_0(x)\,\mathrm{d}x = \sum_{j=0}^{\infty} \frac{r_{0,j}}{j+1}\,\frac{1-z^{j+1}}{1-z} =$$
$$= \sum_{j=0}^{\infty} \frac{r_{0,j}}{j+1} \sum_{k=0}^{j} z^k = \sum_{k=0}^{\infty} \Big(\sum_{j=k}^{\infty} \frac{r_{0,j}}{j+1}\Big)\, z^k,$$

so for (r_k) we get

$$(4.19)\qquad r_k = \sum_{j=k}^{\infty} \frac{r_{0,j}}{j+1} \qquad [\,k \in \mathbb{Z}_+\,].$$

In particular, by Fubini's theorem it follows that $\sum_{k=0}^{\infty} r_k = \sum_{k=0}^{\infty} r_{0,k}$ and

$$(4.20)\qquad \sum_{k=0}^{\infty} \frac{r_k}{k+1} = \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \frac{1}{j+1}\Big)\, \frac{r_{0,k}}{k+1}.$$

Since $\sum_{j=0}^{k} 1/(j+1) \sim \log(k+1)$ as $k \to \infty$, we see that the equivalence noted above indeed holds. Moreover, if X is a random variable with pgf P, then by the last part of (4.6) we have

$$(4.21)\qquad \sum_{k=0}^{\infty} r_{0,k} = \mathbb{E}X.$$

The preceding discussion also leads to the following *characterization* of the self-decomposable distributions on $\mathbb{Z}_+$ among the infinitely divisible ones; cf. Theorem 4.7. From now on it is convenient to use probability distributions $(p_k)_{k\in\mathbb{Z}_+}$ rather than pgf's.

Theorem 4.13. *A distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ is self-decomposable iff it is infinitely divisible having a canonical sequence (r_k) that is nonincreasing. In this case (r_k) can be written as in (4.19), where $(r_{0,k})$ is the second canonical sequence of (p_k).*

PROOF. If (p_k) is self-decomposable, then, as we saw above, (r_k) can be written as in (4.19) with $r_{0,k} \geq 0$ for all k, so (r_k) is nonincreasing.
Turning to the converse, we let (p_k) be infinitely divisible such that (r_k) is nonincreasing. Then by the first part of (4.6) we have $\lim_{k\to\infty} r_k = 0$, so r_k can be written as $r_k = \sum_{j=k}^{\infty}(r_j - r_{j+1})$ for $k \in \mathbb{Z}_+$. Comparing this with (4.19) suggests looking at the sequence $(r_{0,k})_{k\in\mathbb{Z}_+}$ of nonnegative numbers defined by

$$r_{0,k} = (k+1)\,\{r_k - r_{k+1}\} \qquad [\,k \in \mathbb{Z}_+\,]. \tag{4.22}$$

Clearly, the gf of this sequence can be written, for $z \in [0,1)$, as

$$\sum_{k=0}^{\infty} r_{0,k}\, z^k = \frac{\mathrm{d}}{\mathrm{d}z}\Big[(z-1)\sum_{k=0}^{\infty} r_k z^k\Big];$$

since by the middle part of (4.6) and (4.8) this is precisely the R_0-function of (p_k), we conclude from Theorem 4.8 that (p_k) is self-decomposable. □

By Proposition 4.1 a self-decomposable distribution $p = (p_k)$ on $\mathbb{Z}_+$ satisfies $p_0 > 0$. It also has the property that $p_1 > 0$, because $p_1 = p_0 r_0$ and the preceding theorem implies that r_0 must be positive. Corollary II.8.3 now shows that $p_k > 0$ for *all* k.

Corollary 4.14. *The support $S(p)$ of a self-decomposable distribution p on $\mathbb{Z}_+$ is equal to $\mathbb{Z}_+$.*

Before using Theorem 4.13 for proving an interesting property of self-decomposable distributions on $\mathbb{Z}_+$, we return to Examples 4.5 and 4.6 in order to illustrate the preceding results, and prove some useful *closure properties*.

Example 4.15. Consider the *Poisson* (λ) distribution with pgf P given by

$$P(z) = \exp\big[-\lambda\,(1-z)\big].$$

The self-decomposability of P also follows from Theorems 4.8 or 4.13. In fact, the (first) canonical sequence (r_k) was found in Example II.4.6:

$$r_k = \begin{cases} \lambda & \text{, if } k = 0, \\ 0 & \text{, if } k \geq 1, \end{cases}$$

so, indeed, (r_k) is nonincreasing. It follows that the R_0-function of P is given by $R_0(z) = \lambda$ which, indeed, is absolutely monotone. Hence the second canonical sequence equals the first one, and the underlying infinitely divisible pgf P_0 of P is equal to P. ◻◻

Example 4.16. Consider the *negative-binomial* (r, p) distribution; its pgf is given by

$$P(z) = \Big(\frac{1-p}{1-p\,z}\Big)^r.$$

Operating as in the preceding example we first recall from Example II.4.7 that the (first) canonical sequence (r_k) is given by

$$r_k = r\,p^{k+1} \qquad [\,k \in \mathbb{Z}_+\,],$$

so, indeed, (r_k) is nonincreasing. Next, we turn to the R- and R_0-function of P; since $R(z) = rp/(1-p\,z)$, for R_0 we find $R_0(z) = rp\,(1-p)/(1-p\,z)^2$ which, indeed, is absolutely monotone. For the second canonical sequence $(r_{0,k})$ it follows that

$$r_{0,k} = r(1-p)\,(k+1)\,p^{k+1} \qquad [\,k \in \mathbb{Z}_+\,].$$

Moreover, using (4.11) we can compute the underlying infinitely divisible pgf P_0 of P, and find

$$P_0(z) = \exp\big[-r\,\{1-(1-p)/(1-p\,z)\}\big],$$

so P_0 is of the compound-Poisson type. ◻◻

Self-decomposability is preserved under convolutions and taking limits; see Propositions 4.3 and 4.4. We now consider some other operations; they also occur in Section II.6 on closure properties of general infinitely divisible distributions on $\mathbb{Z}_+$.

Proposition 4.17. *Let P be a self-decomposable pgf. Then:*

(i) *For $a > 0$ the a-th power P^a of P is a self-decomposable pgf.*

(ii) *For $\alpha \in (0,1)$ the pgf $P^{(\alpha)}$ with $P^{(\alpha)}(z) := P(1-\alpha+\alpha z)$ is self-decomposable.*

(iii) *For $\alpha \in (0,1)$ the pgf $P^{(\alpha)}$ with $P^{(\alpha)}(z) := P(\alpha z)/P(\alpha)$ is self-decomposable.*

PROOF. Part (i) follows by taking the a-th power in (4.2) and using Proposition II.2.3; P and its factors P_α are infinitely divisible because of Theorem 4.7 and Corollary 4.9. One can also use Theorem 4.8; the R_0-function of P^a is a times the R_0-function of P. Similarly, one easily verifies that the R_0-function $R_0^{(\alpha)}$ of $P^{(\alpha)}$ in (ii) and (iii) can be expressed in terms of the R- and R_0-function of P by

$$R_0^{(\alpha)}(z) = \alpha\, R_0(1-\alpha+\alpha z), \quad \alpha\, R_0(\alpha z) + \alpha\,(1-\alpha)\, R'(\alpha z),$$

respectively; since these functions are absolutely monotone, the pgf's in (ii) and (iii) are self-decomposable. ◫

If P is a self-decomposable pgf, then for $\alpha \in (0,1)$ the pgf's $P^{(\alpha)}$ with

$$P^{(\alpha)}(z) := \frac{P(\alpha)\, P(z)}{P(\alpha z)}, \quad P^{(\alpha)}(z) := 1-\alpha+\alpha P(z),$$

need *not* be self-decomposable, but, as opposed to their counterparts on $\mathbb{R}_+$, sometimes they are. To see this, let P be the pgf of the *geometric* (p) distribution; then both pgf's above are easily shown to be factors of P as computed in Example 4.6. Now, these factors are self-decomposable iff $p \le \frac{1}{2}$; this immediately follows from the fact that the second canonical sequence $(r_{0,k})$ of P is nonincreasing iff $p \le \frac{1}{2}$ (cf. Example 4.16), because of the following result.

Theorem 4.18. *Let P be a self-decomposable pgf. Then its factors P_α with $\alpha \in (0,1)$ are self-decomposable iff the underlying infinitely divisible pgf P_0 of P is self-decomposable. In fact, for $\alpha \in (0,1)$ the P_0-function of P_α equals the P_α-function of P_0: $P_{\alpha,0} = P_{0,\alpha}$.*

PROOF. Recall that P_α and P_0 are defined by

$$P_\alpha(z) = \frac{P(z)}{P(1-\alpha+\alpha z)}, \quad P_0(z) = \exp\big[-(1-z)\, R(z)\big],$$

where R is the R-function of P. Now, as is easily verified and already noted in the proof of Theorem 4.8, the R-function R_α of P_α is related to R by $R_\alpha(z) = R(z) - \alpha R(1-\alpha+\alpha z)$, so

$$P_{\alpha,0}(z) := \exp\big[-(1-z)\,R_\alpha(z)\big] = \frac{P_0(z)}{P_0(1-\alpha+\alpha z)} =: P_{0,\alpha}(z);$$

this proves the final statement of the theorem. The rest is now easy; use Theorem 4.10 for P_α, and (4.2) and Corollary 4.9 for P_0. ◻

Because of (II.3.7), for every pLSt π and every infinitely divisible pgf P_0 the function $P := \pi \circ (-\log P_0)$ is a pgf. Moreover, if π is infinitely divisible, then so is P; cf. (II.6.8). Now, if π is self-decomposable, then P need *not* be self-decomposable, even if P_0 is self-decomposable; see Section 9 for an example. Taking P_0 stable as in Example II.4.8, however, yields a positive result as stated below; just use (2.1) with α replaced by α^γ and s by $(1-z)^\gamma$.

Proposition 4.19. *If π is a self-decomposable pLSt, then $z \mapsto \pi\big((1-z)^\gamma\big)$ is a self-decomposable pgf for every $\gamma \in (0,1\,]$.*

We return to Theorem 4.13; together with Theorem II.4.4 it implies that a self-decomposable distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ satisfies the recurrence relations

$$(4.23) \qquad (n+1)\,p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \qquad [\,n \in \mathbb{Z}_+\,]$$

with $(r_k)_{k\in\mathbb{Z}_+}$ a nonincreasing sequence of nonnegative numbers, the canonical sequence of (p_k). Now, this result enables us to show that any self-decomposable distribution (p_k) is *unimodal*. This means that there are only the following two possibilities: (1) (p_k) is nonincreasing; (2) (p_k) is nondecreasing and not constant on $\{0,\ldots,n_1\}$ and nonincreasing on $\{n_1, n_1+1,\ldots\}$, for some $n_1 \in \mathbb{N}$. The next theorem says a little more; that the case $r_0 = 1$ is critical, is suggested by Example II.11.13.

Theorem 4.20. *A self-decomposable distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ is unimodal. Moreover, (p_k) is nonincreasing iff $r_0 := p_1/p_0 \le 1$, and:*

(i) *If $r_0 < 1$, then (p_k) is (strictly) decreasing.*
(ii) *If $r_0 = 1$, then (p_k) is nonincreasing with $p_1 = p_0$.*
(iii) *If $r_0 > 1$, then (p_k) is nondecreasing and not constant on $\{0,\ldots,n_1\}$ and (strictly) decreasing on $\{n_1, n_1+1,\ldots\}$, for some $n_1 \in \mathbb{N}$.*

PROOF. Let (p_k) be self-decomposable, so by Corollary 4.14 $p_k > 0$ for all k, and (p_k) satisfies the recurrence relations of (4.23) with (r_k) nonincreasing and $r_k \geq 0$ for all k. Define

$$d_k := p_k - p_{k-1} \text{ for } k \in \mathbb{N}, \quad \lambda_k := r_{k+1} - r_k \text{ for } k \in \mathbb{Z}_+,$$

then $\lambda_k \leq 0$ for all k, and by subtraction in (4.23) one sees that $(d_k)_{k\in\mathbb{N}}$ satisfies

$$(4.24) \qquad (n+1)\, d_{n+1} = (r_0 - 1)\, p_n + \sum_{k=0}^{n-1} \lambda_k\, p_{n-1-k} \qquad [\, n \in \mathbb{Z}_+ \,].$$

Now, if $r_0 < 1$, then from (4.24) it follows that $d_k < 0$ for all k, so (p_k) is decreasing. Similarly, if $r_0 = 1$, then $d_k \leq 0$ for all k, so (p_k) is nonincreasing. Conversely, if (p_k) is nonincreasing, then $r_0 = p_1/p_0 \leq 1$.
So now assume that $r_0 > 1$. Then $p_1 > p_0$, so (p_k) starts increasing. Since $p_k \to 0$ as $k \to \infty$, there is at least one change of sign in the sequence $(d_k)_{k\in\mathbb{N}}$: there exists $n_1 \in \mathbb{N}$ such that $d_1 > 0$, $d_2 \geq 0, \ldots, d_{n_1} \geq 0$, $d_{n_1+1} < 0$. In view of part (iii) of the theorem we have to show that

$$d_{n_1+m} < 0 \quad \text{for all } m \in \mathbb{N}.$$

We will do so by induction. For $m = 1$ the inequality holds. So take $m \in \mathbb{N}$, set $n_1 + m =: n_2$, and suppose that $d_{n_1+1} < 0, \ldots, d_{n_2} < 0$. In order to show that then also $d_{n_2+1} < 0$, we apply (4.24) with $n = n_2$, write $r_0 - 1 = (r_m - 1) - \sum_{k=0}^{m-1} \lambda_k$, and note that $p_{n_2} \leq p_{n_2-1-k}$ for all $k = 0, \ldots, m-1$. Thus we get

$$(4.25) \qquad (n_2+1)\, d_{n_2+1} \leq (r_m - 1)\, p_{n_2} + \sum_{k=m}^{n_2-1} \lambda_k\, p_{n_2-1-k}.$$

If $r_m < 1$, then obviously $d_{n_2+1} < 0$. So now assume that $r_m \geq 1$. Put $p_k := 0$ for $k \in \mathbb{Z} \setminus \mathbb{Z}_+$, and note that $p_{n_2} \leq p_{n_1}$ and $p_{n_2-1-k} \geq p_{n_1-1-k}$ for $k = m, \ldots, n_2 - 1$. Then the upperbound in (4.25) can be estimated further to obtain

$$(n_2+1)\, d_{n_2+1} \leq (r_m - 1)\, p_{n_1} + \sum_{k=m}^{n_2-1} \lambda_k\, p_{n_1-1-k}.$$

Next, the procedure above resulting in (4.25) is reversed; use again the relation between $r_0 - 1$ and $r_m - 1$, and note that for $k = 0, \ldots, m-1$ we have $p_{n_1} \geq p_{n_1-1-k}$; then it follows that

$$(n_2+1)\, d_{n_2+1} \leq (r_0 - 1)\; p_{n_1} + \sum_{k=0}^{n_2-1} \lambda_k\, p_{n_1-1-k}.$$

Since in the sum here the upperbound may be replaced by $n_1 - 1$, we can apply (4.24) with $n = n_1$ to conclude that

(4.26) $$(n_2+1)\, d_{n_2+1} \leq (n_1+1)\, d_{n_1+1}.$$

The fact that $d_{n_1+1} < 0$ now shows that $d_{n_2+1} < 0$ also when $r_m \geq 1$. ◻◻

Finally, we briefly return to Theorem 4.11. Let $\mathcal{L}_1$ be the set of non-negative functions on $[0,1]$ of the form $-\log P$ where P is the pgf of an infinitely divisible distribution with finite logarithmic moment, and let $\mathcal{L}_2$ be the set of functions of the form $-\log P$ where P is a self-decomposable pgf. Then Theorem 4.11 says that the following mapping T is 1–1 from $\mathcal{L}_1$ *onto* $\mathcal{L}_2$:

(4.27) $$(Th)(z) := \int_z^1 \frac{h(x)}{1-x}\,\mathrm{d}x \qquad [\,h \in \mathcal{L}_1;\ 0 \leq z \leq 1\,].$$

Now, $\mathcal{L}_1$ and $\mathcal{L}_2$ are *semi-linear spaces* and T is a *semi-linear mapping* in the same sense as at the end of Section 2. So one may ask for the possible *eigenvalues* and *eigenfunctions* of T, i.e., the constants $\tau > 0$ and functions $h \in \mathcal{L}_1$ for which

(4.28) $$Th = \tau h.$$

By differentiation it follows that such a τ and h satisfy the following equation: $\tau\, h'(z)/h(z) = -1/(1-z)$; so for some $\lambda > 0$ we have

(4.29) $$h(z) = \lambda\,(1-z)^{1/\tau} \qquad [\,0 \leq z \leq 1\,].$$

Now, any such function h obviously satisfies (4.28). But by Theorem II.4.3 it is indeed in $\mathcal{L}_1$ iff $-h'$ is absolutely monotone, which is equivalent to saying that $\tau \geq 1$. We conclude that any $\tau \geq 1$ is an eigenvalue of T and that the corresponding eigenfunctions h are given by (4.29) with $\lambda > 0$; see also Example 4.15 for the case $\tau = 1$. For the pgf P with $-\log P = h$ this means that

(4.30) $$P(z) = \exp\bigl[-\lambda\,(1-z)^{1/\tau}\bigr].$$

We already met this infinitely divisible pgf P in Example II.4.8; note, however, that P is even *self-decomposable* because also $h \in \mathcal{L}_2$. In the next section these special self-decomposable pgf's will be recognized as being *stable* with *exponent* $1/\tau$.

5. Stability on the nonnegative integers

Let X be a $\mathbb{Z}_+$-valued random variable. Then X is said to be *discrete stable* if, analogous to (3.1), for every $n \in \mathbb{N}$ there exists $c_n \in (0, 1]$ such that

$$(5.1) \qquad X \stackrel{\mathrm{d}}{=} c_n \odot (X_1 + \cdots + X_n),$$

where $X_1, \ldots, X_n$ are independent with $X_i \stackrel{\mathrm{d}}{=} X$ for all i. Here the operation $\odot$ is the discrete multiplication as introduced in Section A.4 and reviewed in the beginning of the preceding section. Note that $c_1 = 1$, because $1 \odot X := X$. In terms of the pgf P of X equation (5.1) reads as follows:

$$(5.2) \qquad P(z) = \big\{P(1 - c_n + c_n z)\big\}^n.$$

Since here $c_n < 1$ for $n \geq 2$ and hence $P(0) > 0$, we can apply Proposition II.2.3 to state a first result: *A discrete stable distribution on $\mathbb{Z}_+$ is infinitely divisible.* Moreover, it follows that *degenerate* distributions on $\mathbb{Z}_+$, which are stable in the classical sense, can*not* be discrete stable. Therefore, since non-degenerate stable distributions on $\mathbb{R}_+$ are absolutely continuous, it is not confusing when discrete stability is just called *stability*; this will mostly be done. The stable distributions on $\mathbb{Z}_+$ turn out to have many properties analogous to those of their counterparts on $\mathbb{R}_+$ in Section 3. As argued in the preceding section for the self-decomposable distributions on $\mathbb{Z}_+$, we nevertheless give precise statements and proofs.

Let P be a stable pgf, and take $m, n \in \mathbb{N}$. Then a three-fold application of (5.2) shows that the function $\overline{P}$ with $\overline{P}(s) := P(1 - s)$ satisfies

$$\overline{P}(c_{mn} s) = \big\{\overline{P}(c_n c_{mn} s)\big\}^n = \big\{\overline{P}(c_m c_n c_{mn} s)\big\}^{mn} = \overline{P}(c_m c_n s).$$

Since $\overline{P}$ is (strictly) decreasing, it follows that the sequence (c_n) satisfies

$$(5.3) \qquad c_{mn} = c_m c_n \qquad [\, m, n \in \mathbb{N} \,].$$

Now, both (5.2) and (5.3) can be generalized as follows; there exists a continuous function $c : [1,\infty) \to (0,1]$ such that

$$(5.4) \qquad \begin{cases} P(z) = \{P\big(1- c(x) + c(x)\, z\big)\}^x & \text{for } x \geq 1, \\ c(xy) = c(x)\, c(y) \quad \text{for } x \geq 1 \text{ and } y \geq 1. \end{cases}$$

We will show this together with the following consequence.

Theorem 5.1. *A pgf P is stable iff there exists $\gamma > 0$ such that*

$$(5.5) \qquad P(z) = \{P(1- x^{-1/\gamma} + x^{-1/\gamma} z)\}^x \qquad [x \geq 1].$$

Proof. Let P be stable; then we have (5.2) where (c_n) is a sequence of constants in $(0,1]$ satisfying (5.3). From (5.2) it follows that (c_n) is nonincreasing. Therefore, the following function c on $\mathbb{Q} \cap [1,\infty)$, which because of (5.3) is well defined, has values in $(0,1]$:

$$c(x) := c_n/c_k \;\text{ if }\; x = n/k \;\text{ with }\; n,k \in \mathbb{N},\; n \geq k.$$

Clearly, $c(n) = c_n$ for $n \in \mathbb{N}$, and the second part of (5.4) holds for rational x and y in $[1,\infty)$. In proving the first part for $x = n/k$ with $n,k \in \mathbb{N}$ and $n \geq k$, we may restrict ourselves to $z \geq 1 - c_k$, and can then apply (5.2) with z replaced by $1-(1-z)/c_k$. In order to show that c can be continuously extended such that (5.4) holds for all real x and y in $[1,\infty)$, we let $x \geq 1$ and take a sequence (x_n) in $\mathbb{Q} \cap [1,\infty)$ such that $\lim_{n\to\infty} x_n = x$; then we have

$$(5.6) \qquad \lim_{n\to\infty} P\big(1- c(x_n) + c(x_n)\, z\big) = \{P(z)\}^{1/x}.$$

By the continuity and strict monotonicity of P it follows that the sequence $\big(c(x_n)\big)$ can have at most one, non-zero, limit point; since $c(x_n) \in (0,1]$ for all n, this means that $\big(c(x_n)\big)$ has a limit in $(0,1]$. Using (5.6) once more, one sees that the value of this limit does not depend on the approximating sequence (x_n). Therefore, the function c on $\mathbb{Q} \cap [1,\infty)$ can be extended to all of $[1,\infty)$ in the following way:

$$c(x) := \lim_{n\to\infty} c(x_n) \;\text{ if }\; x = \lim_{n\to\infty} x_n \;\text{ with }\; x_n \in \mathbb{Q} \cap [1,\infty) \text{ for all } n.$$

Relation (5.4) now easily follows. Hence we have (5.6) for (x_n) in $[1,\infty)$, and we can proceed as above to show that c is continuous on $[1,\infty)$.
Now, it is well known that a continuous function c with the multiplicative property of (5.4) has the form $c(x) = x^r$ for some $r \in \mathbb{R}$. Here r is necessarily

negative, because $c(x) \le 1$ for all x. Thus we can write $r = -1/\gamma$ for some $\gamma > 0$; this results in (5.5).
Conversely, if P satisfies (5.5) for some $\gamma > 0$, then it also satisfies (5.2) for all n with c_n given by $c_n = n^{-1/\gamma}$, so P is stable. □

For a stable pgf P the positive constant γ for which (5.5) holds, is called the *exponent* (of stability) of P (or of a corresponding random variable X). From Theorem 5.1 and its proof it will be clear that, for given $\gamma > 0$, the stable distributions with exponent γ can be characterized as follows.

Corollary 5.2. *For $\gamma > 0$ an $\mathbb{Z}_+$-valued random variable X is stable with exponent γ iff for every $n \in \mathbb{N}$ it can be written as*

$$X \stackrel{\mathrm{d}}{=} n^{-1/\gamma} \odot (X_1 + \cdots + X_n), \tag{5.7}$$

where $X_1, \ldots, X_n$ are independent with $X_i \stackrel{\mathrm{d}}{=} X$ for all i.

Theorem 5.1 is basic in the sense that it can be used to easily obtain both a useful characterization theorem and a canonical representation for the stable distributions on $\mathbb{Z}_+$. Let the random variable X, with pgf P, be stable with exponent γ. Then for $x \ge 1$ and $y \ge 1$ the function $\overline{P}$ with $\overline{P}(s) := P(1-s)$ satisfies

$$\begin{aligned} \overline{P}(s) &= \big\{\overline{P}\big((x+y)^{-1/\gamma}s\big)\big\}^{x+y} = \\ &= \big\{\overline{P}\big(x^{-1/\gamma}\{x/(x+y)\}^{1/\gamma}s\big)\big\}^{x}\big\{\overline{P}\big(y^{-1/\gamma}\{y/(x+y)\}^{1/\gamma}s\big)\big\}^{y} = \\ &= \overline{P}\Big(\{x/(x+y)\}^{1/\gamma}s\Big)\,\overline{P}\Big(\{y/(x+y)\}^{1/\gamma}s\Big), \end{aligned}$$

and hence

$$X \stackrel{\mathrm{d}}{=} \Big(\frac{x}{x+y}\Big)^{1/\gamma} \odot X \,+\, \Big(\frac{y}{x+y}\Big)^{1/\gamma} \odot X', \tag{5.8}$$

where X' is a random variable with $X' \stackrel{\mathrm{d}}{=} X$ and such that the two summands in the right hand side are independent. Equation (5.8) immediately yields the direct part of the following *characterization* result.

Theorem 5.3. *For $\gamma > 0$ a $\mathbb{Z}_+$-valued random variable X is stable with exponent γ iff, with X' as above, X can be written as*

$$X \stackrel{\mathrm{d}}{=} \alpha \odot X \,+\, \beta \odot X' \tag{5.9}$$

for all $\alpha, \beta \in (0,1)$ with $\alpha^\gamma + \beta^\gamma = 1$.

PROOF. We are left with showing the converse part of the theorem. Let X then satisfy (5.9) for all α and β as indicated. We will use induction to show that (5.7) holds for all n. The equality with $n = 2$ follows from taking $\alpha = \beta = (\frac{1}{2})^{1/\gamma}$. Next, let $n \geq 2$, and suppose that (5.7) holds for this n; for the function $\overline{P}$ with $\overline{P}(s) := P(1-s)$ and with P the pgf of X, this means that $\overline{P}(s) = \{\overline{P}(n^{-1/\gamma}\, s)\}^n$. Now, apply (5.9) with $\alpha = \{n/(n+1)\}^{1/\gamma}$ and $\beta = \{1/(n+1)\}^{1/\gamma}$; then it follows that

$$\overline{P}(s) = \overline{P}(\alpha s)\, \overline{P}(\beta s) = \{\overline{P}(n^{-1/\gamma}\, \alpha s)\}^n\, \overline{P}(\beta s) = \{\overline{P}(\beta s)\}^{n+1},$$

so (5.7) holds with n replaced by $n+1$. From Corollary 5.2 we conclude that X is stable with exponent γ. ◻

An immediate consequence of this theorem is the following important result; just use the definition of self-decomposability.

Theorem 5.4. *A stable distribution on $\mathbb{Z}_+$ is self-decomposable.*

We proceed with showing how Theorem 5.1 gives rise to a *canonical representation* for the stable distributions on $\mathbb{Z}_+$. Let $\gamma > 0$, and let P be a stable pgf with exponent γ, so P satisfies (5.5). Taking first $z = 0$ in this relation and then $x = (1-z)^{-\gamma}$ with $0 \leq z < 1$, we see that

$$\log P(0) = (1-z)^{-\gamma} \log P(z), \quad \text{so} \quad P(z) = \exp\bigl[-\lambda\, (1-z)^{\gamma}\bigr],$$

where we put $\lambda := -\log P(0)$, which is positive. Now, any such function P with $\lambda > 0$ obviously satisfies (5.5). But it is indeed the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$ iff γ is restricted to $(0, 1]$; this immediately follows from Theorem II.4.3 and the fact that the R-function of P is given by

$$R(z) := \frac{\mathrm{d}}{\mathrm{d}z} \log P(z) = \lambda\gamma\, (1-z)^{\gamma-1} \qquad [\,0 \leq z < 1\,]. \tag{5.10}$$

Thus we have proved the following result.

Theorem 5.5 (Canonical representation). *For $\gamma > 0$ a function P on $[\,0,1\,]$ is the pgf of a stable distribution on $\mathbb{Z}_+$ with exponent γ iff $\gamma \leq 1$ and P has the form*

$$P(z) = \exp\bigl[-\lambda\, (1-z)^{\gamma}\bigr] \qquad [\,0 \leq z \leq 1\,], \tag{5.11}$$

where $\lambda > 0$.

So, there are no stable distributions with exponent $\gamma > 1$, and those with exponent $\gamma = 1$ are given by the *Poisson* distributions. The stable distributions with exponent $\gamma \leq 1$ coincide with the infinitely divisible distributions on $\mathbb{Z}_+$ whose canonical sequences (r_k) have gf's R of the form (5.10) with $\lambda > 0$; see the middle part of (4.6). Inversion yields the following characterization of the stable distributions on $\mathbb{Z}_+$ among the *infinitely divisible* ones.

Theorem 5.6. *For $\gamma \in (0,1]$ a distribution on $\mathbb{Z}_+$ is stable with exponent γ iff it is infinitely divisible having a canonical sequence (r_k) of the form*

$$(5.12) \qquad r_k = c\binom{k-\gamma}{k} \qquad [\, k \in \mathbb{Z}_+ \,]$$

with $c > 0$. Here c is given by $c = \lambda\gamma$ if the distribution has pgf (5.11).

Similarly, the stable distributions on $\mathbb{Z}_+$ can be characterized among the *self-decomposable* ones; compute the R_0-function (4.8) of P in (5.11) and invert it to obtain $(r_{0,k})$ with gf R_0, or make use of Theorem 5.6 and the relation (4.22) between (r_k) and $(r_{0,k})$.

Theorem 5.7. *For $\gamma \in (0,1]$ a distribution on $\mathbb{Z}_+$ is stable with exponent γ iff it is self-decomposable having a second canonical sequence $(r_{0,k})$ of the form*

$$(5.13) \qquad r_{0,k} = c\binom{k-\gamma}{k} \qquad [\, k \in \mathbb{Z}_+ \,]$$

with $c > 0$. Here c is given by $c = \lambda\gamma^2$ if the distribution has pgf (5.11).

From Theorems 5.6 and 5.7 it follows that the first and second canonical sequences (r_k) and $(r_{0,k})$ of a stable pgf P with exponent γ are related by

$$(5.14) \qquad r_k = \frac{1}{\gamma}\, r_{0,k} \qquad [\, k \in \mathbb{Z}_+ \,].$$

Since $R(z) = \big(\log P(z)\big)'$ and $R_0(z) = \big(\log P_0(z)\big)'$ with P_0 the underlying infinitely divisible pgf of P, relation (5.14) reflects the observation at the end of the preceding section. It can be formulated as a characterization of the stable pgf's as in the theorem below; recall that $\mathcal{L}_1$ and $\mathcal{L}_2$ are the semi-linear spaces of functions of the form $-\log P$ with P an infinitely divisible pgf with finite logarithmic moment and with P a self-decomposable pgf, respectively.

Theorem 5.8. *For $\gamma \in (0,1]$ a pgf P is stable with exponent γ iff the function $h = -\log P$ is an eigenfunction at the eigenvalue $1/\gamma$ of the semi-linear bijective mapping $T : \mathcal{L}_1 \to \mathcal{L}_2$ given by*

$$(Th)(z) := \int_z^1 \frac{h(x)}{1-x}\,dx \qquad [\,h \in \mathcal{L}_1;\ 0 \le z \le 1\,].$$

The mapping T has no other eigenvalues than $1/\gamma$ with $\gamma \in (0,1]$.

Finally, we pay some attention to the non-Poissonian stable *distributions* on $\mathbb{Z}_+$ themselves, rather than to their transforms or canonical sequences. Let X be stable with exponent $\gamma \in (0,1)$. Then from (5.7) it follows that

$$(5.15) \qquad \mathbb{E}X = \infty;$$

to show this one can also use (4.6) and (4.21). In view of Proposition 5.8, however, X does have a finite logarithmic moment:

$$(5.16) \qquad \mathbb{E}\,\log\,(X+1) < \infty.$$

Combining Theorems II.7.5 and 5.6, and observing that $\binom{k-\gamma}{k} \sim k^{-\gamma}$ as $k \to \infty$, we see that for $r > 0$ we even have

$$(5.17) \qquad \mathbb{E}X^r < \infty \iff r < \gamma.$$

Since X is self-decomposable, we can use the remark around (4.23) and Theorem 5.6 to conclude that the distribution (p_k) of X satisfies

$$(5.18) \qquad (n+1)\,p_{n+1} = c\sum_{k=0}^{n} \binom{k-\gamma}{k}\,p_{n-k} \qquad [\,n \in \mathbb{Z}_+\,]$$

for some $c > 0$. Where there are simple explicit expressions for the pgf and both the canonical sequences of X, the distribution (p_k) itself seems to be generally intractable. Last but not least we mention the following consequence of Theorem 4.20, which also holds when $\gamma = 1$; note that Theorem 5.6 implies that $r_0 = p_1/p_0$ is given by $r_0 = \lambda\gamma$ if X has pgf (5.11).

Theorem 5.9. *A stable distribution on $\mathbb{Z}_+$ is unimodal. Moreover, if its pgf is represented by (5.11), then it is nonincreasing iff $\lambda\gamma \le 1$.*

It is remarkable that, contrary to the $\mathbb{R}_+$-case, stable distributions on $\mathbb{Z}_+$ can be monotone.

6. Self-decomposability on the real line

We return to the classical case and consider general distributions on $\mathbb{R}$. The definition of self-decomposability, as given in the first section, can be rephrased in terms of characteristic functions as follows. A characteristic function ϕ is *self-decomposable* iff for every $\alpha \in (0,1)$ there exists a characteristic function ϕ_α such that

$$\phi(u) = \phi(\alpha u)\, \phi_\alpha(u). \tag{6.1}$$

Here the factors ϕ_α are uniquely determined by ϕ because ϕ turns out to have no zeroes. In fact, we have the following *criterion* for self-decomposability; here for any function ϕ without zeroes the *ϕ_α-function* of ϕ is defined by

$$\phi_\alpha(u) = \frac{\phi(u)}{\phi(\alpha u)}. \tag{6.2}$$

Proposition 6.1. *A characteristic function ϕ is self-decomposable iff ϕ has no zeroes in $\mathbb{R}$ and for all $\alpha \in (0,1)$ the ϕ_α-function of ϕ is a characteristic function.*

PROOF. We only need to show that $\phi(u) \neq 0$ for $u \in \mathbb{R}$ if ϕ is self-decomposable; the rest of the proposition immediately follows from (6.1). So, let ϕ be self-decomposable with factors ϕ_α, and suppose that ϕ has a zero. Since ϕ is continuous with $\phi(-u) = \overline{\phi(u)}$ and $\phi(0) = 1$, there then exists $u_0 > 0$ such that

$$\phi(u_0) = 0, \quad \phi(u) \neq 0 \text{ for } |u| < u_0.$$

In particular, for $\alpha \in (0,1)$ we have $\phi(\alpha u_0) \neq 0$, so $\phi_\alpha(u_0) = 0$. Now, use the well-known inequality $\operatorname{Re}\{1 - \psi(u)\} \geq \frac{1}{4} \operatorname{Re}\{1 - \psi(2u)\}$ for a characteristic function ψ, and replace ψ by $|\phi_\alpha|^2$ and u by $\frac{1}{2}u_0$; then it follows that

$$\big|\phi_\alpha(\tfrac{1}{2}u_0)\big| \leq \tfrac{1}{2}\sqrt{3} \ (< 1) \qquad [0 < \alpha < 1].$$

But on the other hand we have $\phi(\frac{1}{2}\alpha u_0) \neq 0$ for $\alpha \in (0,1)$, so

$$\lim_{\alpha \uparrow 1} \phi_\alpha(\tfrac{1}{2}u_0) = \lim_{\alpha \uparrow 1} \phi(\tfrac{1}{2}u_0)/\phi(\tfrac{1}{2}\alpha u_0) = 1.$$

This contradicts the inequality above; we conclude that ϕ has no zeroes in $\mathbb{R}$. ◻

This criterion can be used to show that the class of self-decomposable distributions is *closed under scale transformation*, *under convolution* and *under weak convergence*; for the last property, however, we first have to repeat the argument in the proof just given.

Proposition 6.2.

(i) *If X is a self-decomposable random variable, then so is aX for every $a \in \mathbb{R}$. Equivalently, if ϕ is a self-decomposable characteristic function, then so is $\phi^{(a)}$ with $\phi^{(a)}(u) := \phi(au)$ for every $a \in \mathbb{R}$. In particular, if ϕ is a self-decomposable characteristic function, then so is $\overline{\phi}$.*

(ii) *If X and Y are independent self-decomposable random variables, then $X+Y$ is self-decomposable. Equivalently, if ϕ_1 and ϕ_2 are self-decomposable characteristic functions, then their pointwise product $\phi_1\phi_2$ is a self-decomposable characteristic function.*

Proposition 6.3. *If a sequence $(X^{(m)})$ of self-decomposable random variables converges in distribution to X, then X is self-decomposable. Equivalently, if a sequence $(\phi^{(m)})$ of self-decomposable characteristic functions converges (pointwise) to a characteristic function ϕ, then ϕ is self-decomposable.*

PROOF. Let ϕ be a characteristic function that satisfies $\phi = \lim_{m\to\infty} \phi^{(m)}$, where $\phi^{(m)}$ is self-decomposable with factors $\phi_\alpha^{(m)}$, $\alpha \in (0,1)$. First, suppose there exists $u_0 > 0$ such that

$$\phi(u_0) = 0, \quad \phi(u) \neq 0 \text{ for } |u| < u_0.$$

Then for $\alpha \in (0,1)$ we have $\phi(\alpha u) \neq 0$ for $|u| \leq u_0$, so $\lim_{m\to\infty} \phi_\alpha^{(m)}(u)$ exists for $|u| \leq u_0$ with value zero for $u = u_0$. As in the proof of Proposition 6.1 it follows that

$$\Big|\lim_{m\to\infty} \phi_\alpha^{(m)}(\tfrac{1}{2}u_0)\Big| \leq \tfrac{1}{2}\sqrt{3} \ (< 1) \qquad [0 < \alpha < 1],$$

which, however, contradicts the fact that

$$\lim_{\alpha\uparrow 1} \lim_{m\to\infty} \phi_\alpha^{(m)}(\tfrac{1}{2}u_0) = \lim_{\alpha\uparrow 1} \phi(\tfrac{1}{2}u_0)/\phi(\tfrac{1}{2}\alpha u_0) = 1.$$

Thus we have shown that ϕ has no zeroes in $\mathbb{R}$. Now, for $\alpha \in (0,1)$ we can consider the ϕ_α-function of ϕ, and see that $\phi_\alpha = \lim_{m\to\infty} \phi_\alpha^{(m)}$. Since ϕ_α

is continuous at zero, by the continuity theorem we conclude that ϕ_α is a characteristic function. Hence ϕ is self-decomposable. □□

The use of Proposition 6.1 is also illustrated by the following simple examples, which were already considered in the context of infinite divisibility; see Section IV.2.

Example 6.4. For $\sigma^2 > 0$, let X have the *normal* $(0, \sigma^2)$ distribution, so its density f and characteristic function ϕ are given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}x^2/\sigma^2}, \quad \phi(u) = \mathrm{e}^{-\frac{1}{2}\sigma^2 u^2}.$$

Then for $\alpha \in (0,1)$ the ϕ_α-function of ϕ is recognized as the characteristic function of the normal $\big(0, (1-\alpha^2)\,\sigma^2\big)$ distribution. We conclude that the normal (μ, σ^2) distribution is *self-decomposable* for $\mu = 0$ and hence for $\mu \in \mathbb{R}$; cf. (1.3). □□

Example 6.5. For $\lambda > 0$, let X have the *Cauchy* (λ) distribution, so its density f and characteristic function ϕ are given by

$$f(x) = \frac{1}{\pi}\,\frac{\lambda}{\lambda^2 + x^2}, \quad \phi(u) = \mathrm{e}^{-\lambda|u|}.$$

Then for $\alpha \in (0,1)$ the ϕ_α-function of ϕ is recognized as the characteristic function of the Cauchy $\big((1-\alpha)\,\lambda\big)$ distribution. We conclude that the Cauchy (λ) distribution is *self-decomposable.* □□

Example 6.6. For $r > 0$, $\lambda > 0$, let X have the *sym-gamma* (r, λ) distribution, so its characteristic function ϕ is given by

$$\phi(u) = \Big(\frac{\lambda^2}{\lambda^2 + u^2}\Big)^r.$$

Then for $\alpha \in (0,1)$ the ϕ_α-function of ϕ can be written as

$$\phi_\alpha(u) = \Big(\frac{\lambda^2 + \alpha^2 u^2}{\lambda^2 + u^2}\Big)^r = \Big\{\alpha^2 + (1-\alpha^2)\,\frac{\lambda^2}{\lambda^2 + u^2}\Big\}^r,$$

so ϕ_α is of the form $\phi_\alpha(u) = \pi(u^2)$ with π a pLSt (cf. Example 2.4), which by (IV.3.8) implies that ϕ_α is a characteristic function. We conclude that the sym-gamma (r, λ) distribution is *self-decomposable.* This can also be shown by using Proposition 6.2 and noting that $\phi = \psi\overline{\psi}$ with ψ the characteristic function of the gamma (r, λ) distribution, which is self-decomposable; cf. Example 2.4. □□

Of course, one of the first things we want to show next, is the *infinite divisibility* of a self-decomposable characteristic function ϕ. Moreover, we are interested in deriving a *canonical representation* for ϕ. In doing so we want to avoid the classical but laborious way of using triangular arrays of random variables. Instead we try to adapt the methods of proof in Section 2 for the $\mathbb{R}_+$-case. Since there seems to be no useful analogue of the concept of complete monotonicity, we at once focus on the basic representation of a self-decomposable pLSt π as given by Theorem 2.9, from which the infinite divisibility of π immediately follows:

$$(6.3)\qquad \pi(s) = \exp\Big[\int_0^s \frac{\log \pi_0(u)}{u}\, \mathrm{d}u\Big] \qquad [\, s \geq 0\,],$$

where π_0 is an infinitely divisible pLSt. In Section 2 this representation was derived by first showing the complete monotonicity of the ρ- and ρ_0-function of π and then proving from this the infinite divisibility of the π_0-function of π. Now, observe that this derivation can be cut short by use of the relation between π and its factors π_α as given by (2.6):

$$(6.4)\qquad \pi_0(s) := \exp\big[s\,\pi'(s)/\pi(s)\big] = \lim_{\alpha\uparrow 1}\, \exp\Big[-\frac{1}{1-\alpha}\,\big\{1-\pi_\alpha(s)\big\}\Big],$$

so π_0 is the limit of compound-Poisson pLSt's and hence is an infinitely divisible pLSt. Here we use the fact that $\pi_0(0+) = 1$ or, equivalently, that

$$(6.5)\qquad \lim_{s\downarrow 0} s\,\pi'(s) = 0.$$

This property holds for functions even more general than pLSt's; it was used for proving the relation (2.8) between the ρ- and ρ_0-function of π. For deriving (6.3), however, it suffices to know (6.5) only for self-decomposable pLSt's, as was the case for proving the direct part of Theorem 2.8.

We now want to use this method of proof for deriving a representation similar to (6.3) for a self-decomposable characteristic function ϕ. So, we would like ϕ to have the following *property*:

(6.6) $\quad$ *ϕ is differentiable on* $\mathbb{R}\setminus\{0\}$ *and satisfies* $\lim_{u\to 0} u\,\phi'(u) = 0$.

Though, ultimately, it will appear that this property can be proved from (6.1), we did not succeed in proving it directly. In fact, (6.6) is *not* true for general characteristic functions, infinitely divisible or not; see Notes. On the other hand, if a random variable X corresponding to ϕ has a finite first moment, then one has, as is well known,

$$\phi'(u) = \mathbb{E}\,(iX)\,\mathrm{e}^{iuX}, \quad \text{so} \;\; \bigl|u\,\phi'(u)\bigr| \leq |u|\,\mathbb{E}|X|,$$

which tends to zero as $u \to 0$. For self-decomposable characteristic functions without finite first moment, however, (6.6) seems very hard to prove, and this leaves us with a *dilemma*; we want to make use of (6.6) without having to prove it in advance. This dilemma will be resolved as follows. For the time being we shall redefine (but not rename) *self-decomposability* of a characteristic function ϕ by requiring that it satisfies both (6.1) and (6.6). Eventually it will appear that the set of self-decomposable characteristic functions, defined this way, coincides with the set satisfying only (6.1). We return to this after Theorem 6.12.

We start with deriving an analogue of Theorem 2.9 for our (restricted) concept of self-decomposability; it immediately yields an analogue of Theorem 2.8. In view of the latter result, for any $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ that is differentiable on $\mathbb{R}\setminus\{0\}$ and has no zeroes, we define the ϕ_0*-function* of ϕ as follows:

$$\phi_0(u) = \exp\bigl[u\,\phi'(u)/\phi(u)\bigr] \;\text{ for } u \neq 0, \quad \phi_0(0) = 1. \tag{6.7}$$

Theorem 6.7. *A $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of a self-decomposable distribution on $\mathbb{R}$ iff ϕ has the form*

$$\phi(u) = \exp\Bigl[\int_0^u \frac{\log\,\phi_0(v)}{v}\,dv\Bigr] \qquad [\,u \in \mathbb{R}\,] \tag{6.8}$$

with ϕ_0 the characteristic function of an infinitely divisible random variable X_0, for which necessarily

$$\mathbb{E}\,\log\bigl(|X_0| + 1\bigr) < \infty. \tag{6.9}$$

Proof. Let ϕ be a self-decomposable characteristic function with factors ϕ_α as in (6.1). Because of property (6.6) the derivative $\phi'(u)$ at $u \neq 0$ exists; it can be obtained in the following way:

$$\phi'(u) = \lim_{\alpha\uparrow 1} \frac{\phi(u) - \phi(\alpha u)}{u - \alpha u} = -\frac{\phi(u)}{u}\,\lim_{\alpha\uparrow 1} \frac{1 - \phi_\alpha(u)}{1 - \alpha}. \tag{6.10}$$

Since by Proposition 6.1 ϕ has no zeroes, we can consider the ϕ_0-function of ϕ; by (6.10) it can be written as

$$\phi_0(u) = \lim_{\alpha \uparrow 1} \exp\Big[-\frac{1}{1-\alpha}\big\{1-\phi_\alpha(u)\big\}\Big] \qquad [u \in \mathbb{R}], \tag{6.11}$$

so ϕ_0 is the limit of compound-Poisson characteristic functions. Now, property (6.6) implies that ϕ_0 is continuous at zero. Hence from the continuity theorem it follows that ϕ_0 is a characteristic function, and from Proposition IV.2.3 that it is infinitely divisible. Finally, rewriting (6.7) yields

$$\frac{\mathrm{d}}{\mathrm{d}u}\log \phi(u) = \frac{\log \phi_0(u)}{u} \qquad [u \neq 0],$$

which implies the desired representation (6.8) (with $\int_0^u = -\int_u^0$ if $u < 0$). Suppose, conversely, that ϕ is of the form (6.8) with ϕ_0 an infinitely divisible characteristic function, and let $\alpha \in (0,1)$. Since ϕ has no zeroes, we can consider the ϕ_α-function of ϕ; it takes the form

$$\phi_\alpha(u) = \exp\Big[\int_{\alpha u}^{u} \frac{\log \phi_0(v)}{v}\,\mathrm{d}v\Big] = \exp\Big[\int_{\alpha}^{1} \frac{\log \phi_0(us)}{s}\,\mathrm{d}s\Big].$$

Now, for every $u \in \mathbb{R}$ the integrand $s \mapsto \big\{\log \phi_0(us)\big\}/s$ is continuous on the interval $[\alpha, 1]$, and hence Riemann-integrable. Therefore, for every $n \in \mathbb{N}$ there exist $s_{n,1}, \ldots, s_{n,n} \in (\alpha, 1)$, independent of u, such that

$$\begin{aligned}\phi_\alpha(u) &= \exp\Big[\lim_{n\to\infty} \frac{1-\alpha}{n} \sum_{k=1}^{n} \frac{\log \phi_0(us_{n,k})}{s_{n,k}}\Big] = \\ &= \lim_{n\to\infty} \prod_{k=1}^{n} \big\{\phi_0(us_{n,k})\big\}^{t_{n,k}},\end{aligned}$$

where $t_{n,k} := (1-\alpha)/(ns_{n,k})$. As ϕ_α is continuous at zero and ϕ_0 is infinitely divisible, from the continuity theorem and several properties from Section IV.2 it follows that ϕ_α is an infinitely divisible characteristic function. And, by letting $\alpha \downarrow 0$ we similarly see that ϕ itself is an infinitely divisible characteristic function as well. We can now apply Proposition 6.1 to conclude that ϕ satisfies (6.1). Since the integrand in (6.8) is continuous on $\mathbb{R} \setminus \{0\}$, we can differentiate to obtain also property (6.6):

$$u\,\phi'(u) = \phi(u)\,\log \phi_0(u) \qquad [u \neq 0].$$

The logarithmic moment condition (6.9) is equivalent to the integral in (6.8) being finite; this concerns a general property of characteristic functions, which is proved in Proposition A.2.7. ∎

Corollary 6.8. *A self-decomposable distribution on $\mathbb{R}$ is infinitely divisible. Also, the components X_α in (1.1) of a self-decomposable random variable X are infinitely divisible; equivalently, the factors ϕ_α in (6.1) of a self-decomposable characteristic function ϕ are infinitely divisible.*

Corollary 6.9. *Let ϕ be a $\mathbb{C}$-valued function on $\mathbb{R}$ with $\phi(0) = 1$ that is differentiable on $\mathbb{R} \setminus \{0\}$ and has no zeroes. Then ϕ is the characteristic function of a self-decomposable distribution on $\mathbb{R}$ iff its ϕ_0-function is an infinitely divisible characteristic function.*

In view of (6.8) the ϕ_0-function of a self-decomposable characteristic function ϕ will be called the *underlying* (infinitely divisible) characteristic function of ϕ.

It is now easy to get a *canonical representation* for self-decomposable characteristic functions similar to the Lévy representation in Theorem IV.4.4 for infinitely divisible characteristic functions. To see this we define for $u \in \mathbb{R}$ and $x \in \mathbb{R}$

$$I(u,x) := \mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2}, \quad J(u,x) := \int_0^u \frac{\mathrm{e}^{ivx} - 1}{v} \,\mathrm{d}v - \frac{iux}{1+x^2}.$$

The underlying characteristic function ϕ_0 of a self-decomposable characteristic function ϕ can then be represented, as all infinitely divisible characteristic functions, in terms of its canonical triple (a_0, σ_0^2, M_0) by

$$\phi_0(u) = \exp\Big[iua_0 - \tfrac{1}{2}u^2\sigma_0^2 + \int_{\mathbb{R}\setminus\{0\}} I(u,x)\,\mathrm{d}M_0(x) \Big], \tag{6.12}$$

where the Lévy function M_0 satisfies $\int_{[-1,1]\setminus\{0\}} x^2 \,\mathrm{d}M_0(x) < \infty$. Inserting this representation for ϕ_0 in (6.8) and changing the order of integration, we obtain a similar representation for the self-decomposable characteristic function ϕ; we only have to replace I by J and the factor $\frac{1}{2}$ in the normal component by $\frac{1}{4}$.

Theorem 6.10 (Canonical representation). *A $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of a self-decomposable distribution on $\mathbb{R}$ iff ϕ has the form*

$$\phi(u) = \exp\Big[iua_0 - \tfrac{1}{4}u^2\sigma_0^2 + \int_{\mathbb{R}\setminus\{0\}} J(u,x)\, dM_0(x) \Big], \tag{6.13}$$

where $a_0 \in \mathbb{R}$, $\sigma_0^2 \geq 0$ and M_0 is a right-continuous function that is nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$ with $M(x) \to 0$ as $x \to -\infty$ or $x \to \infty$ and

$$(6.14) \qquad \int_{[-1,1]\setminus\{0\}} x^2 \, dM_0(x) < \infty, \quad \int_{\mathbb{R}\setminus[-1,1]} \log |x| \, dM_0(x) < \infty.$$

Proof. We only have to show yet that the second part of (6.14) holds if ϕ is of the form (6.13). For this it is sufficient to note that the integral in (6.13) with $u = 1$ is then finite or, equivalently, that $\int_{\mathbb{R}\setminus[-1,1]} A(x) \, \mathrm{d}M_0(x) < \infty$, where for $|x| > \frac{1}{2}\pi$ and with $D := \{y > 0 : \cos y \leq 0\}$:

$$A(x) := \Big| \int_0^1 \frac{\mathrm{e}^{ivx} - 1}{v} \, \mathrm{d}v \Big| \geq \int_0^{|x|} \frac{1 - \cos y}{y} \, \mathrm{d}y \geq$$
$$\geq \int_0^{|x|} \frac{1}{y} 1_D(y) \, \mathrm{d}y \geq \tfrac{1}{2} \int_{\frac{1}{2}\pi}^{|x|} \frac{1}{y} \, \mathrm{d}y = \tfrac{1}{2} \log |x| - \tfrac{1}{2} \log \tfrac{1}{2}\pi.$$

Since M_0 is bounded on $\mathbb{R} \setminus [-1, 1]$, the desired moment condition for M_0 immediately follows; in fact, it is equivalent to the integral in (6.13) being finite. □

The triple (a_0, σ_0^2, M_0) in Theorem 6.10 will be called the *second canonical triple* of ϕ, and of the corresponding distribution function F and of a corresponding random variable X. It is the (first) canonical triple of the underlying characteristic function ϕ_0 of ϕ. The condition on M_0 given by the second part of (6.14) is equivalent to that on ϕ_0 in (6.9) because, as has been shown in the proofs above, both conditions are equivalent to the finiteness of $\log \phi(1)$. The (first) canonical triple of ϕ, which by Corollary 6.8 is infinitely divisible, can be obtained from the second one in the following way.

Proposition 6.11. *Let ϕ be a self-decomposable characteristic function with second canonical triple (a_0, σ_0^2, M_0). Then ϕ is infinitely divisible with Lévy triple (a, σ^2, M) such that*

$$(6.15) \qquad a = a_0 - 2 \int_{\mathbb{R}\setminus\{0\}} \frac{x^2}{(1 + x^2)^2} \, M_0(x) \, dx, \quad \sigma^2 = \tfrac{1}{2}\sigma_0^2,$$

and M is absolutely continuous with density m given by

$$(6.16) \qquad m(x) = -\frac{1}{x} \, M_0(x) \qquad [\, x \neq 0 \,].$$

PROOF. We want to write the canonical representation (6.13) for ϕ in the Lévy form; cf. (6.12). To this end we first rewrite the integrand $J(u,x)$ in the following way:

$$\begin{aligned}
J(u,x) &:= \int_0^u \frac{\mathrm{e}^{ivx}-1}{v}\,\mathrm{d}v - \frac{iux}{1+x^2} = \\
&= \int_0^x \frac{\mathrm{e}^{iuy}-1}{y}\,\mathrm{d}y - iu\int_0^x \frac{1-y^2}{(1+y^2)^2}\,\mathrm{d}y = \\
&= \int_0^x I(u,y)\,\frac{1}{y}\,\mathrm{d}y + iu\int_0^x \frac{2y^2}{(1+y^2)^2}\,\mathrm{d}y =: \\
&=: J_1(u,x) + iu\,J_2(x).
\end{aligned}$$

Since $\left|J_2(x)\right| \le 2x^2$ if $|x| \le 1$ and $\le \pi$ if $|x| > 1$, the integral of J_2 over $\mathbb{R}\setminus\{0\}$ with respect to M_0 is finite; thus we can write

$$\begin{aligned}
&\int_{\mathbb{R}\setminus\{0\}} J(u,x)\,\mathrm{d}M_0(x) = \\
&\qquad = \int_{\mathbb{R}\setminus\{0\}} J_1(u,x)\,\mathrm{d}M_0(x) + iu\int_{\mathbb{R}\setminus\{0\}} J_2(x)\,\mathrm{d}M_0(x).
\end{aligned}$$

Now, splitting up the domain of integration in both integrals into $(-\infty,0)$ and $(0,\infty)$, and changing the order of integration in the resulting four integrals, we get

$$\begin{aligned}
\int_{\mathbb{R}\setminus\{0\}} J_1(u,x)\,\mathrm{d}M_0(x) &= -\int_{\mathbb{R}\setminus\{0\}} I(u,y)\,\frac{1}{y}\,M_0(y)\,\mathrm{d}y, \\
\int_{\mathbb{R}\setminus\{0\}} J_2(x)\,\mathrm{d}M_0(x) &= -\int_{\mathbb{R}\setminus\{0\}} \frac{2y^2}{(1+y^2)^2}\,M_0(y)\,\mathrm{d}y.
\end{aligned}$$

Finally, by inserting these results in (6.13) we see that ϕ takes the Lévy form. From the uniqueness of Lévy representations the assertions of the proposition now immediately follow. □

From this proposition and Fubini's theorem it follows that the Lévy function M itself can be expressed in terms of M_0 in the following way:

$$M(x) = \begin{cases} -\displaystyle\int_{-\infty}^x \frac{1}{y}\,M_0(y)\,\mathrm{d}y = \int_{(-\infty,x]} \log\frac{y}{x}\,\mathrm{d}M_0(y), & \text{if } x<0, \\ \displaystyle\int_x^\infty \frac{1}{y}\,M_0(y)\,\mathrm{d}y = -\int_{(x,\infty)} \log\frac{y}{x}\,\mathrm{d}M_0(y)\,, & \text{if } x>0. \end{cases} \tag{6.17}$$

By taking here $x = -1$ and $x = 1$ we see that the second condition in (6.14) is equivalent to $M(-1)$ and $M(1)$ being finite. In a similar way one shows that the first condition in (6.14) is equivalent to the same condition for M. Moreover, from Theorem IV.7.3 it easily follows that if X is self-decomposable with second canonical function M_0, then for $r > 0$

$$(6.18) \qquad \mathbb{E}\,|X|^r < \infty \iff \int_{\mathbb{R}\setminus[-1,1]} |x|^r \,\mathrm{d}M_0(x) < \infty.$$

Since M_0 is nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$, Proposition 6.11 also shows that the Lévy function M of a self-decomposable distribution has a density m such that the function $x \mapsto x\,m(x)$ is nonincreasing on $(-\infty, 0)$ and on $(0, \infty)$. It turns out that the converse also holds; thus we have the following *characterization* of the self-decomposable distributions among the infinitely divisible ones.

Theorem 6.12. *A distribution on $\mathbb{R}$ is self-decomposable iff it is infinitely divisible having an absolutely continuous Lévy function M with a density m such that $x \mapsto x\,m(x)$ is nonincreasing on $(-\infty, 0)$ and on $(0, \infty)$.*

PROOF. We are left with proving the converse statement. So, let ϕ be the characteristic function of an infinitely divisible distribution with canonical triple (a, σ^2, M), and suppose that M has a density m satisfying the monotonicity property as stated; we may take m right-continuous, of course. In view of (6.16) we consider the function M_0 on $\mathbb{R} \setminus \{0\}$ defined by

$$(6.19) \qquad M_0(x) = -x\,m(x) \qquad [x \neq 0];$$

then M_0 is right-continuous, and nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$. Moreover, since m is integrable over $\mathbb{R} \setminus [-1, 1]$, it is seen that $M_0(x) \to 0$ as $x \to -\infty$ or $x \to \infty$. Also, M_0 satisfies (6.14); cf. the remarks following (6.17). Now, we can read the proof of Proposition 6.11 backward and see that the Lévy representation for ϕ can be rewritten in the form (6.13) with a_0 and σ_0^2 satisfying (6.15). From Theorem 6.10 we conclude that ϕ is self-decomposable. ◻◻

It is now the appropriate moment to return to the *dilemma* regarding property (6.6). It is well known from more classical treatments of self-decomposability (see Notes) that Theorem 6.12 characterizes *all* distributions that are self-decomposable in the original sense, i.e., that satisfy only (6.1).

Hence self-decomposability is *equivalent* to self-decomposability in our restricted sense, i.e., (6.6) is implied by (6.1). We state this as a formal result.

Proposition 6.13. *A self-decomposable characteristic function* ϕ *is differentiable on* $\mathbb{R} \setminus \{0\}$ *and satisfies* $\lim_{u \to 0} u\, \phi'(u) = 0$.

We turn to another use of Theorem 6.12. Let F be a self-decomposable distribution function with Lévy triple (a, σ^2, M). If $\sigma^2 > 0$, so if F has a normal component, then F is *absolutely continuous*. Now, let $\sigma^2 = 0$ and suppose F is non-degenerate. Then M is not identically zero, and since by Theorem 6.12 $m(x) \geq x_0 m(x_0)/x$ if $0 < x < x_0$ and if $x_0 < x < 0$, it follows that

$$M(0-) = \infty \;\text{ or }\; M(0+) = -\infty, \tag{6.20}$$

so M is *unbounded*. Since M is also absolutely continuous, we can apply Theorem IV.4.23 to obtain the following result.

Theorem 6.14. *A non-degenerate self-decomposable distribution on* $\mathbb{R}$ *is absolutely continuous.*

Before looking more closely at self-decomposable densities we return to the three examples from the beginning of this section in order to illustrate the preceding results, and we prove some useful *closure properties*.

Example 6.15. Consider the *normal* $(0, \sigma^2)$ distribution with characteristic function ϕ given by

$$\phi(u) = \mathrm{e}^{-\frac{1}{2}\sigma^2 u^2}.$$

Then ϕ has Lévy triple $(0, \sigma^2, 0)$, so with Lévy function M that vanishes everywhere; cf. Example IV.4.6. From Theorem 6.12 it follows that ϕ is *self-decomposable*, and from Proposition 6.11 that ϕ has second canonical triple $(0, 2\sigma^2, 0)$, so the underlying infinitely divisible distribution is also normal. ◫

Example 6.16. Consider the *Cauchy* (λ) distribution with characteristic function ϕ given by

$$\phi(u) = \mathrm{e}^{-\lambda|u|}.$$

According to Example IV.4.7 the Lévy triple of ϕ is $(0,0,M)$ with M absolutely continuous with density

$$m(x) = \frac{\lambda}{\pi x^2} \qquad [\,x \neq 0\,].$$

From Theorem 6.12 it follows that ϕ is *self-decomposable*, and from Proposition 6.11 that the second canonical triple equals the first one, so the underlying infinitely divisible distribution is the same Cauchy. ◫

Example 6.17. Consider the *sym-gamma* (r,λ) distribution with characteristic function ϕ given by

$$\phi(u) = \Big(\frac{\lambda^2}{\lambda^2+u^2}\Big)^r.$$

According to Example IV.4.8 the Lévy triple of ϕ is $(0,0,M)$ with M absolutely continuous with density

$$m(x) = \frac{r}{|x|}\,\mathrm{e}^{-\lambda|x|} \qquad [\,x \neq 0\,].$$

From Theorem 6.12 it follows that ϕ is *self-decomposable*, and from Proposition 6.11 that the second canonical triple of ϕ is $(0,0,M_0)$ with M_0 absolutely continuous with density

$$m_0(x) = r\lambda\,\mathrm{e}^{-\lambda|x|} \qquad [\,x \neq 0\,].$$

The underlying infinitely divisible characteristic function ϕ_0 of ϕ can now be computed by using Theorem IV.4.18, but it can also be obtained from (6.7); we find

$$\phi_0(u) = \exp\Big[-2r\,\big\{1-\lambda^2/(\lambda^2+u^2)\big\}\Big],$$

so ϕ_0 is of the compound-Poisson type. ◫

Self-decomposability is preserved under scale transformations, convolutions and taking limits; see Propositions 6.2 and 6.3. We now consider some other operations; cf. Section IV.6.

Proposition 6.18. *Let ϕ be a self-decomposable characteristic function. Then:*

(i) *The absolute value $|\phi|$ of ϕ is a self-decomposable characteristic function.*

(ii) *For $a > 0$ the a-th power ϕ^a of ϕ is a self-decomposable characteristic function.*

PROOF. By Corollary 6.8 ϕ and its factors ϕ_α are infinitely divisible. Therefore, Proposition IV.2.2 implies that $|\phi|$ and $|\phi_\alpha|$ are characteristic functions; part (i) now follows by taking absolute values in (6.1). Similarly, part (ii) is proved by taking the a-th power in (6.1) and using Proposition IV.2.5. ◻

If ϕ is a self-decomposable characteristic function, then $1-\alpha+\alpha\,\phi$ with $\alpha \in (0,1)$ is *not* self-decomposable because of Theorem 6.14. In particular, the factors ϕ_α of the *sym-gamma* characteristic function ϕ, which were computed in Example 6.6, are *not* self-decomposable; note that by Example 6.17 the underlying characteristic function ϕ_0 of ϕ is *not* self-decomposable either. In fact, self-decomposability of the ϕ_α turns out to be equivalent to that of ϕ_0.

Theorem 6.19. *Let ϕ be a self-decomposable characteristic function. Then its factors ϕ_α with $\alpha \in (0,1)$ are self-decomposable iff the underlying infinitely divisible characteristic function ϕ_0 of ϕ is self-decomposable. In fact, for $\alpha \in (0,1)$ the ϕ_0-function of ϕ_α equals the ϕ_α-function of ϕ_0: $\phi_{\alpha,0} = \phi_{0,\alpha}$.*

PROOF. Recall that ϕ_α and ϕ_0 are defined by

$$\phi_\alpha(u) = \frac{\phi(u)}{\phi(\alpha u)}, \qquad \phi_0(u) = \exp\big[u\,\phi'(u)/\phi(u)\big] \text{ for } u \neq 0;$$

because of Propositions 6.1 and 6.13 ϕ is differentiable on $\mathbb{R}\setminus\{0\}$ and has no zeroes. It follows that ϕ_α has the same properties, so its ϕ_0-function is well defined; it is given by

$$\phi_{\alpha,0}(u) := \exp\big[u\,\phi_\alpha'(u)/\phi_\alpha(u)\big] = \frac{\exp\big[u\,\phi'(u)/\phi(u)\big]}{\exp\big[\alpha u\,\phi'(\alpha u)/\phi(\alpha u)\big]} =$$

$$= \frac{\phi_0(u)}{\phi_0(\alpha u)} =: \phi_{0,\alpha}(u).$$

This proves the final statement of the theorem. The rest is now easy; use Corollary 6.9 for ϕ_α, and (6.1) and Corollary 6.8 for ϕ_0. ◻

For every pLSt π and every infinitely divisible characteristic function ϕ_0 the function $\phi := \pi \circ (-\log \phi_0)$ is a characteristic function; cf. (IV.3.8).

Moreover, if π is infinitely divisible, then so is ϕ; cf. (IV.6.4). Now, if π is self-decomposable, then ϕ need *not* be self-decomposable, even if ϕ_0 is self-decomposable; see Section 9 for an example. Taking ϕ_0 stable as in Example IV.4.9, however, yields the positive result stated below; just use (2.1) with α replaced by α^γ and s by $|u|^\gamma$.

Proposition 6.20. *If π is a self-decomposable pLSt, then $u \mapsto \pi(|u|^\gamma)$ is a self-decomposable characteristic function for every $\gamma \in (0,2]$.*

A self-decomposable distribution function F with $\ell_F = 0$ has a density f that is continuous and positive on $(0,\infty)$. This result is contained in Theorem 2.16 and can be used to prove a similar property for self-decomposable distributions on $\mathbb{R}$.

Theorem 6.21. *A non-degenerate self-decomposable distribution on $\mathbb{R}$ has a density which is either continuous and bounded on $\mathbb{R}$ or is continuous and monotone on $(-\infty,b)$ and on (b,∞) for some $b \in \mathbb{R}$.*

Proof. We will use several times the easily verified fact that if g and h are probability densities on $\mathbb{R}$ and one of them is continuous and bounded on $\mathbb{R}$, then so is the density $f := g * h$ for which

$$f(x) = \int_{-\infty}^{\infty} g(x-y)\,h(y)\,\mathrm{d}y \qquad [\,x \in \mathbb{R}\,].$$

Let F be a non-degenerate self-decomposable distribution function with Lévy triple (a,σ^2,M). As a normal distribution has a continuous bounded density, it follows that F has such a density if it has a normal component, i.e., if $\sigma^2 > 0$. So, further let $\sigma^2 = 0$. Then we write M as the sum of two canonical functions M_+ and M_- with $M_+(0-) = 0$ and $M_-(0+) = 0$. By Proposition IV.4.5 (iv) we have $F = G \star H$, where G and H are infinitely divisible with Lévy functions M_+ and M_-, respectively. Since by Theorem 6.12 M has a density m such that $x \mapsto x\,m(x)$ is nonincreasing on $(-\infty,0)$ and on $(0,\infty)$, the same holds for M_+ and M_-, so G and H are self-decomposable and, if non-degenerate, have densities g and h, say, because of Theorem 6.14. The function f as defined above is then a density of F. Now, set $k(x) := x\,m(x)$ for $x > 0$ and $\ell(x) := -x\,m(x)$ for $x < 0$; then we distinguish the following two cases: I. $k(0+) > 1$ or $\ell(0-) > 1$; II. $k(0+) \le 1$ and $\ell(0-) \le 1$.

We start with case I; without restriction we may assume that $k(0+) > 1$. First, suppose that $\int_0^1 k(x)\,\mathrm{d}x < \infty$. Then by Theorem IV.4.13 we have $\ell_G > -\infty$, and as k is the canonical density (in the $\mathbb{R}_+$-sense) of $G(\,\cdot + \ell_G)$, we can apply Theorem 2.16 and Corollary 2.18 (vi) to conclude that the density g of G may be taken continuous and bounded on $\mathbb{R}$. Hence F has a continuous bounded density, both in case $\ell(0-) = 0$ and in case $\ell(0-) > 0$. Next, suppose that $\int_0^1 k(x)\,\mathrm{d}x = \infty$, so $k(0+) = \infty$. Then on $(0,\infty)$ we write m as the sum of two canonical densities m_1 and m_2 with

$$m_1(x) := \min\{2/x,\, m(x)\} \qquad [\,x > 0\,].$$

As above it follows that G can be written as the convolution of two distribution functions G_1 and G_2, where G_1 is self-decomposable with canonical density (in the $\mathbb{R}_+$-sense) k_1 given by $k_1(x) = x\,m_1(x) = \min\{2,\, k(x)\}$ for $x > 0$. Now, k_1 satisfies $k_1(0+) = 2 > 1$ and $\int_0^1 k_1(x)\,\mathrm{d}x < \infty$; hence, as we have seen above, G_1 has a continuous bounded density. But then so has G and hence F.

In case II we trivially have $\int_0^1 k(x)\,\mathrm{d}x < \infty$ and $\int_0^1 \ell(x)\,\mathrm{d}x < \infty$, so G and H are restricted to half-lines; without loss of generality we may take $\ell_G = r_H = 0$. Since k is the canonical density (in the $\mathbb{R}_+$-sense) of G and ℓ that of H, we can apply Theorems 2.16 and 2.17 to conclude that the density g of G may be taken continuous and nonincreasing on $(0,\infty)$ and the density h of H continuous and nondecreasing on $(-\infty,0)$. Now, observe that the density f of F can be written as

$$f(x) = \begin{cases} \int_0^\infty h(x-y)\,g(y)\,\mathrm{d}y & \text{, if } x < 0, \\ \int_{-\infty}^0 g(x-y)\,h(y)\,\mathrm{d}y & \text{, if } x > 0. \end{cases}$$

From this it is immediately clear that f is nondecreasing on $(-\infty,0)$ and nonincreasing on $(0,\infty)$. Moreover, by the monotone convergence theorem it now follows that f is continuous on $\mathbb{R}\setminus\{0\}$. ◻

In case II of the preceding proof we expect f to be continuous on all of $\mathbb{R}$ if at least one of g and h is bounded; then f will be bounded as well. If g and h are both discontinuous at zero, then both continuity of f and discontinuity of f are possible; in the latter case g and h must be unbounded near zero. Examples of both situations are provided by the *sym-gamma* distribution of Example 6.6.

A self-decomposable distribution obeying the second alternative of Theorem 6.21 is obviously *unimodal*; see Section A.2 for definition and properties. Actually, all self-decomposable distributions turn out to be unimodal. In showing this we can restrict ourselves to distributions without normal component because the *normal* distribution has a *log-concave* density and hence (cf. Theorem A.2.11) is *strongly unimodal*, i.e., its convolution with any unimodal distribution is again unimodal. So, let F be a self-decomposable distribution function without normal component, and suppose first that the Lévy function M of F satisfies $M(0-) = 0$. Then one can show (we will do so in the proof of Theorem 6.23) that F is the weak limit of shifted self-decomposable distribution functions on $\mathbb{R}_+$. Since unimodality is preserved under weak convergence, the unimodality of F immediately follows from Theorem 2.17. Similarly, F is unimodal if $M(0+) = 0$. Now, in order to deal with the general case we have to consider *convolutions* of these two types of distribution functions. In general, however, unimodality is *not* preserved under convolutions. But it is if one imposes some extra conditions as given by the following lemma.

Lemma 6.22. *Let G be an absolutely continuous distribution function on $\mathbb{R}_+$ with density g that is continuous on $(0, \infty)$, and let H be an absolutely continuous distribution function on $\mathbb{R}_-$ with density h that is continuous on $(-\infty, 0)$. Suppose G and H are unimodal with modes a and b, respectively, and have the following property: If $a > 0$, then $g(0+) = 0$ and g is positive and log-concave on $(0, a)$; if $b < 0$, then $h(0-) = 0$ and h is positive and log-concave on $(b, 0)$. Then $G \star H$ is unimodal.*

Since this lemma has nothing to do with infinite divisibility, we don't give a proof and refer to Section 10 for further information. We will use it now to prove the following important result.

Theorem 6.23. *A self-decomposable distribution on $\mathbb{R}$ is unimodal.*

PROOF. Let F be a self-decomposable distribution function. Let (a, σ^2, M) be its Lévy triple; as argued above, we may take $\sigma^2 = 0$. According to Theorem 6.12 the function M has a density m such that $x \mapsto x\, m(x)$ is nonincreasing on $(-\infty, 0)$ and on $(0, \infty)$; we take m *right-continuous*. First we show that we may restrict ourselves to functions m with some further nice properties.

Suppose that m can be obtained as the monotone (pointwise) limit of a sequence $(m_n)_{n\in\mathbb{N}}$ of canonical densities with the same monotonicity property as m has. Then, letting $I(u,x)$ be the 'Lévy kernel' as before, by the monotone convergence theorem we have

$$\lim_{n\to\infty}\int_{\mathbb{R}\setminus\{0\}} I(u,x)\,m_n(x)\,\mathrm{d}x = \int_{\mathbb{R}\setminus\{0\}} I(u,x)\,m(x)\,\mathrm{d}x \quad [\,u\in\mathbb{R}\,],$$

and hence $\lim_{n\to\infty}\widetilde{F}_n(u) = \widetilde{F}(u)$ for $u\in\mathbb{R}$, where F_n is the infinitely divisible distribution function with Lévy triple $(a,0,M_n)$ and M_n has density m_n. From the continuity theorem it follows that (F_n) converges weakly to F. Since unimodality is preserved under weak convergence, we conclude that F is unimodal as soon as all F_n are unimodal.

This observation will now be used repeatedly for showing that m may be taken special, as indicated below. Set $k(x) := x\,m(x)$ for $x>0$ and $\ell(x) := -x\,m(x)$ for $x<0$; then k is nonincreasing and ℓ is nondecreasing. We approximate k and ℓ as in step 1 in the proof of Theorem 2.17; define $k_n(x) := \min\{k(1/n), k(x)\}$ for $n\in\mathbb{N}$ and $x>0$, and similarly $\ell_n(x)$ for $n\in\mathbb{N}$ and $x<0$. For the resulting Lévy density m_n this amounts to

$$m_n(x) = \min\Big\{\frac{1}{n|x|}\,m\Big(\frac{x}{n|x|}\Big), m(x)\Big\} \qquad [\,n\in\mathbb{N};\ x\neq 0\,].$$

Each m_n is a canonical density having the same monotonicity property as m has; moreover, m_n is such that k_n and ℓ_n are bounded. Since $m_n\uparrow m$ as $n\to\infty$, we may further assume that m is like m_n, i.e., that k and ℓ are *bounded.* Now, observe that M has the following property:

$$\int_{[-1,1]\setminus\{0\}} |x|\,\mathrm{d}M(x) = \int_0^1 k(x)\,\mathrm{d}x + \int_{-1}^0 \ell(x)\,\mathrm{d}x < \infty.$$

From Theorem IV.4.15 it follows that, up to a shift over some constant which may be taken zero, of course, F can be written as $F = G\star H$, where G is a self-decomposable distribution function with $\ell_G = 0$ and with canonical density k, and H is a self-decomposable distribution function with $r_H = 0$ and with canonical density ℓ. According to Theorem 2.17 G and H are unimodal; moreover, both have modes zero iff $k(0+)\le 1$ and $\ell(0-)\le 1$. So, in this case F is unimodal as well, because of Lemma 6.22 (with $a=0$ and $b=0$); see also case II in the proof of Theorem 6.21.

When $k(0+)>1$ or $\ell(0-)>1$, we first have to approximate m (monotonically) further. We do so by approximating k and/or ℓ as in steps 2, 3 and 4 in the proof of Theorem 2.17. Thus, if e.g. $k(0+) > 1$, we may further assume that m is such that k is bounded and has a bounded, continuous, negative derivative on $(0, \infty)$. We can then apply Proposition 2.19 to conclude that the component G of F has a continuous density g that satisfies $g(0+) = 0$ and is positive and log-concave on $(0, a)$, where a is a (positive) mode of G. Since H can be handled similarly, Lemma 6.22 yields the unimodality of $F = G \star H$ also in the present case. ◻◻

Finally, we return to Theorem 6.7. Let $\mathcal{L}_1$ be the set of $\mathbb{C}$-valued functions on $\mathbb{R}$ of the form $-\log \phi$ where ϕ is the characteristic function of an infinitely divisible distribution on $\mathbb{R}$ with finite logarithmic moment, and let $\mathcal{L}_2$ be the set of functions of the form $-\log \phi$ where ϕ is a self-decomposable characteristic function. Then Theorem 6.7 says that the following mapping T is 1–1 from $\mathcal{L}_1$ *onto* $\mathcal{L}_2$:

$$(6.21) \qquad (Th)(u) := \int_0^u \frac{h(v)}{v}\, \mathrm{d}v \qquad [\, h \in \mathcal{L}_1;\ u \in \mathbb{R}\,].$$

Now, $\mathcal{L}_1$ and $\mathcal{L}_2$ are *semi-linear spaces* and T is a *semi-linear mapping* in the same sense as at the end of Section 2. So one may ask for the possible *eigenvalues* and *eigenfunctions* of T, i.e., the constants $\tau > 0$ and functions $h \in \mathcal{L}_1$ for which

$$(6.22) \qquad Th = \tau h.$$

Proposition 6.24. *Consider the bijective mapping $T : \mathcal{L}_1 \to \mathcal{L}_2$ as given by (6.21); it has the following properties:*

- (i) *A constant $\tau > 0$ is an eigenvalue of T iff $\tau \geq \frac{1}{2}$.*
- (ii) *The functions h of the form $h(u) = iua$ with $a \in \mathbb{R}$ are eigenfunctions of T at the eigenvalue $\tau = 1$.*
- (iii) *For $\tau \geq \frac{1}{2}$ a function $h \in \mathcal{L}_1$, not of the form $h(u) = iua$ with $a \in \mathbb{R}$, is an eigenfunction of T at the eigenvalue τ iff h has the form*

$$h(u) = \lambda\, |u|^{1/\tau}\, (1 \pm_u \theta i) \qquad [\, u \in \mathbb{R}\,],$$

where $\lambda > 0$, $\theta \in \mathbb{R}$ and $\pm_u := +$ if $u \geq 0$ and $:= -$ if $u < 0$.

PROOF. Clearly, a function h of the form $h(u) = iua$ with $a \in \mathbb{R}$ belongs to $\mathcal{L}_1$ and satisfies (6.22) with $\tau = 1$. Exclude further such functions h, and suppose that $\tau > 0$ is an eigenvalue of T with eigenfunction $h \in \mathcal{L}_1$.

Then also $h \in \mathcal{L}_2$, so $h = -\log \phi$ with ϕ the characteristic function of a non-degenerate self-decomposable, and hence absolutely continuous, distribution; cf. Theorem 6.14. As is well known, ϕ then satisfies $\phi(u) \neq 1$ for $u \neq 0$, so $h(u) \neq 0$ for $u \neq 0$. Since h is continuous, we can now differentiate in (6.22) to obtain $\tau\, h'(u)/h(u) = 1/u$ for $u \neq 0$; as $h(-u) = \overline{h(u)}$, this differential equation is solved by

$$h(u) = (\lambda \pm_u \mu i)\, |u|^{1/\tau} \qquad [u \in \mathbb{R}],$$

where $\lambda, \mu \in \mathbb{R}$. As $h \in \mathcal{L}_1$, the characteristic function ϕ above has the property that $|\phi|$ is a characteristic function as well; cf. Proposition IV.2.2. Now, $|\phi|$ is given by

$$(6.23) \qquad \big|\phi(u)\big| = \exp\big[-\lambda\, |u|^{1/\tau}\big] \qquad [u \in \mathbb{R}],$$

and, as has been shown in Example IV.4.9, this function of u is a characteristic function (if and) only if $\lambda > 0$ and $\tau \geq \frac{1}{2}$; see also Example 9.20. Thus, only $\tau \geq \frac{1}{2}$ can be an eigenvalue of T, and h can be rewritten as in (iii); take $\theta := \mu/\lambda$. Conversely, any such function h obviously satisfies (6.22). But only when $\theta = 0$ we know that $h \in \mathcal{L}_1$ because then ϕ with $-\log \phi = h$ satisfies $\phi = |\phi|$ as in (6.23). Nevertheless, this is sufficient for any $\tau \geq \frac{1}{2}$ being indeed an eigenvalue of T. ◫

Actually, in Examples 6.15 and 6.16 we already saw that the normal and Cauchy distributions lead to eigenfunctions of T (as in (iii), with $\theta = 0$) at the eigenvalues $\tau = \frac{1}{2}$ and $\tau = 1$, respectively. For general h as in (iii) the function ϕ with $-\log \phi = h$ has the form

$$(6.24) \qquad \phi(u) = \exp\big[-\lambda\, |u|^{1/\tau}\, (1 \pm_u \theta i)\big].$$

Note that *if* ϕ is a characteristic function, which so far we only know when $\theta = 0$, then ϕ is infinitely divisible and hence $h \in \mathcal{L}_1$ and h is an eigenfunction of T; it then follows that also $h \in \mathcal{L}_2$, so ϕ is *self-decomposable.* In the next section these special self-decomposable characteristic functions will be recognized as being *stable* with *exponent* $1/\tau$.

7. Stability on the real line

We shall mainly be concerned with (strictly) stable distributions on $\mathbb{R}$ for reasons that have been made clear in Section 1; only at the end of the present section we briefly turn to the more general weakly stable distributions on $\mathbb{R}$. Recall from Section 1 that a random variable X is called *stable* if for every $n \in \mathbb{N}$ there exists $c_n > 0$ such that

$$X \stackrel{\text{d}}{=} c_n\,(X_1 + \cdots + X_n), \tag{7.1}$$

where $X_1, \ldots, X_n$ are independent with $X_i \stackrel{\text{d}}{=} X$ for all i. In terms of the characteristic function ϕ of X equation (7.1) reads as follows:

$$\phi(u) = \big\{\phi(c_n u)\big\}^n. \tag{7.2}$$

From (7.1) or (7.2) one immediately obtains the following important observation: *A stable characteristic function is infinitely divisible, and hence has no zeroes.* This result will be frequently and often tacitly used, as will be the following property of a general characteristic function ϕ: If $a > 0$ and $b > 0$, then

$$\phi(au) = \phi(bu) \ \text{ for all } u \in \mathbb{R} \implies a = b. \tag{7.3}$$

We are interested in a canonical representation for the stable characteristic functions and in a characterization of the stable distributions among the infinitely divisible and self-decomposable ones. In deriving these results we will only partly be able to proceed along the lines of Section 3 for the $\mathbb{R}_+$-case. Let ϕ be a stable characteristic function. Then from (7.2) it follows that for $m, n \in \mathbb{N}$

$$\phi(c_{mn}u) = \big\{\phi(c_n c_{mn}u)\big\}^n = \big\{\phi(c_m c_n c_{mn}u)\big\}^{mn} = \phi(c_m c_n u),$$

and hence, because of (7.3),

$$c_{mn} = c_m c_n \qquad [m, n \in \mathbb{N}]. \tag{7.4}$$

Now, in view of the observations above, in exactly the same way as in the proof of Theorem 3.1, we can generalize (7.2) and (7.4) as follows; there exists a continuous function $c : (0,\infty) \to (0,\infty)$ such that

$$\begin{cases} \phi(u) = \big\{\phi\big(c(x)u\big)\big\}^x \ \text{ for } x > 0, \\ c(xy) = c(x)\,c(y) \ \text{ for } x > 0 \text{ and } y > 0. \end{cases} \tag{7.5}$$

Moreover, such a function c necessarily has the form $c(x) = x^r$ for some $r \in \mathbb{R}$; and by letting $x \downarrow 0$ in the first part of (7.5) we see that r must be negative. Thus we can write $r = -1/\gamma$ for some $\gamma > 0$. This results in the direct part of the following basic theorem; the converse part follows by taking $x = n \in \mathbb{N}$.

Theorem 7.1. *A characteristic function ϕ is stable iff there exists $\gamma > 0$ such that*

$$(7.6) \qquad \phi(u) = \{\phi(x^{-1/\gamma}u)\}^x \qquad [x > 0].$$

For a stable characteristic function ϕ the positive constant γ for which (7.6) holds, is called the *exponent* (of stability) of ϕ (or of a corresponding random variable X). From Theorem 7.1 and its proof it will be clear that for $\gamma > 0$ a characteristic function ϕ is stable with exponent γ iff the constants c_n in (7.2) have the form $c_n = n^{-1/\gamma}$. This yields the following characterization in terms of random variables.

Corollary 7.2. *For $\gamma > 0$ a random variable X is stable with exponent γ iff for every $n \in \mathbb{N}$ it can be written as*

$$(7.7) \qquad X \stackrel{d}{=} n^{-1/\gamma}(X_1 + \cdots + X_n),$$

where $X_1, \ldots, X_n$ are independent with $X_i \stackrel{d}{=} X$ for all i.

From Theorem 7.1 it also easily follows that a stable random variable X with exponent γ satisfies

$$(7.8) \qquad (x+y)^{1/\gamma}X \stackrel{d}{=} x^{1/\gamma}X + y^{1/\gamma}X' \qquad [x > 0,\ y > 0],$$

where X' denotes a random variable with $X' \stackrel{d}{=} X$ and independent of X. Dividing both sides by $(x+y)^{1/\gamma}$ leads to the direct part of the following *characterization* result; the converse part follows, as in the $\mathbb{R}_+$-case, from Corollary 7.2 by using induction, and taking $\alpha = \{n/(n+1)\}^{1/\gamma}$ and $\beta = \{1/(n+1)\}^{1/\gamma}$ with $n \in \mathbb{N}$.

Theorem 7.3. *For $\gamma > 0$ a random variable X is stable with exponent γ iff, with X' as above, X can be written as*

$$(7.9) \qquad X \stackrel{d}{=} \alpha X + \beta X'$$

for all $\alpha, \beta \in (0,1)$ with $\alpha^\gamma + \beta^\gamma = 1$.

An immediate consequence of this theorem is the following important result; just use the definition of self-decomposability.

Theorem 7.4. *A stable distribution on $\mathbb{R}$ is self-decomposable.*

We proceed with showing how Theorem 7.1 gives rise to a *canonical representation* for the stable distributions on $\mathbb{R}$. We further exclude the degenerate distributions; they are stable with exponent $\gamma = 1$. Let $\gamma > 0$, and let ϕ be a stable characteristic function with exponent γ, so ϕ satisfies (7.6). Since by Theorems 7.4 and 6.14 ϕ corresponds to a non-degenerate self-decomposable, and hence absolutely continuous, distribution, we have, as is well known, $\phi(u) \neq 1$ for $u \neq 0$, so in particular $\phi(1) \neq 1$. Therefore, we can first take $u = 1$ in (7.6) and then $x = u^{-\gamma}$ with $u > 0$ to see that $\phi(u)$ with $u > 0$ satisfies

$$\log \phi(1) = u^{-\gamma} \log \phi(u), \quad \text{so } \phi(u) = \exp\bigl[-\zeta\, u^{\gamma}\bigr],$$

where we put $\zeta := -\log \phi(1)$, which is non-zero. Because $\phi(u) = \overline{\phi(-u)}$, it follows that $\phi(u)$ with $u < 0$ can be represented similarly; replace ζ by $\overline{\zeta}$ and u by $|u|$. Any function ϕ of the resulting form with $\zeta \in \mathbb{C} \setminus \{0\}$ obviously satisfies (7.6). But for ϕ to be an infinitely divisible characteristic function it is necessary that $|\phi|$ is a characteristic function as well; cf. Proposition IV.2.2. Now, $|\phi|$ is given by

$$\text{(7.10)} \qquad \bigl|\phi(u)\bigr| = \exp\bigl[-\lambda\, |u|^{\gamma}\bigr] \qquad [u \in \mathbb{R}],$$

where $\lambda := \operatorname{Re} \zeta$, and, as has been shown in Example IV.4.9, this function of u is a characteristic function iff $\lambda > 0$ and $\gamma \leq 2$; see also Example 9.20. Thus, putting $\theta := (\operatorname{Im} \zeta)/\lambda$, we arrive at the following result; here we set, as before, $\pm_u := +$ if $u \geq 0$ and $:= -$ if $u < 0$.

Proposition 7.5. *For $\gamma > 0$ a characteristic function ϕ corresponds to a non-degenerate stable distribution with exponent γ iff $\gamma \leq 2$ and ϕ has the form*

$$\text{(7.11)} \qquad \phi(u) = \exp\bigl[-\lambda\, |u|^{\gamma}\, (1 \pm_u \theta i)\bigr],$$

where $\lambda > 0$ and $\theta \in \mathbb{R}$.

This result is not as nice as it may seem. It does show that there are no stable distributions with exponent $\gamma > 2$, but for $\gamma \leq 2$ not all functions ϕ of the form (7.11) are characteristic functions; the set of *admissible* constants θ, i.e., the θ's for which (7.11) is a characteristic function, turns out to depend on γ. Of course, when we restrict attention to *symmetric* distributions, we must take $\theta = 0$; since, as noted above, this θ is admissible, Proposition 7.5 does yield a *canonical representation* for *symmetric stable* distributions on $\mathbb{R}$.

Theorem 7.6 (Canonical representation; symmetric case). *For $\gamma \in (0, 2\,]$ a $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of a symmetric stable distribution with exponent γ iff ϕ has the form*

$$\phi(u) = \exp\left[-\lambda\,|u|^{\gamma}\right] \qquad [\,u \in \mathbb{R}\,], \tag{7.12}$$

where $\lambda > 0$.

When $\gamma = 1$, the function ϕ in (7.12) describes the symmetric *Cauchy* distributions and, as $\pm_u 1 = u/|u|$ for $u \neq 0$, the more general ϕ in (7.11) can be rewritten as

$$\phi(u) = \exp\left[-\lambda\,|u| - iu\lambda\theta\right]; \tag{7.13}$$

so any $\theta \in \mathbb{R}$ is admissible and the set of non-degenerate stable distributions with exponent $\gamma = 1$ is given by the family of Cauchy distributions. In case $\gamma = 2$ only $\theta = 0$ is admissible, so the symmetric *normal* distributions are the only stable distributions with exponent $\gamma = 2$. This can be shown with some difficulty by inverting (7.11) with $\gamma = 2$, but we will not do so here; see, however, Theorem 7.13. Instead we first turn to the Lévy triple of a stable distribution, and then show how the resulting Lévy representation leads to a precise determination of the admissible θ and hence to a canonical representation for stable distributions also in the non-symmetric case.

In order to obtain the Lévy triple of a stable distribution we observe that, up to a $(-\log)$-transformation, the functions in Proposition 7.5 and those in Proposition 6.24 are exactly the same. This immediately leads to a characterization of the stable characteristic functions as in the theorem below; recall that $\mathcal{L}_1$ and $\mathcal{L}_2$ are the semi-linear spaces of functions of the form $-\log\phi$ with ϕ an infinitely divisible characteristic function with finite logarithmic moment and with ϕ a self-decomposable characteristic function, respectively.

Theorem 7.7. *For $\gamma \in (0, 2\,]$ a characteristic function ϕ without zeroes is stable with exponent γ iff the function $h = -\log\phi$ is an eigenfunction at the eigenvalue $1/\gamma$ of the semi-linear bijective mapping $T : \mathcal{L}_1 \to \mathcal{L}_2$ given by*

$$(Th)(u) := \int_0^u \frac{h(v)}{v}\,dv \qquad [\,h \in \mathcal{L}_1;\; u \in \mathbb{R}\,].$$

The mapping T has no other eigenvalues than $1/\gamma$ with $\gamma \in (0,2\,]$.

We next combine this theorem with Theorem 6.7 and recall that the second canonical triple of a self-decomposable characteristic function ϕ equals the Lévy triple of the underlying infinitely divisible characteristic function ϕ_0 of ϕ; by Proposition IV.4.5 (iii) we then arrive at the following preliminary result.

Proposition 7.8. *For $\gamma \in (0,2\,]$ a characteristic function ϕ is stable with exponent γ iff it is self-decomposable with underlying infinitely divisible characteristic function ϕ_0 given by $\phi_0 = \phi^\gamma$ or, equivalently, with second canonical triple (a_0, σ_0^2, M_0) given by*

(7.14) $$(a_0, \sigma_0^2, M_0) = (\gamma a, \gamma\sigma^2, \gamma M),$$

where (a, σ^2, M) is the Lévy triple of ϕ.

Equation (7.14) can be solved by using the relation between the triples (a, σ^2, M) and (a_0, σ_0^2, M_0) as given by Proposition 6.11, which shows that (7.14) is equivalent to

(7.15) $$\begin{cases} (\gamma - 1)\, a = 2\gamma \int_{\mathbb{R}\setminus\{0\}} \big\{x/(1+x^2)\big\}^2\, M(x)\, dx, \\ (\gamma - 2)\, \sigma^2 = 0, \\ m(x) = -\gamma\, M(x)/x \qquad [\,x \neq 0\,], \end{cases}$$

where m is a density of M. Thus we are led to the following *characterization* of the stable distributions among the infinitely divisible ones.

Theorem 7.9. *For $\gamma \in (0,2\,]$ a non-degenerate probability distribution on $\mathbb{R}$ is stable with exponent γ iff it is infinitely divisible with Lévy triple (a, σ^2, M) of the following form:*

(i) *If $\gamma = 2$, then $a = 0$, $\sigma^2 > 0$ and $M = 0$ everywhere on $\mathbb{R}\setminus\{0\}$.*

(ii) *If $\gamma = 1$, then $a \in \mathbb{R}$, $\sigma^2 = 0$, and M is absolutely continuous with density m, for some $c > 0$ given by*

$$m(x) = c\,\frac{1}{x^2} \qquad [\,x \neq 0\,].$$

(iii) *If* $\gamma \notin \{1, 2\}$, *then* $\sigma^2 = 0$, M *is absolutely continuous with density*

$$m(x) = \begin{cases} c_1\,|x|^{-1-\gamma} & \text{, if } x < 0, \\ c_2\,x^{-1-\gamma} & \text{, if } x > 0, \end{cases}$$

for some $c_1 \geq 0$ *and* $c_2 \geq 0$ *with* $c_1 + c_2 > 0$, *and* a *is given by*

$$a = \begin{cases} (c_2 - c_1)\int_0^\infty x^{-\gamma}/(1+x^2)\,dx & \text{, if } 0 < \gamma < 1, \\ (c_1 - c_2)\int_0^\infty x^{2-\gamma}/(1+x^2)\,dx & \text{, if } 1 < \gamma < 2. \end{cases}$$

Proof. Consider a non-degenerate stable distribution; then, as argued above, its Lévy triple (a, σ^2, M) satisfies (7.15). First, look at the equation for the density m; from this it follows that m has a continuous derivative on $\mathbb{R} \setminus \{0\}$ with

$$x\,m'(x) = (-1 - \gamma)\,m(x) \qquad [\,x \neq 0\,].$$

Clearly, the only solutions m that are continuous on $\mathbb{R} \setminus \{0\}$, are given by the functions m in (iii) with $c_1 \geq 0$ and $c_2 \geq 0$.

Now, let $\gamma = 2$. Then $c_1 = c_2 = 0$, because a function m with the property that $m(x) = 1/|x|^3$ for $x < 0$ or $x > 0$ cannot be a canonical density; it violates the condition $\int_{[-1,1]\setminus\{0\}} x^2\,m(x)\,dx < \infty$. So $M = 0$ everywhere, and hence by (7.15) also $a = 0$. So $\sigma^2 > 0$, and (i) is proved.

Next, let $\gamma = 1$. Then the second equation in (7.15) shows that $\sigma^2 = 0$, and the first one that $c_1 = c_2 =: c$, which is positive because we exclude degenerate distributions. Thus also (ii) is proved; cf. Example IV.4.7.

Finally, let $\gamma \notin \{1, 2\}$. Then again $\sigma^2 = 0$, and necessarily $c_1 > 0$ or $c_2 > 0$, so $c_1 + c_2 > 0$. Since $\gamma\,M(x) = c_1\,|x|^{-\gamma}$ if $x < 0$ and $= -c_2\,x^{-\gamma}$ if $x > 0$, the first equation in (7.15) shows that a satisfies

$$(\gamma - 1)\,a = 2(c_1 - c_2)\int_0^\infty \frac{x^{2-\gamma}}{(1+x^2)^2}\,dx.$$

Using integration by parts and viewing $2x/(1+x^2)^2$ as the derivative of $-1/(1+x^2)$ if $0 < \gamma < 1$ and of $1 - 1/(1+x^2) = x^2/(1+x^2)$ if $1 < \gamma < 2$, one easily shows that a can be written as in (iii).

Conversely, an infinitely divisible distribution with Lévy triple (a, σ^2, M) as given in (i), (ii) or (iii) is self-decomposable on account of Theorem 6.12, and by use of Proposition 6.11 its second canonical triple is easily shown to satisfy (7.14). From Proposition 7.8 we then obtain the desired stability of the distribution. ◻

This theorem enables us to rewrite the Lévy representation of a stable characteristic function such that it takes the form (7.11) for some $\gamma \in (0,2\,]$, $\lambda > 0$ and $\theta \in \mathbb{R}$; this will immediately yield the admissible θ. For instance, part (i) of the theorem shows that the stable characteristic functions with exponent $\gamma = 2$ are given by the functions ϕ of the form

$$\phi(u) = \exp\left[-\tfrac{1}{2}\sigma^2 u^2\right] \qquad [\,u \in \mathbb{R}\,] \tag{7.16}$$

with $\sigma^2 > 0$, so ϕ satisfies (7.11) with $\lambda = \frac{1}{2}\sigma^2$ and $\theta = 0$. When $\gamma = 1$, we already know from (7.13) that any $\theta \in \mathbb{R}$ is admissible, so we need not rewrite the Lévy representation in this case. But when we do so by using part (ii) of the theorem, as in the proof of Theorem 7.15 below, we can express (λ, θ) in terms of the constants $a \in \mathbb{R}$ and $c > 0$; we find:

$$\phi(u) = \exp\left[-c\pi|u| + iua\right], \tag{7.17}$$

and hence $\lambda = c\pi$ and $\theta = -a/(c\pi)$, which relations also easily follow from Example IV.4.7. In rewriting the Lévy representation when $\gamma \notin \{1,2\}$ we need the following technical lemma; see Section A.5.

Lemma 7.10. *In the following statement the first equality holds for every $\gamma \in (0,1)$ and the second one for every $\gamma \in (1,2)$:*

$$\int_0^\infty \frac{e^{it}-1}{t^{1+\gamma}}\,dt = -\frac{\Gamma(2-\gamma)}{\gamma\,(1-\gamma)}\,e^{-\frac{1}{2}\gamma\pi i} = \int_0^\infty \frac{e^{it}-1-it}{t^{1+\gamma}}\,dt.$$

Theorem 7.11 (Canonical representation; general case). *For $\gamma \in (0,2\,]$ a $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of a non-degenerate stable distribution with exponent γ iff ϕ has the form*

$$\phi(u) = \exp\left[-\lambda\,|u|^\gamma\,(1 \pm_u \theta i)\right], \tag{7.18}$$

where $\lambda > 0$ and where $\theta \in \mathbb{R}$ is such that

$$|\theta| \le |\tan\tfrac{1}{2}\gamma\pi| \qquad [:= \infty \ \text{ if } \gamma = 1\,]. \tag{7.19}$$

Proof. In view of the discussion above we only need consider $\gamma \notin \{1,2\}$. According to Theorem 7.9 the stable characteristic functions with exponent γ are then given by the functions ϕ that satisfy

$$\log\,\phi(u) = iua + c_1 \int_{-\infty}^0 \frac{I(u,x)}{|x|^{1+\gamma}}\,\mathrm{d}x + c_2 \int_0^\infty \frac{I(u,x)}{x^{1+\gamma}}\,\mathrm{d}x,$$

where $c_1 \geq 0$ and $c_2 \geq 0$ with $c_1 + c_2 > 0$ and a is determined by c_1 and c_2 as indicated in part (iii) of the theorem; here $I(u,x)$ is the 'Lévy kernel' as defined in Section 6. In showing that these functions ϕ are of the form (7.18), we only need consider $\phi(u)$ with $u > 0$ because $\phi(u) = \overline{\phi(-u)}$. Then one easily verifies that for $0 < \gamma < 1$

$$\int_{-\infty}^{0} \frac{I(u,x)}{|x|^{1+\gamma}}\,\mathrm{d}x = u^\gamma \int_0^\infty \frac{\mathrm{e}^{-it} - 1}{t^{1+\gamma}}\,\mathrm{d}t + iu \int_0^\infty \frac{x^{-\gamma}}{1+x^2}\,\mathrm{d}x,$$

$$\int_{0}^{\infty} \frac{I(u,x)}{x^{1+\gamma}}\,\mathrm{d}x = u^\gamma \int_0^\infty \frac{\mathrm{e}^{it} - 1}{t^{1+\gamma}}\,\mathrm{d}t - iu \int_0^\infty \frac{x^{-\gamma}}{1+x^2}\,\mathrm{d}x,$$

and for $1 < \gamma < 2$

$$\int_{-\infty}^{0} \frac{I(u,x)}{|x|^{1+\gamma}}\,\mathrm{d}x = u^\gamma \int_0^\infty \frac{\mathrm{e}^{-it} - 1 + it}{t^{1+\gamma}}\,\mathrm{d}t - iu \int_0^\infty \frac{x^{2-\gamma}}{1+x^2}\,\mathrm{d}x,$$

$$\int_{0}^{\infty} \frac{I(u,x)}{x^{1+\gamma}}\,\mathrm{d}x = u^\gamma \int_0^\infty \frac{\mathrm{e}^{it} - 1 - it}{t^{1+\gamma}}\,\mathrm{d}t + iu \int_0^\infty \frac{x^{2-\gamma}}{1+x^2}\,\mathrm{d}x.$$

Hence in both cases the linear term in $\log \phi(u)$ disappears, and because of Lemma 7.10

$$\log \phi(u) = -\frac{\Gamma(2-\gamma)}{\gamma\,(1-\gamma)}\, u^\gamma \big\{c_1\,\mathrm{e}^{\frac{1}{2}\gamma\pi i} + c_2\,\mathrm{e}^{-\frac{1}{2}\gamma\pi i}\big\} =$$

$$= -\frac{\Gamma(2-\gamma)}{\gamma\,(1-\gamma)}\, u^\gamma \big\{(c_1 + c_2)\cos\tfrac{1}{2}\gamma\pi + i\,(c_1 - c_2)\sin\tfrac{1}{2}\gamma\pi\big\}.$$

This shows that ϕ is of the form (7.18) with λ and θ given by

$$\text{(7.20)} \qquad \lambda = (c_1 + c_2)\,\frac{\Gamma(2-\gamma)}{\gamma\,(1-\gamma)}\,\cos\tfrac{1}{2}\gamma\pi, \quad \theta = \frac{c_1 - c_2}{c_1 + c_2}\,\tan\tfrac{1}{2}\gamma\pi.$$

It follows that $\lambda > 0$ and, since by the nonnegativity of c_1 and c_2 we have $|c_1 - c_2| \leq c_1 + c_2$, that θ satisfies the inequality (7.19). Moreover, any such λ and θ can be obtained by choosing c_1 and c_2 appropriately. □

Sometimes it is convenient to write the canonical representation (7.18) for $\gamma \notin \{1,2\}$ in the following form:

$$\text{(7.21)} \qquad \phi(u) = \exp\big[-\lambda\,|u|^\gamma\,(1 \pm_u i\,\beta\,\tan\tfrac{1}{2}\gamma\pi)\big],$$

where $\lambda > 0$ and $\beta \in \mathbb{R}$ with $|\beta| \leq 1$. In fact, by (7.20) we have

$$\text{(7.22)} \qquad \beta = \frac{c_1 - c_2}{c_1 + c_2},$$

where c_1 and c_2 are the nonnegative constants that determine both the Lévy density m and the Lévy constant a as indicated in Theorem 7.9 (iii).

Combining this with Theorem IV.4.13 leads to the following identification of the *one-sided* stable distributions among the general ones; cf. Theorem 3.5.

Proposition 7.12. *Let $\gamma \in (0,2\,]$, and let F be a non-degenerate distribution function with characteristic function ϕ that is stable with exponent γ. Then $\ell_F > -\infty$ iff $\gamma < 1$ and, in representation (7.21) for ϕ, $\beta = -1$. In this case $\ell_F = 0$ and the pLSt π of F is given by*

$$(7.23) \qquad \pi(s) = \exp\left[-\lambda_0\, s^{\gamma}\right] \quad \textit{with } \lambda_0 := \lambda \big/ \cos \tfrac{1}{2}\gamma\pi.$$

Similarly, $r_F < \infty$ iff $\gamma < 1$ and $\beta = 1$, in which case $r_F = 0$.

PROOF. Let $\gamma \notin \{1,2\}$, otherwise $\ell_F = -\infty$. Then F has canonical triple $(a,0,M)$ with a and M as in Theorem 7.9 (iii). By Theorem IV.4.13 $\ell_F > -\infty$ iff $M(0-) = 0$ and $\int_{(0,1]} x \,\mathrm{d}M(x) < \infty$. This means that $c_1 = 0$ and $\int_{(0,1]} x^{-\gamma}\,\mathrm{d}x < \infty$, i.e., $\beta = -1$ and $\gamma < 1$. In this case $\ell_F = 0$ because of the expression for ℓ_F given by Theorem IV.4.13. Moreover, $\phi(u)$ in (7.21) with $u > 0$ can then be rewritten, with λ_0 as in (7.23):

$$\begin{aligned} -\log \phi(u) &= \lambda_0\, u^{\gamma} \,(\cos \tfrac{1}{2}\gamma\pi - i \sin \tfrac{1}{2}\gamma\pi) = \lambda_0\, u^{\gamma}\, \mathrm{e}^{-\frac{1}{2}\gamma\pi i} = \\ &= \lambda_0 \exp\left[\gamma\left\{\log|-iu| + i \arg(-iu)\right\}\right] = \lambda_0\,(-iu)^{\gamma}. \end{aligned}$$

Since this function of u is analytic on the set of $u \in \mathbb{C}$ for which $\mathrm{Re}\,(-iu) > 0$, i.e., for which $\mathrm{Im}\, u > 0$, we can take $u = is$ with $s > 0$ to get the corresponding pLSt π; this results in (7.23). ◻◻

We now briefly look at the stable *distributions* themselves, rather than to their transforms or canonical triples. Let X be a non-degenerate stable random variable with exponent $\gamma < 2$. Then by combining Theorems IV.7.3 and 7.9 one immediately sees that for $r > 0$

$$(7.24) \qquad \mathbb{E}\,|X|^r < \infty \iff r < \gamma.$$

Next, since X is self-decomposable, we can apply Theorems 6.14, 6.21 and 6.23 to conclude that X has an absolutely continuous, unimodal distribution with a density that is continuous 'outside the mode'. This density, however, may be taken to be continuous everywhere because by Theorem 7.11 the corresponding characteristic function ϕ is absolutely integrable; cf. Section A.2. Moreover, by Fourier inversion one obtains the following integral representation for the density, also when $\gamma = 2$.

Theorem 7.13. *For $\gamma \in (0,2]$ a non-degenerate stable distribution with exponent γ is absolutely continuous and unimodal with a bounded continuous density which, up to a scale transformation, is of the form f_θ with*

$$f_\theta(x) = \frac{1}{\pi} \int_0^\infty e^{-u^\gamma} \cos(xu + \theta u^\gamma)\, du \qquad [x \in \mathbb{R}], \tag{7.25}$$

where $\theta \in \mathbb{R}$ is such that $|\theta| \le |\tan \frac{1}{2}\gamma\pi|$.

PROOF. Since λ in (7.18) is a scale parameter, we may take $\lambda = 1$. As noted in (A.2.15), the bounded continuous density is then of the form f_θ with

$$\begin{aligned} 2\pi\, f_\theta(x) &= \int_{-\infty}^\infty e^{-iux} \phi(u)\, du = \\ &= \int_{-\infty}^0 e^{-iux}\, e^{-|u|^\gamma(1-i\theta)}\, du + \int_0^\infty e^{-iux}\, e^{-u^\gamma(1+i\theta)}\, du = \\ &= \int_0^\infty e^{-u^\gamma} \big\{ e^{i(xu+\theta u^\gamma)} + e^{-i(xu+\theta u^\gamma)} \big\}\, du, \end{aligned}$$

which reduces to (7.25); here $\theta \in \mathbb{R}$ satisfies (7.19). □

Whereas there are simple explicit expressions for the characteristic function and the canonical density of a stable distribution, its density itself seems to be generally intractable. Theorem 7.13, together with Proposition 7.5 and Theorem 7.11, implies the following curiosity for integrable functions of the form f_θ in (7.25) with $\theta \in \mathbb{R}$:

$$f_\theta(x) \ge 0 \text{ for all } x \in \mathbb{R} \iff |\theta| \le |\tan \tfrac{1}{2}\gamma\pi|; \tag{7.26}$$

this result seems very hard to prove directly.

Finally, we pay some attention to the, more general, weakly stable distributions on $\mathbb{R}$. Recall from Section 1 that a random variable X is said to be *weakly stable* if for every $n \in \mathbb{N}$ it can be written as

$$X \stackrel{d}{=} c_n (X_1 + \cdots + X_n) + d_n, \tag{7.27}$$

where $c_n > 0$, $d_n \in \mathbb{R}$, and $X_1, \ldots, X_n$ are independent with $X_i \stackrel{d}{=} X$ for all i. Putting $X_s := X - X'$ with X and X' independent and $X' \stackrel{d}{=} X$, we see that if X satisfies (7.27) then so does X_s with $d_n = 0$, so if X is weakly stable then X_s is (strictly) stable with the same norming constants c_n.

It follows that also in weak stability we have $c_n = n^{-1/\gamma}$ with *exponent* $\gamma \in (0, 2]$. Since for $\gamma \neq 1$ weakly stable random variables and stable ones only differ by a shift (see Section 1), we restrict attention to the case $\gamma = 1$. So, let X be non-degenerate and satisfy

$$(7.28) \qquad X \stackrel{d}{=} \frac{1}{n}(X_1 + \cdots + X_n) + d_n \qquad [n \in \mathbb{N}],$$

where $d_n \in \mathbb{R}$. Then for the characteristic function ϕ of X we have

$$\phi(u) = \{\phi(u/n)\}^n \, \mathrm{e}^{iud_n} \qquad [n \in \mathbb{N}],$$

and repeated use of this shows that the sequence (d_n) satisfies

$$d_{mn} = d_m + d_n \qquad [m, n \in \mathbb{N}].$$

In a similar way as (c_n) was extended to a continuous function c on $(0, \infty)$, the sequence (d_n) can be extended to a continuous function d on $(0, \infty)$ such that

$$(7.29) \qquad \begin{cases} \phi(u) = \{\phi(u/t)\}^t \, \mathrm{e}^{iud(t)} & \text{for } t > 0, \\ d(st) = d(s) + d(t) & \text{for } s > 0 \text{ and } t > 0. \end{cases}$$

Now, the only continuous functions d satisfying the second part of (7.29) are those of the form $d(t) = c \log t$ for $t > 0$ and some $c \in \mathbb{R}$. So for ϕ we obtain

$$(7.30) \qquad \phi(u) = \{\phi(u/t)\}^t \, \mathrm{e}^{iuc \log t} \qquad [t > 0].$$

Taking here first $u = 1$ and then setting $t = 1/u$, we see that for $u > 0$

$$\phi(u) = \{\phi(1)\}^u \, \mathrm{e}^{iuc \log u} = \exp[-\lambda u + i\mu u + ic\, u \log u],$$

where we put $\phi(1) = \exp[-\lambda + i\mu]$, with $\lambda > 0$ as necessarily $|\phi(u)| < 1$. Using the fact that $\phi(u) = \overline{\phi(-u)}$ we conclude that the characteristic function ϕ of a non-degenerate weakly stable distribution with exponent $\gamma = 1$ has the form

$$(7.31) \qquad \phi(u) = \exp\Big[i\mu u - \lambda|u| \left(1 \pm_u i\,\theta \log|u|\right)\Big],$$

where $\mu \in \mathbb{R}$, $\lambda > 0$ and $\theta \in \mathbb{R}$. Clearly, any function ϕ of this form is weakly stable *if* it is a characteristic function, and we have to establish for what values of θ this is the case.

To this end we return to the mapping T considered in Theorem 7.7, and we look for functions h satisfying

$$(7.32) \qquad (Th)(u) = h(u) + icu$$

for some $c \in \mathbb{R}$. It is easily verified that $h = -\log \phi$ with ϕ a characteristic function of the form (7.31) satisfies this relation with $c = -\lambda\theta$. Conversely, if (7.32) holds for $h = -\log \phi$ with ϕ a characteristic function without zeroes, then we can differentiate to obtain $h(u) = u\,h'(u) + icu$, and hence $h''(u) = -ic/u$, which implies that for $u > 0$ and some $\delta \in \mathbb{C}$

$$\begin{aligned} h(u) &= -ic\,\{u \log u - u\} + \delta u = \\ &= -ic\,u \log u + i\,(c + \nu)\,u + \lambda u, \end{aligned}$$

where we put $\delta = \lambda + i\nu$, with $\lambda > 0$ as necessarily $|\phi(u)| < 1$; so, by varying c, for $\phi = \exp[-h]$ we obtain exactly the functions of the form (7.31). We conclude that a characteristic function ϕ without zeroes is weakly stable with exponent $\gamma = 1$ iff the function $h = -\log \phi$ satisfies (7.32) for some $c \in \mathbb{R}$ or, equivalently, iff ϕ is *self-decomposable* with underlying infinitely divisible characteristic function ϕ_0 given by $\phi_0(u) = \phi(u)\,e^{icu}$ for some $c \in \mathbb{R}$; here we used Theorem 6.7. Now, proceed as in the stable case and use Proposition 6.11; then it follows that the weakly stable distributions with exponent $\gamma = 1$ correspond to the *infinitely divisible* distributions with Lévy triple $(a, 0, M)$ such that M is absolutely continuous with density m satisfying

$$(7.33) \qquad m(x) = -\frac{1}{x}\,M(x) \qquad [x \neq 0].$$

Solving this equation, as in the beginning of the proof of Theorem 7.9, leads to the following result.

Theorem 7.14. *A non-degenerate probability distribution on $\mathbb{R}$ is weakly stable with exponent $\gamma = 1$ iff it is infinitely divisible with Lévy triple $(a, 0, M)$ where $a \in \mathbb{R}$ and M is absolutely continuous with density m given by*

$$(7.34) \qquad m(x) = \begin{cases} c_1\,x^{-2} & , \text{ if } x < 0, \\ c_2\,x^{-2} & , \text{ if } x > 0, \end{cases}$$

for some constants $c_1 \geq 0$ and $c_2 \geq 0$ satisfying $c_1 + c_2 > 0$.

This theorem enables us to rewrite the Lévy representation of a weakly stable characteristic function with $\gamma = 1$ such that it takes the form (7.31) for some $\mu \in \mathbb{R}$, $\lambda > 0$, $\theta \in \mathbb{R}$; this will immediately yield the admissible θ's. In doing so we need the following technical lemma; see Section A.5 for the second part.

Lemma 7.15. *For $u > 0$ the following equalities hold:*

$$A(u) := \int_0^\infty \frac{1 - \cos ux}{x^2}\, dx = \tfrac{1}{2}\pi u;$$

$$B(u) := \int_0^\infty \Big(\sin ux - \frac{ux}{1 + x^2}\Big) \frac{1}{x^2}\, dx = b\, u - u \log u,$$

with $b := B(1) = 1 - \gamma$, where γ is Euler's constant.

We thus arrive at the following *canonical representation* for weakly stable characteristic functions with exponent $\gamma = 1$.

Theorem 7.16. *A $\mathbb{C}$-valued function ϕ on $\mathbb{R}$ is the characteristic function of a non-degenerate weakly stable distribution with exponent $\gamma = 1$ iff ϕ has the form*

(7.35) $$\phi(u) = \exp\Big[iu\mu - \lambda\,|u|\,\big(1 \pm_u i\,\theta \log|u|\big)\Big],$$

where $\mu \in \mathbb{R}$ and $\lambda > 0$, and where $\theta \in \mathbb{R}$ is such that

(7.36) $$|\theta| \le \frac{2}{\pi}.$$

Proof. According to Theorem 7.14 the characteristic functions of the non-degenerate weakly stable distributions with $\gamma = 1$ are given by the functions ϕ that satisfy

$$\log\, \phi(u) = iua + c_1 \int_{-\infty}^0 \frac{I(u,x)}{x^2}\, dx + c_2 \int_0^\infty \frac{I(u,x)}{x^2}\, dx,$$

where $a \in \mathbb{R}$, and $c_1 \ge 0$ and $c_2 \ge 0$ satisfy $c_1 + c_2 > 0$; here $I(u,x)$ is the 'Lévy kernel'. In showing that these functions ϕ are of the form (7.35), we only need consider $\phi(u)$ with $u > 0$ because $\phi(u) = \overline{\phi(-u)}$. Then using Lemma 7.15 (and its notation) we see that

$$\log\, \phi(u) = iua - c_1\,\big\{A(u) + i\,B(u)\big\} - c_2\,\big\{A(u) - i\,B(u)\big\} =$$

$$= iua - (c_1 + c_2)\,\tfrac{1}{2}\pi u - i\,(c_1 - c_2)\,\{b\,u - u\,\log\, u\}.$$

This shows that ϕ is of the form (7.35) with μ, λ and θ given by

$$\mu = a - (c_1 - c_2)\, b, \quad \lambda = \tfrac{1}{2}\pi\,(c_1 + c_2), \quad \theta = -\frac{2}{\pi}\,\frac{c_1 - c_2}{c_1 + c_2}. \tag{7.37}$$

It follows that $\mu \in \mathbb{R}$, $\lambda > 0$ and, since by the nonnegativity of c_1 and c_2 we have $|c_1 - c_2| \le c_1 + c_2$, that θ satisfies the inequality (7.36). Moreover, any such μ, λ, θ can be obtained by choosing a, c_1, c_2 appropriately. □

Sometimes it is convenient to write the canonical representation (7.35) in the following form:

$$\phi(u) = \exp\Big[iu\mu - \lambda\,|u|\,\big(1 \pm_u i\,\beta\,(2/\pi)\log|u|\big)\Big], \tag{7.38}$$

where $\mu \in \mathbb{R}$, $\lambda > 0$ and $\beta \in \mathbb{R}$ with $|\beta| \le 1$. In fact, by (7.37) we have

$$\beta = -\frac{c_1 - c_2}{c_1 + c_2}, \tag{7.39}$$

where c_1 and c_2 are the nonnegative constants that determine the Lévy density m as indicated in Theorem 7.14. Note that this β has the opposite sign of β in (7.22). Further note that $\beta = 0$ iff $c_1 = c_2$, so by Theorems 7.9 (ii) and 7.14 a non-degenerate weakly stable distribution with exponent $\gamma = 1$ is (strictly) stable iff $\beta = 0$; this also follows by comparing representations (7.13) and (7.38).

8. Self-decomposability and stability induced by semigroups

The results obtained in Sections 4 and 5 concerning self-decomposability and stability on $\mathbb{Z}_+$ can be generalized by using a multiplication more general than the *standard* one used in (4.1) and (5.1). In order to show this we observe that this standard multiplication can be rewritten in the following way:

$$\alpha \odot X = Z_X(t) \qquad [0 < \alpha < 1;\ t = -\log\alpha], \tag{8.1}$$

where $Z_X(\cdot)$ is a continuous-time pure-death process starting with X individuals at time zero. This means that each of these individuals has probability e^{-t} of surviving during a period of length t, so $Z_X(t)$ is the number of survivors at time t. It is now easy to generalize (8.1) by replacing the death process by a more general continuous-time Markov *branching process* $Z_X(\cdot)$ with $Z_X(0) = X$. As is well known, such a branching process is governed by a family $\mathcal{F} = (F_t)_{t \geq 0}$ of pgf's that is a *composition semigroup*:

$$(8.2) \qquad F_{s+t} = F_s \circ F_t \qquad [\, s, t \geq 0\,];$$

in fact, F_t is the pgf of $Z_1(t)$, and the transition matrix $\big(p_{ij}(t)\big)$ of the Markov process is determined by $\mathcal{F}$ as follows: $\sum_{j=0}^{\infty} p_{ij}(t)\, z^j = \big\{F_t(z)\big\}^i$. The resulting multiplication will be denoted by $\odot_{\mathcal{F}}$; now (8.1), with $\odot$ replaced by $\odot_{\mathcal{F}}$, can be written in terms of pgf's as

$$(8.3) \qquad P_{\alpha \odot_{\mathcal{F}} X}(z) = P_X\big(F_t(z)\big) \qquad [\,0 < \alpha < 1;\, t = -\log\, \alpha\,].$$

Note that the standard multiplication is obtained by using the semigroup $\mathcal{F}$ given by $F_t(z) = 1 - \mathrm{e}^{-t} + \mathrm{e}^{-t}\, z$ for $t \geq 0$.

The multiplications of the form (8.3) turn out to be the only ones satisfying certain minimal requirements. To show this we denote the mapping $P_X \mapsto P_{\alpha \odot X}$ by T_α, where now $\odot$ is any reasonable discrete multiplication, i.e., a multiplication mimicking the ordinary multiplication αX. Therefore, the following five properties are indispensible for the operators T_α with $\alpha \in (0, 1)$:

$$(8.4) \qquad \begin{cases} T_\alpha P \;\text{ is a pgf}, \\ T_\alpha(P_1 P_2) = (T_\alpha P_1)\,(T_\alpha P_2), \\ T_\alpha(p\, P_1 + (1-p)\, P_2) = p\, T_\alpha P_1 + (1-p)\, T_\alpha P_2 \;\text{ for } 0 < p < 1, \\ T_\alpha \;\text{ is continuous: } P = \lim_{n \to \infty} P_n \implies T_\alpha P = \lim_{n \to \infty} T_\alpha P_n, \\ T_\alpha T_\beta = T_{\alpha\beta}. \end{cases}$$

Now, it is not difficult to show that $(T_\alpha)_{0<\alpha<1}$ satisfies (8.4) iff T_α operates on pgf's P in the following way:

$$(8.5) \qquad T_\alpha P = P \circ F_t \qquad [\,0 < \alpha < 1;\, t := -\log\, \alpha\,],$$

where $\mathcal{F} = (F_t)_{t>0}$ is a composition semigroup of pgf's. Since such semigroups need not be very well-behaved, we add the following two regularity conditions which, like (8.4), are very natural for analogues of scalar multiplication:

$$(8.6) \qquad \lim_{\alpha \uparrow 1} T_\alpha P = P, \quad (T_\alpha P)'(1) = \alpha\, P'(1) \;\text{ for } 0 < \alpha < 1.$$

For the corresponding semigroup $\mathcal{F}$, i.e., $\mathcal{F}$ given by (8.5), this means that

$$\lim_{t\downarrow 0} F_t(z) = z, \quad F_t'(1) = \mathrm{e}^{-t} \text{ for } t > 0. \tag{8.7}$$

By putting $F_0(z) := z$ it follows that $\mathcal{F} = (F_t)_{t\geq 0}$ is a *continuous* semigroup; it is continuous at $t = 0$, and hence at all $t \geq 0$ because of the semigroup property. Similarly, the second property in (8.7) is equivalent to just requiring that $F_1'(1) = 1/\mathrm{e}$; it implies that the corresponding branching process $Z_1(\cdot)$ is *subcritical*, i.e., $\mathbb{E}Z_1(t) = m^t$ with $m < 1$, and hence

$$\lim_{t\to\infty} F_t(z) = 1. \tag{8.8}$$

So we only consider multiplications of the form (8.3) with $\mathcal{F} = (F_t)_{t\geq 0}$ a continuous composition semigroup of pgf's satisfying $F_1'(1) = 1/\mathrm{e}$. Before turning to self-decomposability and stability according to these multiplications we need some more information about $\mathcal{F}$, such as can be found in the literature on branching processes. First of all, it can be shown that $F_t(z)$ is a differentiable function of $t \geq 0$ with

$$\frac{\partial}{\partial t} F_t(z) = U\big(F_t(z)\big) = U(z)\, F_t'(z) \qquad [\,t \geq 0;\ 0 \leq z \leq 1\,], \tag{8.9}$$

where U is the infinitesimal *generator* of $\mathcal{F}$ defined by

$$U(z) := \frac{\partial}{\partial t} F_t(z)\Big|_{t=0} = \lim_{t\downarrow 0} \frac{F_t(z) - z}{t} \qquad [\,0 \leq z \leq 1\,]; \tag{8.10}$$

U is continuous with $U(z) > 0$ for $0 \leq z < 1$ and $U(z) \sim 1 - z$ as $z \uparrow 1$. We further need the function A with

$$A(z) := \exp\Big[-\int_0^z \frac{1}{U(x)}\,\mathrm{d}x\Big] = 1 - B(z) \qquad [\,0 \leq z \leq 1\,], \tag{8.11}$$

where $B(z) := \lim_{t\to\infty} \{F_t(z) - F_t(0)\}/\{1 - F_t(0)\}$ is a pgf with $B(0) = 0$. Moreover, from the first equality in (8.9) it follows that A satisfies

$$A\big(F_t(z)\big) = \mathrm{e}^{-t} A(z) \qquad [\,t \geq 0;\ 0 \leq z \leq 1\,]. \tag{8.12}$$

Unfortunately, there are very few semigroups $\mathcal{F}$ with explicit expressions for F_t, U and A. We give two examples where such expressions are available.

Example 8.1. For $t \geq 0$ let the function F_t on $[\,0, 1\,]$ be given by

$$F_t(z) = 1 - \mathrm{e}^{-t}\,(1 - z) = 1 - \mathrm{e}^{-t} + \mathrm{e}^{-t} z \qquad [\,0 \leq z \leq 1\,].$$

Clearly, $\mathcal{F} = (F_t)_{t \geq 0}$ is a continuous composition semigroup of pgf's with $F_1'(1) = 1/\mathrm{e}$; its generator U and function A in (8.11) are given by

$$U(z) = A(z) = 1 - z.$$

This semigroup induces the *standard* multiplication used before. ◫

Example 8.2. For $t \geq 0$ let the function F_t on $[0,1]$ be given by

$$F_t(z) = 1 - \frac{2\,\mathrm{e}^{-t}\,(1-z)}{2 + (1-\mathrm{e}^{-t})\,(1-z)} = (1-\gamma_t) + \gamma_t\, z\, \frac{1-p_t}{1-p_t z},$$

where $\gamma_t := 2\,\mathrm{e}^{-t}/(3-\mathrm{e}^{-t})$ and $p_t := \frac{1}{3}\,(1-\gamma_t)$. From the second expression for F_t it follows that F_t is a pgf with $F_t'(1) = \mathrm{e}^{-t}$, and from the first one that $\mathcal{F} = (F_t)_{t \geq 0}$ is a continuous composition semigroup with U and A given by

$$U(z) = \tfrac{1}{2}\,(1-z)\,(3-z), \quad A(z) = 3\,\frac{1-z}{3-z}.$$

Note that $B = 1 - A$ is the pgf of the geometric $(\frac{1}{3})$ distribution on $\mathbb{N}$. ◫

Let $\mathcal{F}$ be a semigroup and $\odot_{\mathcal{F}}$ the corresponding multiplication as considered above. Then a $\mathbb{Z}_+$-valued random variable X is said to be *$\mathcal{F}$-self-decomposable* if for every $\alpha \in (0,1)$ it can be written as

$$X \stackrel{\mathrm{d}}{=} \alpha \odot_{\mathcal{F}} X + X_\alpha, \tag{8.13}$$

where in the right-hand side the random variables $\alpha \odot_{\mathcal{F}} X$ and X_α are independent. In terms of pgf's relation (8.13) becomes

$$P(z) = P\big(F_t(z)\big)\, P_t(z), \tag{8.14}$$

where we have put P for P_X, P_t for P_{X_α} and $t = -\log \alpha$. As in the standard case (cf. Proposition 4.1) one shows that an $\mathcal{F}$-self-decomposable pgf P satisfies $P(0) > 0$. Moreover, we can derive a *representation theorem* analogous to Theorem 4.11.

Theorem 8.3. *A function P on $[0,1]$ is the pgf of an $\mathcal{F}$-self-decomposable distribution on $\mathbb{Z}_+$ iff it has the form*

$$P(z) = \exp\Big[\int_z^1 \frac{\log P_0(x)}{U(x)}\,dx\Big] \qquad [0 \leq z \leq 1], \tag{8.15}$$

where U is the generator of $\mathcal{F}$ and P_0 is the pgf of an infinitely divisible random variable X_0, for which necessarily $\mathbb{E}\log(X_0+1) < \infty$.

PROOF. Let P be an $\mathcal{F}$-self-decomposable pgf with factors P_t as in (8.14). In order to show that P has the form (8.15), we proceed as in (6.4) for the $\mathbb{R}_+$-case and as in the first part of the proof of Theorem 6.7 for the $\mathbb{R}$-case. By the first property of $\mathcal{F}$ in (8.7) and the definition of U in (8.10) the derivative P' of P can be obtained in the following way:

$$P'(z) = \lim_{t\downarrow 0} \frac{P\big(F_t(z)\big) - P(z)}{F_t(z) - z} = \frac{P(z)}{U(z)} \lim_{t\downarrow 0} \frac{1 - P_t(z)}{t}. \tag{8.16}$$

It follows that

$$P_0(z) := \exp\big[-U(z)\, P'(z)/P(z)\big] = \lim_{t\downarrow 0} \exp\big[-t^{-1}\{1 - P_t(z)\}\big], \tag{8.17}$$

so the function P_0 is the limit of compound-Poisson pgf's. Now, since $U(z) \sim 1 - z$ and $(1 - z)\, P'(z) \to 0$ as $z \uparrow 1$, P_0 satisfies $P_0(1-) = 1$. Hence from the continuity theorem it follows that P_0 is a pgf, and from Proposition II.2.2 that it is infinitely divisible. Next, we express P in terms of P_0; rewriting (8.17) yields

$$\frac{\mathrm{d}}{\mathrm{d}z} \log P(z) = \frac{-\log P_0(z)}{U(z)} \qquad [0 \le z < 1],$$

which implies the desired representation (8.15). Finally, the logarithmic moment condition on P_0 follows from the fact that the integral in (8.15) is finite; since $U(z) \sim 1 - z$ as $z \uparrow 1$, this concerns a general property of pgf's, which is proved in Proposition A.4.2.

Conversely, suppose that P is of the form (8.15) with P_0 infinitely divisible, so the R-function R_0 of P_0 is absolutely monotone. We now proceed as in the second part of the proof of Theorem 4.8. Because of (8.7), (8.8) and the second equality in (8.9), the R-function of P can be written as

$$R(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log P(z) = \frac{-\log P_0(z)}{U(z)} = \frac{1}{U(z)} \int_z^1 R_0(x)\, \mathrm{d}x =$$
$$= \int_0^\infty R_0\big(F_s(z)\big)\, F_s'(z)\, \mathrm{d}s = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}z} \log P_0\big(F_s(z)\big)\, \mathrm{d}s,$$

so R is a mixture of absolutely monotone functions, and hence is itself absolutely monotone. From Theorem II.4.3 it follows that P is a pgf and, in fact, an infinitely divisible pgf. To show that P is $\mathcal{F}$-self-decomposable, we take $t > 0$ and consider the function P_t defined by (8.14). By the semigroup property of $\mathcal{F}$ the R-function R_t of P_t can be written as

$$R_t(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \frac{P(z)}{P\big(F_t(z)\big)} = R(z) - R\big(F_t(z)\big)\, F_t'(z) =$$

$$= \int_0^t R_0\big(F_s(z)\big)\, F_s'(z)\,\mathrm{d}s = \int_0^t \frac{\mathrm{d}}{\mathrm{d}z} \log\, P_0\big(F_s(z)\big)\,\mathrm{d}s.$$

As above for P, we conclude that P_t is a pgf and, in fact, an infinitely divisible pgf, so P is $\mathcal{F}$-self-decomposable. ◻◻

The expressions for R and R_t in the preceding proof suggest similar expressions for P and P_t; in fact, using the semigroup property of $\mathcal{F}$ again together with (8.8), and now the first equality in (8.9) rather than the second one, one easily proves the following additional result.

Corollary 8.4. *Let P be an $\mathcal{F}$-self-decomposable pgf. Then P is infinitely divisible, and its representation (8.15) can be rewritten as*

$$P(z) = \exp\Big[\int_0^\infty \log\, P_0\big(F_s(z)\big)\, ds\Big]. \tag{8.18}$$

Moreover, the factors P_t of P in (8.14) are infinitely divisible, and

$$P_t(z) = \exp\Big[\int_0^t \log\, P_0\big(F_s(z)\big)\, ds\Big] \qquad [\, t > 0\,]. \tag{8.19}$$

The expression for R in the proof of Theorem 8.3 also shows that $R(0) > 0$. On the other hand, $R(0) = p_1/p_0$, if $p = (p_k)_{k\in\mathbb{Z}_+}$ is the distribution with pgf P. Therefore, Corollary II.8.3 implies the following property.

Corollary 8.5. *The support $S(p)$ of an $\mathcal{F}$-self-decomposable distribution p on $\mathbb{Z}_+$ is equal to $\mathbb{Z}_+$.*

In Chapter VII we will return to the $\mathcal{F}$-self-decomposable distributions on $\mathbb{Z}_+$, and show that they correspond to limit distributions of subcritical $\mathcal{F}$-branching processes with immigration.

We turn to stability with respect to the semigroup $\mathcal{F}$. A $\mathbb{Z}_+$-valued random variable X is said to be $\mathcal{F}$-*stable* if for every $n \in \mathbb{N}$ there exists $c_n \in (0, 1\,]$ such that

$$X \overset{\mathrm{d}}{=} c_n \odot_{\mathcal{F}} (X_1 + \cdots + X_n), \tag{8.20}$$

where $X_1, \ldots, X_n$ are independent with $X_i \overset{\mathrm{d}}{=} X$ for all i. In terms of the pgf P of X equation (8.20) reads as follows:

$$P(z) = \big\{P\big(F_{-\log c_n}(z)\big)\big\}^n, \tag{8.21}$$

so an $\mathcal{F}$-stable distribution on $\mathbb{Z}_+$ is *infinitely divisible* (with positive mass at zero). It is not hard to verify, as in the standard case of Section 5, that for an $\mathcal{F}$-stable pgf P the constants c_n have to be of the form $c_n = n^{-1/\gamma}$ with $\gamma > 0$ and that equation (8.21) for P with $n \in \mathbb{N}$ generalizes to

$$(8.22) \qquad P(z) = \left\{P\big(F_{(\log x)/\gamma}(z)\big)\right\}^x \qquad [x \geq 1],$$

in which case P is called $\mathcal{F}$-stable *with exponent* γ; we omit the details. Now we are ready to prove the following analogue of Theorem 5.5.

Theorem 8.6. *For $\gamma > 0$ a function P on $[0,1]$ is the pgf of an $\mathcal{F}$-stable distribution on $\mathbb{Z}_+$ with exponent γ iff $\gamma \leq 1$ and P has the form*

$$(8.23) \qquad P(z) = \exp\left[-\lambda\, A(z)^\gamma\right],$$

where $\lambda > 0$ and A is determined by $\mathcal{F}$ as in (8.11).

Proof. Let P be an $\mathcal{F}$-stable pgf with exponent γ, so P satisfies (8.22). Differentiation in this equation leads to

$$\lim_{z\uparrow 1} \frac{P'(z)}{P'\big(F_{(\log x)/\gamma}(z)\big)} = x\, F'_{(\log x)/\gamma}(1-) = x^{1-1/\gamma} \qquad [x \geq 1].$$

Now, this limit does not exceed 1, because P' is nondecreasing and $F_t(z) \geq z$ for all t and z; hence $1 - 1/\gamma \leq 0$, so $\gamma \leq 1$. Next, proceeding as in the standard case, we first take $z = 0$ in (8.22) and then $x \geq 1$ such that $F_{(\log x)/\gamma}(0) = z$ with $0 \leq z < 1$; this can be done because by (8.7), (8.8) and (8.9) the function $t \mapsto F_t(0)$ is continuous and increasing on $\mathbb{R}_+$ with $F_t(0) \to 0$ as $t \downarrow 0$ and $F_t(0) \to 1$ as $t \to \infty$. We thus obtain

$$\log P(0) = x \log P(z), \quad \text{so} \quad P(z) = \exp\left[-\lambda\, x^{-1}\right],$$

where we put $\lambda := -\log P(0)$, which is positive. In order to express x in z we use relation (8.12) for A with $z = 0$; then we get $A\big(F_t(0)\big) = \mathrm{e}^{-t}$ for $t \geq 0$, and hence for x and z as above

$$A(z) = \mathrm{e}^{-(\log x)/\gamma} = x^{-1/\gamma}, \quad \text{so} \quad x = A(z)^{-\gamma}.$$

Inserting this in the expression for $P(z)$ above yields representation (8.23). Conversely, let P be of the form (8.23) with $\lambda > 0$ and $\gamma \leq 1$. Then P can be written as

$$(8.24) \qquad P(z) = \pi\big(A(z)\big), \quad \text{with} \quad \pi(s) := \exp\left[-\lambda\, s^\gamma\right],$$

i.e., π is a stable pLSt with exponent γ; cf. Theorem 3.5. Since by (8.11) we have $A = 1-B$ with B a pgf, from (II.3.7) it follows that P is a pgf. Finally, using (8.12) one easily verifies that P satisfies (8.22), so P is $\mathcal{F}$-stable with exponent γ. ◻

From Theorem 8.6 and the last part of its proof, especially (8.24), it follows that $\mathcal{F}$-stability of a pgf P can be characterized by means of stability on $\mathbb{R}_+$:

(8.25) $$P \ \mathcal{F}\text{-stable} \iff P = \pi \circ A \ \text{ with } \pi \text{ a stable pLSt.}$$

Since an LSt is determined by its values on a finite interval, we can reverse matters and characterize stability of a pLSt π by means of $\mathcal{F}$-stability:

(8.26) $$\pi \text{ stable} \iff P := \pi \circ A \ \ \mathcal{F}\text{-stable.}$$

Similarly, using just the definitions and (8.12) one easily shows that

(8.27) $$\pi \text{ self-decomposable} \implies P := \pi \circ A \ \ \mathcal{F}\text{-self-decomposable,}$$

but the converse of this is *not* true; see Section 9 for a counter-example and Section VI.6 for an adapted converse, both in the standard case. Moreover, not every $\mathcal{F}$-self-decomposable pgf is of the form $\pi \circ A$ with π a self-decomposable pLSt; see Section 9 for an example. Since on $\mathbb{R}_+$ stability implies self-decomposability, by combining (8.25) and (8.27) we obtain the following counterpart on $\mathbb{Z}_+$.

Theorem 8.7. *An $\mathcal{F}$-stable distribution is $\mathcal{F}$-self-decomposable.*

As in the standard case one shows that the $\mathcal{F}$-stable pgf's with exponent γ correspond to the $\mathcal{F}$-self-decomposable pgf's P for which the underlying infinitely divisible pgf P_0 in Theorem 8.3 is given by $P_0 = P^\gamma$. Letting the classes $\mathcal{L}_1$ and $\mathcal{L}_2$ be as before, we can reformulate this as follows.

Theorem 8.8. *For $\gamma \in (0,1]$ a pgf P is $\mathcal{F}$-stable with exponent γ iff the function $h := -\log P$ is an eigenfunction at the eigenvalue $1/\gamma$ of the semi-linear bijective mapping $T : \mathcal{L}_1 \to \mathcal{L}_2$ given by*

(8.28) $$(Th)(z) = \int_z^1 \frac{h(x)}{U(x)}\,dx \qquad [\,h \in \mathcal{L}_1;\ 0 \le z \le 1\,].$$

The mapping T has no other eigenvalues than $1/\gamma$ with $\gamma \in (0,1]$.

The theory developed in this section for distributions on $\mathbb{Z}_+$ can be mimicked for distributions on $\mathbb{R}_+$. In order to do this the multiplication of an $\mathbb{R}_+$-valued random variable X by a scalar is generalized in a similar way. We define $\alpha \odot_{\mathcal{C}} X$ by

$$(8.29) \qquad \alpha \odot_{\mathcal{C}} X := Z_X(t) \qquad [\,0 < \alpha < 1;\, t = -\log\,\alpha\,],$$

where $Z_X(\cdot)$ is a continuous-time Markov *branching process* having values in $\mathbb{R}_+$ with $Z_X(0) = X$ and $\mathcal{C} = (C_t)_{t\geq 0}$ the composition semigroup of cumulant generating functions that governs $Z_X(\cdot)$:

$$(8.30) \qquad C_t = -\log\,\eta_t, \text{ with } \eta_t \text{ the pLSt of } Z_1(t) \qquad [\,t \geq 0\,].$$

Here η_t is necessarily *infinitely divisible* because the branching property implies that for $a > 0$ the a-th power η_t^a of η_t is the pLSt of $Z_a(t)$. Again, we assume continuity of $\mathcal{C}$ and $-\eta_1'(0) = 1/\mathrm{e}$. In terms of pLSt's the multiplication reads

$$(8.31) \qquad \pi_{\alpha \odot_{\mathcal{C}} X}(s) = \pi_X\big(C_t(s)\big) \qquad [\,0 < \alpha < 1;\, t = -\log\,\alpha\,].$$

Note that the ordinary multiplication is obtained by using the semigroup $\mathcal{C}$ with

$$(8.32) \qquad C_t(s) = \mathrm{e}^{-t}s, \text{ so } \eta_t(s) = \exp\big[-\mathrm{e}^{-t}s\big] \qquad [\,t \geq 0\,].$$

Starting from (8.31) the theory can be developed almost verbatim as in the discrete case. We return to this briefly in Chapter VII.

9. Examples

Apart from the *stable* distributions, the only explicit self-decomposable distributions we have seen so far, are the *gamma* distributions on $\mathbb{R}_+$, the *negative-binomial* distributions on $\mathbb{Z}_+$ and the *sym-gamma* distributions on $\mathbb{R}$. We now examine the possible self-decomposability of some other infinitely divisible distributions from the previous chapters, using criteria developed in the present chapter. Also some examples with special features and curiosities are presented.

We start with a series of examples of distributions on $\mathbb{R}_+$. The first three concern *completely monotone* densities; some of these densities turn out to be self-decomposable, others do not. The fourth example exhibits another simple class of infinitely divisible distributions not all of which are self-decomposable.

Example 9.1. For $\lambda > 0$ the probability density f_λ on $(0, \infty)$ given by

$$f_\lambda(x) = \tfrac{1}{2}\,\mathrm{e}^{-x} + \tfrac{1}{2}\,\lambda\,\mathrm{e}^{-\lambda\,x} \qquad [\,x > 0\,],$$

is *completely monotone* and hence infinitely divisible by Theorem III.10.7. Is it also self-decomposable? The Lt π_λ of f_λ can be written as

$$\pi_\lambda(s) = \tfrac{1}{2}\,\frac{1}{1+s} + \tfrac{1}{2}\,\frac{\lambda}{\lambda+s} = \frac{1}{1+s}\,\frac{\lambda}{\lambda+s}\Big/\frac{\mu}{\mu+s},$$

with $\mu := 2\lambda/(\lambda+1)$. Since from Example III.4.8 it is known that an *exponential* (λ) distribution has canonical density k given by $k(x) = \mathrm{e}^{-\lambda\,x}$ for $x > 0$, it follows, as in Example III.11.1, that for the canonical density k_λ of f_λ we have

$$k_\lambda(x) = \mathrm{e}^{-x} + \mathrm{e}^{-\lambda\,x} - \mathrm{e}^{-\mu\,x} \qquad [\,x > 0\,],$$

and the question is whether k_λ is nonincreasing; cf. Theorem 2.11. As $\lambda \to \infty$, however, we have $\pi_\lambda(s) \to \tfrac{1}{2}\,\big\{1/(1+s)\big\} + \tfrac{1}{2}$; this function of s is a pLSt, but corresponds to a distribution that is not absolutely continuous. From Proposition 2.3 and Theorem 2.16 it follows that π_λ can*not* be self-decomposable for arbitrarily large λ. For definiteness, it is easily verified that k_3 is nonincreasing, whereas k_{20} is not. So f_3 is *self-decomposable*, but f_{20} is *not*, though both are completely monotone. ◻◻

Example 9.2. Let π be the function on $\mathbb{R}_+$ from Example III.11.9, i.e.,

$$\pi(s) = 1 + s - \sqrt{(1+s)^2 - 1} \qquad [\,s \geq 0\,].$$

There it was noted that π is the Lt of a *completely monotone* density f and that its ρ-function is given by

$$\rho(s) = \frac{1}{\sqrt{(1+s)^2 - 1}} = \frac{1}{\sqrt{s}}\,\frac{1}{\sqrt{2+s}} \qquad [\,s > 0\,],$$

so ρ is completely monotone; hence π is an infinitely divisible pLSt. In order to determine whether π is self-decomposable, we compute its ρ_0-function and find

$$\rho_0(s) := \frac{\mathrm{d}}{\mathrm{d}s}\big[s\,\rho(s)\big] = \frac{1}{\sqrt{s}}\,\Big(\frac{1}{2+s}\Big)^{3/2} \qquad [\,s > 0\,].$$

Clearly, ρ_0 is completely monotone, so by Theorem 2.6 the pLSt π is *self-decomposable*. To show this one can also use Theorem 2.11 and prove that the canonical density k, as given in Example III.11.9, is monotone. In Example VII.7.1 the pLSt π will be interpreted as the transform of a first-passage time. ◻

Example 9.3. Let π be the function on $(0,\infty)$ from Example III.11.10, i.e.,

$$\pi(s) = 2\,\frac{\sqrt{1+s}-1}{s} = \frac{2}{1+\sqrt{1+s}} \qquad [\,s>0\,].$$

With some difficulty it was established that π is the Lt of a probability density f on $(0,\infty)$ given by

$$f(x) = \frac{1}{\sqrt{\pi}} \int_x^\infty y^{-3/2}\,\mathrm{e}^{-y}\,\mathrm{d}y \qquad [\,x>0\,],$$

so f is *completely monotone*; hence π is an infinitely divisible pLSt. In order to determine whether π is self-decomposable, we look at the canonical function K of π; its LSt $\widehat{K}$ is given by

$$\widehat{K}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log \pi(s) = \tfrac{1}{2}\,\frac{1}{1+s+\sqrt{1+s}}.$$

Expanding $\widehat{K}(s)$ in a series of powers of $(1+s)^{-1/2}$ and using Laplace inversion leads to a formula for a density k of K in exactly the same way as the formula for f was obtained; we find:

$$k(x) = \frac{1}{2\sqrt{\pi}} \int_x^\infty y^{-1/2}\,\mathrm{e}^{-y}\,\mathrm{d}y \qquad [\,x>0\,],$$

so k is nonincreasing. From Theorem 2.11 it follows that π is *self-decomposable*. A more direct way of proving this, is observing that π can be written as

$$\pi(s) = \frac{\pi_1(1+s)}{\pi_1(1)}, \quad \text{with } \pi_1(s) := \frac{1}{1+\sqrt{s}} = \pi_0\big(\sqrt{s}\big),$$

where π_0 is the pLSt of the standard exponential distribution. Now apply Proposition 2.14 (ii), (iii). ◻

Example 9.4. Let f_λ be the probability density on $(0,\infty)$ given by

$$f_\lambda(x) = \tfrac{1}{2}\, x\, \mathrm{e}^{-x} + \tfrac{1}{2}\, \lambda\, \mathrm{e}^{-\lambda\, x} \qquad [\, x > 0\,],$$

where $\lambda > 0$. Then its Lt π_λ can be written as

$$\begin{aligned}\pi_\lambda(s) &= \tfrac{1}{2}\Big(\frac{1}{1+s}\Big)^2 + \tfrac{1}{2}\,\frac{\lambda}{\lambda+s} = \\ &= \Big(\frac{1}{1+s}\Big)^2 \frac{\lambda}{\lambda+s}\Big/\Big(\frac{\mu}{\mu+s}\,\frac{\nu}{\nu+s}\Big),\end{aligned}$$

where $\mu := \big(1+2\lambda+\sqrt{d_\lambda}\big)/(2\lambda)$ and $\nu := \big(1+2\lambda-\sqrt{d_\lambda}\big)/(2\lambda)$ with d_λ given by $d_\lambda := 1+4\lambda-4\lambda^2$, so $\nu = \overline{\mu}$ if $d_\lambda < 0$. As in Examples 9.1 and III.11.1 it follows that the ρ-function of π_λ is the Lt of the function k_λ given by

$$k_\lambda(x) = 2\,\mathrm{e}^{-x} + \mathrm{e}^{-\lambda\, x} - \mathrm{e}^{-\mu\, x} - \mathrm{e}^{-\nu\, x} \qquad [\, x > 0\,],$$

which can be seen to be nonnegative, also when μ and ν are not real; we will return to this in Section VI.4. We conclude that f_λ is *infinitely divisible.* Is it also self-decomposable? If so, then f_λ should be nonincreasing because of Theorem 2.17 and the fact that $k_\lambda(0+) = 1$. Now, $f_\lambda(0+) = \frac{1}{2}\lambda \in (0,\infty)$, and

$$f_\lambda'(x) = \tfrac{1}{2}\,\mathrm{e}^{-x}\,\big(1 - x - \lambda^2\,\mathrm{e}^{-(\lambda-1)\,x}\big) \qquad [\, x > 0\,];$$

by letting $x \downarrow 0$ it follows that f_λ is *not* self-decomposable when $\lambda < 1$. On the other hand, f_1 is nonincreasing and, as is easily verified, k_1 is nonincreasing as well. So Theorem 2.11 implies that f_1 is *self-decomposable.* Finally, we can proceed as in Example 9.1; letting $\lambda \to \infty$ shows that f_λ cannot be self-decomposable for arbitrarily large λ. For instance, from the expression for $f_\lambda'(x)$ above it follows that $f_{10}'(x) > 0$ for $x = 0.7$, so f_{10} is not monotone, and hence *not* self-decomposable. ◻◻

We proceed with three examples around stable distributions on $\mathbb{R}_+$; the third one is just a curiosity. Recall from Theorem 3.5 that the stable pLSt's with exponent $\gamma \in (0,1\,]$ are given by the functions π of the form $\pi(s) = \exp\,[-\lambda\, s^\gamma]$ with $\lambda > 0$; this π will be referred to as stable$\,(\lambda)$.

Example 9.5. Let π be the pLSt that is *stable*$\,(2)$ with exponent $\gamma = \frac{1}{2}$:

$$\pi(s) = \exp\big[-2\sqrt{s}\,\big].$$

According to Theorem 3.9 the distribution corresponding to π has a continuous unimodal density f with $f(0+) = 0$. We shall compute f by using Theorems 2.16 and 3.6; the canonical density k of π is given by $k(x) = 1/\sqrt{\pi x}$ for $x > 0$, so f satisfies

$$x\, f(x) = \frac{1}{\sqrt{\pi}} \int_0^x f(u)\, \frac{1}{\sqrt{x-u}}\, \mathrm{d}u \qquad [\, x > 0\,].$$

To solve this functional equation we put $g(y) := y^{-3/2} f(1/y)$ for $y > 0$; by use of the substitution $u = 1/(y+t)$ it follows that g satisfies

$$g(y) = \frac{1}{\sqrt{\pi}} \int_0^\infty g(y+t)\, \frac{1}{\sqrt{t}}\, \mathrm{d}t \qquad [\, y > 0\,].$$

Now, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, it is easily seen that any function g of the form $g(y) = c\, \mathrm{e}^{-y}$ with $c \in \mathbb{R}_+$ provides a solution of this equation. For f, which has to integrate to one, this implies that

$$f(x) = \frac{1}{\sqrt{\pi}}\, x^{-3/2}\, \mathrm{e}^{-1/x} \qquad [\, x > 0\,].$$

This is one of the very few explicitly known stable densities.

We make some further remarks. Since stability implies self-decomposability, we can use the closure property of Proposition 2.14 (ii) to conclude that the function π_1 defined by

$$\pi_1(s) = \exp\big[2\,(1 - \sqrt{s+1})\big] \qquad [\, s \geq 0\,],$$

is a *self-decomposable* pLSt with corresponding density f_1 given by

$$f_1(x) = \frac{\mathrm{e}^2}{\sqrt{\pi}}\, x^{-3/2}\, \mathrm{e}^{-(x+1/x)} \qquad [\, x > 0\,].$$

By computing the ρ-function of π_1 one sees that π_1 has the standard gamma $(\frac{1}{2})$ probability density as a (nonincreasing) canonical density. The distribution with density f_1 is sometimes called *inverse-Gaussian*. For a more natural form of inversion we return to stability and mention the following property: If X is stable (λ) with exponent $\gamma = \frac{1}{2}$, then

$$X \stackrel{\mathrm{d}}{=} \frac{1}{U^2}, \quad \text{with } U \text{ normal}\,(0, 2/\lambda^2).$$

This is easily verified (for $\lambda = 2$) by using the expression for f above. Note that U^2 has a gamma $(\frac{1}{2}, \frac{1}{4}\lambda^2)$ distribution, so a stable distribution with exponent $\gamma = \frac{1}{2}$ can be viewed as an *inverse-gamma* distribution; cf. Section B.3. ◻◻

Example 9.6. Since stability implies self-decomposability, we can use the closure property of Proposition 2.2 to conclude that functions π of the form

$$\pi(s) = \exp\Big[-\sum_{j=1}^{n} \lambda_j\, s^{\gamma_j}\Big] \qquad [\,s \geq 0\,],$$

where $\lambda_j > 0$, $\gamma_j \in (0,1\,]$ and $n \in \mathbb{N}$, are *self-decomposable* pLSt's. For the underlying infinitely divisible pLSt π_0 of π we have

$$\pi_0(s) = \exp\Big[s\,\frac{\mathrm{d}}{\mathrm{d}s}\log\,\pi(s)\Big] = \exp\Big[-\sum_{j=1}^{n} \lambda_j \gamma_j\, s^{\gamma_j}\Big],$$

so π_0 has the same form as π; it is not only infinitely divisible, but even *self-decomposable.* Similarly, the factors π_α of π with $\alpha \in (0,1)$ can be written as

$$\pi_\alpha(s) = \frac{\pi(s)}{\pi(\alpha s)} = \exp\Big[-(1-\alpha)\sum_{j=1}^{n} \lambda_j\, s^{\gamma_j}\Big],$$

so the π_α are *self-decomposable* as well. This phenomenon (of self-decomposability of all factors) is called *multiple self-decomposability.* In the special case at hand we even have *infinite self-decomposability*; the procedure can be repeated indefinitely. This example illustrates Theorem 2.15. □

Example 9.7. Let Y and Z be independent $\mathbb{R}_+$-valued random variables, and suppose that Y is *stable* (λ) with exponent γ and Z is *stable* (μ) with exponent δ. Consider X such that

$$X \stackrel{\mathrm{d}}{=} Y\, Z^{1/\gamma}.$$

Then the pLSt of X can be computed as follows:

$$\pi_X(s) = \int_{\mathbb{R}_+^2} \exp\big[-s\,y\,z^{1/\gamma}\big]\, f_Y(y)\, f_Z(z)\,\mathrm{d}y\,\mathrm{d}z =$$

$$= \int_{\mathbb{R}_+} \pi_Y(s\,z^{1/\gamma})\, f_Z(z)\,\mathrm{d}z = \int_{\mathbb{R}_+} \exp\big[-\lambda\, s^\gamma z\big]\, f_Z(z)\,\mathrm{d}z =$$

$$= \pi_Z(\lambda\, s^\gamma) = \exp\big[-(\lambda^\delta \mu)\, s^{\gamma\delta}\big],$$

so X is *stable* with exponent $\gamma\delta$. □

The final example of a distribution on $\mathbb{R}_+$ shows that a pLSt of the form $\pi_1 \circ (-\log \pi_0)$ with π_0 and π_1 self-decomposable pLSt's need *not* be self-decomposable; it is infinitely divisible, of course.

Example 9.8. Consider the function π on $\mathbb{R}_+$ defined by

$$\pi(s) = \frac{1}{1 + \log(1+s)} \qquad [\, s \geq 0\,].$$

Clearly, π is a compound-exponential pLSt and hence *infinitely divisible*; cf. Section III.3. Moreover, the compound-exponentiality implies that the corresponding distribution is absolutely continuous with density f given by

$$f(x) = \frac{1}{x}\, \mathrm{e}^{-x} \int_0^\infty \frac{1}{\Gamma(t)}\, (x/\mathrm{e})^t \, \mathrm{d}t \qquad [\, x > 0\,].$$

In order to determine whether π is self-decomposable, we consider the canonical function K of π and compute its LSt:

$$\widehat{K}(s) = -\frac{\mathrm{d}}{\mathrm{d}s} \log \pi(s) = \frac{1}{1+s}\, \pi(s).$$

It follows that K is absolutely continuous with density k given by

$$k(x) = \mathrm{e}^{-x} \int_0^x \mathrm{e}^u\, f(u)\, \mathrm{d}u \qquad [\, x > 0\,].$$

Now, letting $x \downarrow 0$, we see that $k(0+) = 0$, so k cannot be monotone. From Theorem 2.11 we conclude that π is *not* self-decomposable. ◻◻

We next turn to infinitely divisible distributions on $\mathbb{Z}_+$, and start with an analogue of the example just given and of Example 9.1.

Example 9.9. For $p \in (0,1)$ consider the function P on $[\,0,1\,]$ defined by

$$P(z) = \frac{1}{\lambda + \log(1 - p\,z)} \qquad [\, 0 \leq z \leq 1\,],$$

where $\lambda := 1 - \log(1-p)$, so $\lambda > 1$. Clearly, P is a compound-exponential pgf and hence *infinitely divisible*; cf. Section II.3. Moreover, the compound-exponentiality implies that the corresponding distribution $(p_k)_{k\in\mathbb{Z}_+}$ can be written as

$$p_k = p^k \int_0^\infty \binom{t+k-1}{k}\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t \qquad [\, k \in \mathbb{Z}_+\,].$$

For $k \in \mathbb{N}$ let $c_{k1}, \ldots, c_{kk} \in \mathbb{N}$ be such that $t\,(t+1) \cdots (t+k-1) = \sum_{j=1}^{k} c_{kj}\, t^j$; then using the fact that $\int_0^\infty t^j\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t = j!/\lambda^{j+1}$ for $j \in \mathbb{Z}_+$, we see that (p_k) can be rewritten as

$$p_0 = \frac{1}{\lambda}, \qquad p_k = \frac{p^k}{k!} \sum_{j=1}^{k} c_{kj}\, \frac{j!}{\lambda^{j+1}} \quad \text{for } k \in \mathbb{N}.$$

In order to determine whether P is self-decomposable, we compute its R-function:

$$R(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log P(z) = \frac{p}{1-p\,z}\, P(z).$$

Rewriting this as $(1-p\,z)\, R(z) = p\, P(z)$ shows that the canonical sequence $(r_k)_{k \in \mathbb{Z}_+}$ of (p_k) satisfies

$$r_0 = p\, p_0, \qquad r_k - p\, r_{k-1} = p\, p_k \quad \text{for } k \in \mathbb{N},$$

and the question is whether (r_k) is nonincreasing; cf. Theorem 4.13. If this is the case, then by Theorem 4.20 (p_k) is monotone; note that $r_0 \le 1$. Now, there is a partial converse: If (p_k) is monotone and $r_0 \ge r_1$, then (r_k) is nonincreasing; this follows by noting that the monotonicity of (p_k) implies that $r_k - p\, r_{k-1} \ge r_{k+1} - p\, r_k$ for $k \in \mathbb{N}$, i.e.,

$$r_k - r_{k+1} \ge p\, (r_{k-1} - r_k) \qquad [\, k \in \mathbb{N}\,].$$

So, let us determine whether (p_k) is monotone. Note that $p_0 \ge p_1$ because $r_0 = p_1/p_0$ and $r_0 \le 1$. Using the expressions for p_k above one sees that p_{k+1} with $k \in \mathbb{N}$ can be estimated as follows:

$$\begin{aligned} p_{k+1} &= \frac{p^{k+1}}{(k+1)!} \int_0^\infty \sum_{j=1}^{k} c_{kj}\, t^j\, (t+k)\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t = \\ &= \frac{p^{k+1}}{(k+1)!} \sum_{j=1}^{k} c_{kj} \Big(\frac{(j+1)!}{\lambda^{j+2}} + k\, \frac{j!}{\lambda^{j+1}} \Big) \le p \Big(\frac{1}{\lambda} + 1 \Big)\, p_k. \end{aligned}$$

Now, since $r_1 = p\, r_0 + p\, p_1 = p\, r_0\, (1 + p_0)$, it follows that (p_k) satisfies the following inequality (also when $k = 0$):

$$p_{k+1} \le (r_1/r_0)\, p_k \qquad [\, k \in \mathbb{Z}_+\,],$$

so (p_k) is monotone if $r_0 \ge r_1$. Combining this with the partial converse above we conclude that (r_k) is nonincreasing iff $r_0 \ge r_1$ or, equivalently,

$$p \le (1-p)\, \big(1 - \log\,(1-p)\big),$$

i.e., $p \le p^*$ with $p^* \approx 0.6822$. It follows that P is *self-decomposable* iff $p \le p^*$. For instance, taking $p = \frac{1}{2}$ leads to the self-decomposability of the pgf P given by

$$P(z) = \frac{1}{1 + \log(2 - z)}.$$

Note that this P is of the form $P(z) = \pi(1 - z)$ with π the pLSt from the preceding example, which is *not* self-decomposable; cf. (8.27). On the other hand, taking $p = \frac{3}{4}$ leads to a pgf P that is *not* self-decomposable, though P is of the form $P = \pi \circ (-\log P_0)$ with π a pLSt and P_0 a pgf, both self-decomposable. ◫

Example 9.10. From Example 4.6 we know that the *geometric* (p) distribution is *self-decomposable*. This is reflected by its canonical sequence $(r_k)_{k \in \mathbb{Z}_+}$, which is given by $r_k = p^{k+1}$ and hence is nonincreasing; cf. Example 4.16 and Theorem 4.13. What about *mixtures* of geometric distributions? Such mixtures are *completely monotone*, and hence infinitely divisible by Theorem II.10.4. For fixed $p, q \in (0, 1)$ let us consider the pgf P_α defined by

$$P_\alpha(z) = \alpha\, \frac{1 - q}{1 - q\, z} + (1-\alpha)\, \frac{1 - p}{1 - p\, z},$$

where $\alpha \in (0, 1)$. Then P_α can be written as

$$P_\alpha(z) = \frac{1 - q}{1 - q\, z}\, \frac{1 - p}{1 - p\, z} \Big/ \frac{1 - r_\alpha}{1 - r_\alpha z},$$

where r_α satisfies $(1 - r_\alpha)\, \{\alpha\, (1 - q) + (1-\alpha)\, (1 - p)\} = (1 - q)\, (1 - p)$; note that $r_\alpha \in (0, 1)$. It follows that the canonical sequence $\big(r_k(\alpha)\big)$ of P_α is given by

$$r_k(\alpha) = q^{k+1} + p^{k+1} - r_\alpha^{k+1} \qquad [\, k \in \mathbb{Z}_+\,].$$

Now, if $\alpha = \frac{1}{2}$, then similarly to Example 9.1 it can be shown that $\big(r_k(\alpha)\big)$ is nonincreasing, so that P_α is *self-decomposable*, if p and q are not too far apart, but not nonincreasing if one is much smaller than the other, in which case P_α is *not* self-decomposable. To be more concrete, let $q \downarrow 0$, so consider

$$P_\alpha(z) = \alpha + (1-\alpha)\, \frac{1 - p}{1 - p\, z}.$$

Note that now P_α is precisely the factor 'of order α' of the geometric (p) distribution; cf. Example 4.6. Hence, as observed just before Theorem 4.18, P_α is *self-decomposable* for all α iff $p \le \frac{1}{2}$. So we further let $p > \frac{1}{2}$. Because of Theorem 4.13 P_α is self-decomposable iff for every $k \in \mathbb{N}$

$$p^k - r_\alpha^k \ge p^{k+1} - r_\alpha^{k+1}, \text{ i.e., } (1-r_\alpha)\,(r_\alpha/p)^k \le 1-p.$$

Observing that $r_\alpha/p = \alpha/(1-p+\alpha p) < 1$, we conclude that P_α is *self-decomposable* iff α satisfies $\alpha \le \{(1-p)/p\}^2$ or, equivalently, p satisfies $p \le 1/(1+\sqrt{\alpha})$. ◻◻

In the next three examples some more *completely monotone* distributions occur, which are well known from Chapter II, viz. the *logarithmic-series* distribution, the distribution of the 'reduced' *first-passage time* $\frac{1}{2}(T_1 - 1)$ from 0 to 1 in the symmetric Bernoulli walk, and the *Borel* distribution.

Example 9.11. For $p \in (0,1)$ let the distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ and its pgf P be given by

$$p_k = c_p\,\frac{p^{k+1}}{k+1}, \quad P(z) = c_p\,\frac{-\log(1-pz)}{z},$$

where $c_p := 1/\{-\log(1-p)\}$. Then (p_k) is *completely monotone* and hence infinitely divisible; cf. Example II.11.7. To show that it is also self-decomposable, we use the fact, which was mentioned but not proved in Chapter II, that a completely monotone distribution on $\mathbb{Z}_+$ is *compound-geometric*; according to Theorem II.5.2 this means that the function S defined by

$$S(z) = \frac{1}{z}\left\{1 - \frac{p_0}{P(z)}\right\} \qquad [0 \le z < 1],$$

is absolutely monotone. Now, compute the R-function of P:

$$R(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log P(z) = \frac{1}{1-p\,z}\,\frac{p_0}{z\,P(z)} - \frac{1}{z} = \frac{p - S(z)}{1-p\,z}.$$

Rewrite this as $p - (1-pz)R(z) = S(z)$; then it follows that the canonical sequence $(r_k)_{k \in \mathbb{Z}_+}$ of (p_k) satisfies

$$p\,r_{k-1} - r_k = s_k \qquad [k \in \mathbb{N}],$$

where s_k is the coefficient of z^k in the power-series expansion of S. By the absolute monotonicity of S we conclude that (r_k) is nonincreasing, so (p_k) is *self-decomposable* by Theorem 4.13. ◻◻

Example 9.12. For $\gamma \in (0,1]$ let P be the function on $[0,1)$ given by

$$P(z) = \frac{1}{1+(1-z)^\gamma} \qquad [0 \le z < 1].$$

Since $P(z) = \pi\big((1-z)^\gamma\big)$ with π the pLSt of the standard exponential distribution, from Proposition 4.19 it follows that P is a *self-decomposable* pgf. When $\gamma = \frac{1}{2}$, the corresponding distribution $(p_k)_{k\in\mathbb{Z}_+}$ is given by

$$p_k = \frac{1}{k+1}\binom{2k}{k}\left(\tfrac{1}{2}\right)^{2k+1} \qquad [k \in \mathbb{Z}_+],$$

it is *completely monotone*, and its canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ is not only monotone but even completely monotone; see Example II.11.11. ◻

Example 9.13. For $\lambda \in (0,1]$ let $(p_k)_{k\in\mathbb{Z}_+}$ be the distribution on $\mathbb{Z}_+$ with

$$p_k = e^{-\lambda}\,(\lambda\,e^{-\lambda})^k\,\frac{(k+1)^{k-1}}{k!} \qquad [k \in \mathbb{Z}_+].$$

Then (p_k) is *infinitely divisible*, even *completely monotone*, and its canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ is given by $r_k = \lambda\,(k+1)\,p_k$; this was proved in Example II.11.16. It follows that

$$r_k/r_{k-1} = \lambda\,e^{-\lambda}\,(1+1/k)^k \qquad [k \in \mathbb{N}];$$

since $\lambda\,e^{-\lambda} \le 1/e$ and $(1+1/k)^k \uparrow e$ as $k \to \infty$, we conclude that (r_k) is nonincreasing, so (p_k) is *self-decomposable* by Theorem 4.13. According to Proposition 4.17 (i), for $a > 0$ the distribution $(p_k^{*a})_{k\in\mathbb{Z}_+}$ is *self-decomposable* as well; by applying 'Bürmann-Lagrange' (see Section A.5) with $G = G_\lambda$, the Poisson (λ) pgf (as in Example II.11.16), and $H = G_\lambda^a = G_{a\lambda}$, it is seen that

$$p_k^{*a} = a\,e^{-a\lambda}\,(\lambda\,e^{-\lambda})^k\,\frac{(k+a)^{k-1}}{k!} \qquad [k \in \mathbb{Z}_+].$$

Writing $a = \theta/\lambda$ with $\theta > 0$ we get

$$p_k^{*(\theta/\lambda)} = \theta\,e^{-\theta}\,e^{-\lambda k}\,\frac{(\lambda k+\theta)^{k-1}}{k!} \qquad [k \in \mathbb{Z}_+],$$

which can be considered as a *generalized-Poisson* distribution; let $\lambda \downarrow 0$. Finally we note that (p_k^{*a}) is *log-convex* iff $a \le 2$. ◻

The last example of a distribution on $\mathbb{Z}_+$ concerns generalized self-decomposability as considered in Section 8.

Example 9.14. Let $\mathcal{F}$ be the semigroup from Example 8.2, so with U- and A-functions given by

$$U(z) = \tfrac{1}{2}\,(1-z)\,(3-z), \quad A(z) = 3\,\frac{1-z}{3-z}.$$

According to (8.27) $\mathcal{F}$-self-decomposable pgf's can be constructed from self-decomposable pLSt's: If π is a self-decomposable pLSt, then $\pi \circ A$ is an $\mathcal{F}$-self-decomposable pgf. For instance, taking for π the pLSt of the standard exponential distribution, one obtains the *$\mathcal{F}$-self-decomposability* of the pgf P_1 given by

$$P_1(z) = \frac{1}{1 + A(z)} = \tfrac{1}{4} + \tfrac{3}{4}\,P_2(z),$$

with P_2 the pgf of the *geometric* $(\frac{2}{3})$ distribution. On the other hand, not every $\mathcal{F}$-self-decomposable pgf is of the form $\pi \circ A$ with π a self-decomposable pLSt. To show this we recall from Theorem 8.3 that the $\mathcal{F}$-self-decomposable pgf's are given by the functions P of the form

$$P(z) = \exp\Big[\int_z^1 \frac{2\log P_0(x)}{(1-x)(3-x)}\,\mathrm{d}x\Big]$$

with P_0 an infinitely divisible pgf. Now, take for P_0 the pgf of the Poisson (2) distribution; then it follows that P is given by

$$P(z) = \frac{2}{3-z},$$

so the *geometric* $(\frac{1}{3})$ distribution is *$\mathcal{F}$-self-decomposable.* Suppose that $P = \pi \circ A$ for some π; since $A^{-1} = A$, on $[\,0,1\,]$ the function π is then given by

$$\pi(s) = P\big(A(s)\big) = \frac{2}{3 - A(s)} = 1 - \tfrac{1}{3}s \qquad [\,0 \le s \le 1\,],$$

which cannot be a pLSt. So P is *not* of the form $\pi \circ A$ with π a pLSt, let alone with π a self-decomposable pLSt. □□

Finally, we present a series of examples of distributions on $\mathbb{R}$. The first two show the self-decomposability of two well-known distributions, the next three concern extensions of these involving stable distributions on $\mathbb{R}_+$ and symmetric stable distributions on $\mathbb{R}$.

Example 9.15. Let X have the *Gumbel* distribution with density f on $\mathbb{R}$ and characteristic function ϕ given by

$$f(x) = \exp\big[-(x + \mathrm{e}^{-x})\big], \quad \phi(u) = \Gamma(1 - iu).$$

As noted in Example IV.11.1, we have $X \stackrel{\mathrm{d}}{=} -\log Y$ with Y standard *exponentially* distributed; moreover, if $Y_1, Y_2, \ldots$ are independent with $Y_i \stackrel{\mathrm{d}}{=} Y$ for all i, then

$$\sum_{k=1}^{n} \frac{1}{k}\, Y_k - \log n \xrightarrow{\mathrm{d}} X \qquad [\, n \to \infty \,].$$

By using (1.3) and Propositions 6.2 and 6.3 it follows that X is *self-decomposable.* This is also reflected by the Lévy density m of X which was found in Example IV.11.10: $m(x) = 0$ for $x < 0$, and

$$x\, m(x) = \frac{1}{\mathrm{e}^x - 1} \qquad [\, x > 0 \,],$$

which is a nonincreasing function of x; cf. Theorem 6.12.
Now, for $\alpha \in (0,1)$ consider the component X_α of X in (1.1). By Corollary 6.8 it is *infinitely divisible*, and from Proposition IV.4.5 (ii), (iv) it follows that X_α has Lévy density m_α satisfying $m_\alpha(x) = 0$ for $x < 0$ and

$$x\, m_\alpha(x) = x\, m(x) - \frac{x}{\alpha}\, m\Big(\frac{x}{\alpha}\Big) = \frac{1}{\mathrm{e}^x - 1} - \frac{1}{\mathrm{e}^{x/\alpha} - 1} \qquad [\, x > 0 \,],$$

which can be shown to be nonincreasing in x. From Theorem 6.12 we conclude that X_α is *self-decomposable.* This can also be proved from Theorem 6.19; by Proposition 6.11 the underlying infinitely divisible distribution of X has Lévy density m_0 satisfying $m_0(x) = 0$ for $x < 0$ and

$$x\, m_0(x) = -x\, \frac{\mathrm{d}}{\mathrm{d}x}\big\{x\, m(x)\big\} = \frac{x\, \mathrm{e}^x}{(\mathrm{e}^x - 1)^2} \qquad [\, x > 0 \,],$$

which is easily seen to be nonincreasing in x.
Finally, we note that in Example 9.17 the self-decomposability of X will be proved by using just the definition and the representation $X \stackrel{\mathrm{d}}{=} -\log Y$ mentioned above. ◻◻

Example 9.16. Let X have the *logistic* distribution with density f on $\mathbb{R}$ and characteristic function ϕ given by

$$f(x) = \tfrac{1}{4}\, \frac{1}{\cosh^2 \frac{1}{2}x} = \tfrac{1}{2}\, \frac{1}{\cosh x + 1}, \quad \phi(u) = \frac{\pi u}{\sinh \pi u}.$$

As noted in Example IV.11.2, we have $X \overset{\mathrm{d}}{=} Y_1 - Y_2$ with Y_1 and Y_2 independent and both having the *Gumbel* distribution of the preceding example. So X is *self-decomposable* by Proposition 6.2. This is also apparent from the Lévy density m of X which was found in Example IV.11.11:

$$|x|\, m(x) = \frac{1}{\mathrm{e}^{|x|} - 1} \qquad [\, x \neq 0\,];$$

cf. Theorem 6.12. Clearly, as in Example 9.15, the components X_α and the underlying infinitely divisible distribution of X are *self-decomposable.* ◻◻

Example 9.17. Take $\lambda > 0$. For $\gamma \in (0,1)$ let T_γ be a random variable that is positive and *stable* (λ) with exponent γ, i.e., T_γ has pLSt π_γ given by $\pi_\gamma(s) = \exp[-\lambda\, s^\gamma]$; cf. Theorem 3.5. Consider X such that

$$X \overset{\mathrm{d}}{=} \log T_\gamma .$$

Then X is *self-decomposable.* To show this we take $\lambda = 1$; this is no essential restriction because of (1.3). Let Z be standard *exponential* and independent of T_γ; then for $x > 0$

$$\mathbb{P}\big((Z/T_\gamma)^\gamma > x\big) = \int_0^\infty \mathbb{P}(Z > x^{1/\gamma} t)\, \mathrm{d}F_{T_\gamma}(t) =$$
$$= \int_0^\infty \exp[-x^{1/\gamma} t]\, \mathrm{d}F_{T_\gamma}(t) = \pi_\gamma(x^{1/\gamma}) = \mathrm{e}^{-x} = \mathbb{P}(Z > x),$$

so we have

$$(Z/T_\gamma)^\gamma \overset{\mathrm{d}}{=} Z.$$

Now, taking logarithms one sees that $-\log Z$ is *self-decomposable* and that its component in (1.1) of order γ, say $(-\log Z)_\gamma$, has the same distribution as $\gamma \log T_\gamma$, so for X we have

$$X \overset{\mathrm{d}}{=} \frac{1}{\gamma}\, (-\log Z)_\gamma .$$

Since $-\log Z$ has a Gumbel distribution, we can apply the results of Example 9.15 to conclude that X is self-decomposable.
Before considering a special case, we note that the relation between T_γ and Z above easily yields an explicit expression, in case $\lambda = 1$, for the *moment* of T_γ of order $r < \gamma$ $\big($cf. (3.17)$\big)$:

$$\mathbb{E}(T_\gamma)^r = \frac{\mathbb{E} Z^{-r/\gamma}}{\mathbb{E} Z^{-r}} = \frac{\Gamma(1 - r/\gamma)}{\Gamma(1 - r)}.$$

Consider the special case with $\lambda = 2$ and $\gamma = \frac{1}{2}$, as in Example 9.5; so we start from the stable density g on $(0, \infty)$ given by

$$g(x) = \frac{1}{\sqrt{\pi}}\, x^{-3/2}\, \mathrm{e}^{-1/x} \qquad [\, x > 0\,].$$

A simple calculation then shows that X has density f given by

$$f(x) = \mathrm{e}^{x}\, g(\mathrm{e}^{x}) = \frac{1}{\sqrt{\pi}} \exp\left[-\left(\tfrac{1}{2}x + \mathrm{e}^{-x}\right)\right] \qquad [\, x \in \mathbb{R}\,],$$

which is *self-decomposable* by the result proved above. As noted at the end of Example 9.5, the stable density g can be viewed as a density of $1/Y$ with Y standard *gamma* $(\frac{1}{2})$ distributed, so X can be written as $X \stackrel{\mathrm{d}}{=} -\log Y$. This explains the similarity of f to the density of the Gumbel distribution in Example 9.15. See also the next example, where we consider a generalization of both densities. □□

Example 9.18. For $r > 0$ let Y have the standard *gamma* (r) distribution. Consider X such that

$$X \stackrel{\mathrm{d}}{=} -\log Y.$$

Then one easily verifies that X has density f on $\mathbb{R}$ and characteristic function ϕ given by

$$f(x) = \frac{1}{\Gamma(r)} \exp\left[-(r\,x + \mathrm{e}^{-x})\right], \quad \phi(u) = \frac{\Gamma(r - iu)}{\Gamma(r)}.$$

Now, by using Euler's formula for the gamma function (see Section A.5) it follows that ϕ can be obtained as

$$\phi(u) = \lim_{n \to \infty} n^{-iu} \prod_{k=1}^{n} \frac{k + r - 1}{k + r - 1 - iu}.$$

Hence, if $Y_1, Y_2, \ldots$ are independent and standard *exponential*, then

$$\sum_{k=1}^{n} \frac{1}{k + r - 1}\, Y_k - \log n \xrightarrow{\mathrm{d}} X \qquad [\, n \to \infty\,].$$

As in Example 9.15 we conclude that X is *self-decomposable*. Moreover, by considering differences as in Example 9.16 it follows that for $r > 0$ the distribution with density g_r on $\mathbb{R}$ and characteristic function χ_r given by

$$g_r(x) = \frac{1}{B(r,r)} \left(\frac{1}{2\cosh \frac{1}{2}x}\right)^{2r}, \quad \chi_r(u) = \frac{\Gamma(r - iu)\, \Gamma(r + iu)}{\Gamma(r)^2},$$

is *self-decomposable* as well. For instance, taking $r = \frac{1}{2}$ one obtains a distribution with the curious property that its density g and characteristic function χ are essentially the same functions:

$$g(x) = \frac{1}{2\pi} \frac{1}{\cosh \frac{1}{2}x}, \qquad \chi(u) = \frac{1}{\cosh \pi u}.$$

Here the expression for χ is obtained by using some properties of the gamma function as given in Section A.5; cf. the case $r = 1$ which was treated in Example IV.11.11. Of course, the distribution with density $h := g^{*2}$ and characteristic function $\psi := \chi^2$ also is *self-decomposable*; we find

$$h(x) = \frac{1}{2\pi^2} \frac{x}{\sinh \frac{1}{2}x}, \qquad \psi(u) = \frac{1}{\cosh^2 \pi u} = \frac{2}{\cosh 2\pi u + 1}.$$

Here the expression for h can be computed directly, or by using the inversion result of (A.2.15) together with the fact that, save for norming constants, h can be viewed as the Ft of the 'density' ψ; see Example 9.16. ◫

Example 9.19. Take $\lambda > 0$. For $\gamma \in (0, 2)$ let S_γ be a random variable that is symmetric *stable* (λ) with exponent γ, i.e., S_γ has characteristic function ϕ_γ given by $\phi_\gamma(u) = \exp\left[-\lambda\, |u|^\gamma\right]$; cf. Theorem 7.6. Consider X such that

$$X \stackrel{\mathrm{d}}{=} \log |S_\gamma|.$$

Then X is *self-decomposable*. To show this we note that $\phi_\gamma(u) = \pi_{\gamma/2}(u^2)$, where $\pi_{\gamma/2}$ is the pLSt of a positive stable (λ) random variable $T_{\gamma/2}$ with exponent $\gamma/2$; cf. Example 9.17. It follows (cf. Sections VI.2 and VI.9) that S_γ can be written as

$$S_\gamma \stackrel{\mathrm{d}}{=} \sqrt{T_{\gamma/2}}\, U,$$

where U is *normal* $(0, 2)$ and independent of $T_{\gamma/2}$. Now, taking absolute values and logarithms one sees that

$$X \stackrel{\mathrm{d}}{=} \tfrac{1}{2} \log T_{\gamma/2} + \log |U| = \tfrac{1}{2} \log T_{\gamma/2} + \tfrac{1}{2} \log V,$$

with V *gamma* $(\frac{1}{2}, \frac{1}{4})$; applying the results of Examples 9.17 and 9.18 we conclude that X is self-decomposable. Finally, we note that, in case $\lambda = 1$, for $r < \gamma$ (cf. (7.24)):

$$\mathbb{E}\, |S_\gamma|^r = \big(\mathbb{E}\, (T_{\gamma/2})^{r/2}\big)\, \big(\mathbb{E}\, V^{r/2}\big) = 2^r\, \frac{\Gamma(1 - r/\gamma)}{\Gamma(1 - r/2)}\, \frac{\Gamma\big((r+1)/2\big)}{\sqrt{\pi}},$$

where we again used Example 9.17. ◫

The last two examples also concern stable distributions on $\mathbb{R}$. The first one shows an alternative way of proving that for $\lambda > 0$ and $\gamma \in (0, 2]$ the function $u \mapsto \exp\left[-\lambda\,|u|^{\gamma}\right]$ is a (stable) characteristic function. The second presents a simple function that turns out to be a weakly stable characteristic function.

Example 9.20. Let $\gamma \in (0, 2]$, and for $n \in \mathbb{N}$ let $Y_{n,1}, \ldots, Y_{n,n}$ be independent and uniformly distributed on $(-n, n)$. Define X_n by

$$X_n = \sum_{j=1}^{n} \frac{\operatorname{sgn}\{Y_{n,j}\}}{|Y_{n,j}|^{1/\gamma}},$$

where $\operatorname{sgn}\{r\} := +1$ if $r \geq 0$, and $:= -1$ if $r < 0$. Then the characteristic function ϕ_n of X_n is given by

$$\phi_n(u) = \Big\{1 - \frac{1}{n} \int_0^n \big(1 - \cos\,(u/y^{1/\gamma})\big)\, \mathrm{d}y\Big\}^n,$$

so $\lim_{n\to\infty} \phi_n(u)$ exists with

$$\lim_{n\to\infty} \phi_n(u) = \exp\Big[-\int_0^\infty \big(1 - \cos\,(u/y^{1/\gamma})\big)\, \mathrm{d}y\Big] =$$

$$= \exp\Big[-\gamma\,|u|^{\gamma} \int_0^\infty \frac{1 - \cos\,t}{t^{1+\gamma}}\, \mathrm{d}t\Big] = \exp\left[-\lambda\,|u|^{\gamma}\right] =: \phi(u),$$

with $\lambda > 0$. Since ϕ is continuous at zero, from the continuity theorem it follows that ϕ is a characteristic function and, as we know from Section 7, a stable one. ◻◻

Example 9.21. Consider the function ϕ on $\mathbb{R}$ defined by

$$\phi(u) = (iu)^{iu} \qquad [\,\text{principal value, } u \neq 0;\ \phi(0) = 1\,].$$

Then one easily verifies that ϕ can be rewritten as

$$\phi(u) = \exp\Big[-(\pi/2)\,|u|\,\big(1 \pm_u i\,(-2/\pi)\,\log|u|\big)\Big] \qquad [\,u \in \mathbb{R}\,],$$

so ϕ has the form (7.38) with $\mu = 0$, $\lambda = \pi/2$ and $\beta = -1$. From Theorem 7.16 it follows that ϕ is the characteristic function of a *weakly stable* distribution with exponent $\gamma = 1$. Note that by (7.39) the corresponding canonical function M vanishes everywhere on $(0, \infty)$. Nevertheless, the distribution is *not* concentrated on a half-line; compare Theorems 7.14 and IV.4.13. ◻◻

10. Notes

Contrary to the order in this chapter, the more special, stable distributions were introduced earlier than the self-decomposable ones. Lévy (1923) considered stable distributions in the context of the central limit theorem; some of them had been known for some time. Self-decomposable distributions were introduced by Lévy (1937) in the same context; see Section I.5. Classical books on stability and self-decomposability are Gnedenko and Kolmogorov (1968), and Zolotarev (1986); there are many textbooks where the subject is treated in more or less detail, e.g., Breiman (1968), Lukacs (1970, 1983), Feller (1971), Loève (1977), Petrov (1995), Lamperti (1996). The book by Samorodnitsky and Taqqu (1994) presents an overview of stable distributions and processes on $\mathbb{R}^n$.

The canonical representations of stable characteristic functions, as given in Theorems 7.11 and 7.16, are due to Khintchine and Lévy (1936). A quite different approach is presented in Geluk and de Haan (2000). Hall (1981) gives an account of the many sign errors in these formulas that occur in the literature. A treatment of stable distributions on $\mathbb{R}_+$ can be found in Feller (1971). Stable and self-decomposable distributions on $\mathbb{Z}_+$ were introduced and studied in Steutel and van Harn (1979); here also the basic representation of a self-decomposable characteristic function as given in Theorem 6.7 was presented without proof. A very different representation is given in Urbanik (1968). The characterization of self-decomposable distributions in Theorem 6.12 in terms of the canonical function is due to Lévy (1937). It leads to the differentiability property of Proposition 6.13, which is not true for general (infinitely divisible) characteristic functions, as was shown by Looijenga (1990). An integral representation for self-decomposable random variables is given by Wolfe (1982); see also Jurek and Vervaat (1983). Part of the results on self-decomposability on $\mathbb{R}_+$ in Section 2 can be found in Bondesson (1992). Here also the class of generalized gamma convolutions introduced by Thorin (1977a,b) is studied in detail; this class 'interpolates' between the class of stable distributions and that of self-decomposable distributions on $\mathbb{R}_+$; see also Section VI.5. Bondesson (1987) gives a sufficient condition for self-decomposability which is satisfied by any density on $\mathbb{R}_+$ that is proportional to a self-decomposable pLSt.

Explicit expressions for stable densities are unavailable except for the few examples we have given; series expansions are given in Feller (1971), and numerical results in Nolan (1997). Gawronski (1984) proves a detailed result on the (bell-) shape of stable densities. Hoffmann-Jørgensen (1993) expresses these densities in terms of special functions.

The problem of unimodality of self-decomposable distributions has a long and rather amusing history; see Lukacs (1970) and Wolfe (1971a). It was eventually solved by Wolfe (1971a) for distributions on $\mathbb{R}_+$, and finally on $\mathbb{R}$ by Yamazato (1978) who succeeded to extend Wolfe's result by proving Lemma 6.22; for more specific results see Sato and Yamazato (1978, 1981). The location of modes is a rather intractable problem; some information on this is contained in Hall (1984) and in Shkol'nik (1992); the mode of the stable distribution on $\mathbb{Z}_+$ with exponent $\gamma = \frac{1}{2}$ is considered by Barendregt (1979). Sato (1994) studies the surprising multimodality of convolution powers of infinitely divisible distributions.

Lévy (1937) considers a class of distributions that he calls 'semi-stable'. Generalizations of self-decomposability are studied by O'Connor (1979a,b), Jurek (1985), Hansen (1988a) and Bunge (1997). The extensions related to the multiplication induced by semigroups in Section 8 can be found in van Harn et al. (1982), and in van Harn and Steutel (1993); this multiplication is used by Hansen (1988a) and (1996), where the eigenfunction properties for logarithms of stable characteristic functions also turn up.

Many known distributions are self-decomposable; cf. Jurek (1997). On the other hand, Examples 9.1, 9.4 and 9.8 show that there also are simple classes of counter-examples. Results on multiple self-decomposability, which was mentioned in Example 9.6, are given in Urbanik (1972), Kumar and Schreiber (1978), and in Berg and Forst (1983), who use a mapping which is the inverse of our T in Section 2; see also Forst (1984), where a notion of self-decomposability on $\mathbb{Z}$ is discussed. Most results in Examples 9.17, 9.18 and 9.19 occur in Shanbhag and Sreehari (1977), or Shanbhag et al. (1977). The construction of the symmetric stable characteristic functions in Example 9.20 can be found in Lamperti (1996). Finally, as an example of how difficult it is to decide on the self-decomposability of a given density, we mention the self-decomposability of the half-Cauchy distribution due to Diédhiou (1998); he continued an investigation by Bondesson (1987), who proved that the half-Cauchy distribution is infinitely divisible (cf. Example VI.12.4).

Chapter VI

INFINITE DIVISIBILITY AND MIXTURES

1. Introduction

There is no special reason why mixtures of infinitely divisible distributions should be infinitely divisible; in fact, it is quite easy to destroy infinite divisibility by mixing, such as by randomizing a parameter. On the other hand, mixing is such an ordinary phenomenon that the question whether or when infinite divisibility is preserved under mixing arises very naturally.

Let F_1 and F_2 be distribution functions, and let $\alpha \in (0,1)$. Then the following function F is a new distribution function:

$$F(x) = \alpha\, F_1(x) + (1-\alpha)\, F_2(x) \qquad [\, x \in \mathbb{R}\,];$$

it is called a *mixture* of F_1 and F_2. The probability distribution corresponding to F is then also called a *mixture*. If both F_1 and F_2 are infinitely divisible, then F may or may not be infinitely divisible. Examples of both situations are easily given: A mixture of two different normal distributions is not infinitely divisible because its tail is too thin (cf. Theorem IV.9.8 and Example IV.11.14); and a mixture of two exponential distributions is infinitely divisible since it has a completely monotone density (cf. Theorem III.10.7). Still, as we shall see, there is some structure in the occurrence of infinite divisibility in mixtures.

The distribution function F above is the very simplest example of a mixture. In this chapter we consider *mixtures* of the following more general kind:

(1.1) $$F(x) = \int_{\Theta} F_\theta(x)\, \nu(\mathrm{d}\theta) \qquad [\, x \in \mathbb{R}\,].$$

Here ν is a probability measure on a measurable space $(\Theta, \mathcal{T})$ and F_θ is a distribution function for every θ such that $\theta \mapsto F_\theta(x)$ is $\mathcal{T}$-measurable for all x. In most cases Θ will be (part of) $\mathbb{R}_+$, and ν will be the Stieltjes measure m_G induced by a distribution function G. In isolated cases ν will be replaced by a signed measure with $\nu(\Theta) = 1$; we will then speak of a *generalized* mixture. We shall be interested in infinite divisibility of F, and look for conditions on the F_θ and/or ν (or G) for this to occur.

As is well known, mixtures as in (1.1) can be written in terms of characteristic functions in the following way:

$$(1.2) \qquad \phi(u) = \int_\Theta \phi_\theta(u)\, \nu(\mathrm{d}\theta) \qquad [\,u \in \mathbb{R}\,],$$

and similarly for probability generating functions (pgf's) and probability Laplace-Stieltjes transforms (pLSt's). If, for every θ, F_θ is absolutely continuous with density f_θ, then the mixture F is absolutely continuous with density f given by

$$(1.3) \qquad f(x) = \int_\Theta f_\theta(x)\, \nu(\mathrm{d}\theta) \qquad [\,x \in \mathbb{R}\,];$$

a similar relation holds for discrete distributions. Also the transforms and densities of mixtures will be called *mixtures*.

In Section 2 we introduce some general types of frequently used mixtures; special cases are then treated in the subsequent sections. Mixtures of *exponential* distributions are dealt with in Section 3; mixtures of *gamma* distributions are studied in Section 4. Then in Section 5 we briefly consider the important class of *generalized gamma convolutions*, which can be viewed as (limits of) mixtures of gamma distributions. Since this class is extensively studied in a recent monograph by Bondesson, we do not give all the, sometimes very technical, details. Section 6 deals with mixtures of *Poisson* distributions; these mixtures make it possible to easily relate $\mathbb{R}_+$-results to results on $\mathbb{Z}_+$, and this leads in a natural way to results on mixtures of *negative-binomial* distributions in Section 7 and *generalized negative-binomial convolutions* in Section 8. Similarly, mixtures of zero-mean *normal* distributions, considered in Section 9, lead to results on mixtures of *sym-gamma* distributions in Section 10 and *generalized sym-gamma convolutions* in Section 11. The results of the present chapter give rise to many interesting examples; they are given in Section 12. Finally, Section 13 contains notes and bibliographical information.

2. Two important types of mixtures

In this section we consider two types of mixtures that are frequently used. First, in (1.2) we let $\Theta = \mathbb{R}_+$ and take $\phi_\theta = \phi_1^\theta$ with ϕ_1 a characteristic function. This leads to the so-called *power mixtures*, of the following form:

$$\phi(u) = \int_{\mathbb{R}_+} \{\phi_1(u)\}^\theta \, \mathrm{d}G(\theta), \tag{2.1}$$

where G is a distribution function on $\mathbb{R}_+$. When the support $S(G)$ of G is not contained in $\mathbb{Z}_+$, we require *infinite divisibility* of ϕ_1; this ensures ϕ_1^θ to be a characteristic function for all θ. In this case (2.1) can be written as

$$\phi(u) = \widehat{G}\big(-\log \phi_1(u)\big), \tag{2.2}$$

with $\widehat{G}$ the LSt of G; a random variable X with characteristic function ϕ can be expressed as

$$X \stackrel{\mathrm{d}}{=} S(T), \tag{2.3}$$

where $S(\cdot)$ is an sii-process with $\phi_{S(1)} = \phi_1$ and T is a random variable that is independent of $S(\cdot)$ with $F_T = G$. In fact, the power mixtures of (2.1) correspond exactly to the *compound distributions* considered earlier in the third sections of Chapters I, II, III and IV. Still, there sometimes is a difference in accent, as we shall see when considering mixtures of normal distributions. We now rephrase Proposition IV.3.6 in the present context; its proof reflects the fact that

$$\{\phi(u)\}^t = \int_{\mathbb{R}_+} \{\phi_1(u)\}^\theta \, \mathrm{d}G^{\star t}(\theta) \qquad [t > 0]. \tag{2.4}$$

Proposition 2.1. *If both the characteristic function ϕ_1 and the distribution function G are infinitely divisible, then the power mixture ϕ in (2.1) is infinitely divisible.*

When the support $S(G)$ of G is contained in $\mathbb{Z}_+$, then ϕ_1^θ is a characteristic function for all $\theta \in S(G)$ and hence we need not require infinite divisibility of ϕ_1. Now (2.1) can be rephrased as

$$\phi(u) = P\big(\phi_1(u)\big), \tag{2.5}$$

with P the pgf corresponding to G, and to get an analogue of (2.3) we have to replace $S(\cdot)$ by a discrete-time sii-process $(S_n)_{n\in\mathbb{Z}_+}$ (so $S_n = Y_1 + \cdots + Y_n$ for all n with $Y_1, Y_2, \ldots$ independent and distributed as S_1):

$$X \stackrel{\mathrm{d}}{=} S_N \quad (\text{so } X \stackrel{\mathrm{d}}{=} Y_1 + \cdots + Y_N), \tag{2.6}$$

where N is a random variable that is independent of the Y_i and has pgf P. Proposition IV.3.1 now yields a less restrictive result; we only have to take G such that in (2.4) $S(G^{\star t}) \subset \mathbb{Z}_+$ for all $t > 0$.

Proposition 2.2. *If the distribution function G is infinitely divisible with support satisfying $0 \in S(G) \subset \mathbb{Z}_+$, then for any characteristic function ϕ_1 the power mixture ϕ in (2.1) is infinitely divisible.*

Consider the special case of (2.1) where ϕ_1 is the characteristic function of the standard *normal* distribution: $\phi_1(u) = \exp[-\frac{1}{2}u^2]$. Then for ϕ we have

$$\phi(u) = \int_{\mathbb{R}_+} \exp[-\tfrac{1}{2}\theta u^2]\,\mathrm{d}G(\theta) = \widehat{G}(\tfrac{1}{2}u^2), \tag{2.7}$$

so ϕ corresponds to a *variance mixture of normal distributions* (with zero mean). From Proposition 2.1 it is clear that such a variance mixture is infinitely divisible if the mixing function G is infinitely divisible. Further, a random variable X corresponding to ϕ can not only be written as in (2.3) but also as

$$X \stackrel{\mathrm{d}}{=} \sqrt{T}\,Y, \tag{2.8}$$

with $Y := S(1)$. This observation leads to a second kind of much used mixtures; the distribution of a random variable X is said to be a *scale mixture* if

$$X \stackrel{\mathrm{d}}{=} Z\,Y, \tag{2.9}$$

where Z and Y are independent and Z is nonnegative. In terms of characteristic functions we now get the following counterpart to (2.1):

$$\phi(u) = \int_{\mathbb{R}_+} \phi_1(\theta u)\,\mathrm{d}G(\theta), \tag{2.10}$$

where $\phi_1 = \phi_Y$ and $G = F_Z$. When Y is nonnegative, we sometimes allow Z to have negative values as well; in (2.10) we then have to integrate over $\mathbb{R}$.

Not very much in general can be said about the infinite divisibility of X in (2.9): ZY may be infinitely divisible though neither of Z and Y are, and ZY may be not infinitely divisible though both Z and Y are; cf. Section 12. Still, as we shall see in the next sections, large classes of infinitely divisible scale mixtures can be identified.

Finally we briefly discuss a discrete analogue of the scale mixtures in (2.9) that is suggested by the discrete multiplication $\odot$ introduced in Section A.4. Let X satisfy

$$X \stackrel{\mathrm{d}}{=} Z \odot Y, \tag{2.11}$$

where Z and Y are independent, Z has values in $[\,0,1\,]$ and Y is $\mathbb{Z}_+$-valued. Then X is $\mathbb{Z}_+$-valued as well, and from (A.4.13) it is seen that the pgf P of X is given by

$$P(z) = \int_{[0,1]} P_1(1-\theta+\theta z)\,\mathrm{d}G(\theta), \tag{2.12}$$

where $P_1 = P_Y$ and $G = F_Z$. In the case where Y is *Poisson*, Z in (2.11) may be taken arbitrarily nonnegative; the pgf's of the resulting *mixtures of Poisson distributions* then take the form

$$P(z) = \int_{\mathbb{R}_+} \exp\big[-\lambda\,\theta\,(1-z)\big]\,\mathrm{d}G(\theta) = \widehat{G}\big(\lambda\,\{1-z\}\big), \tag{2.13}$$

with $\lambda > 0$. Note that P in (2.13) can also be viewed as a power mixture; so, from Proposition 2.1 it follows that a mixture of Poisson distributions with infinitely divisible mixing function G is infinitely divisible. A converse to this statement will be given in Section 6, where we shall see that also other properties of P and G in (2.13) are closely related.

3. Mixtures of exponential distributions

The exponential distribution with parameter $\lambda > 0$ is a probability distribution on $\mathbb{R}_+$ with density f_λ and pLSt π_λ given by

$$f_\lambda(x) = \lambda\,\mathrm{e}^{-\lambda x}, \quad \pi_\lambda(s) = \frac{\lambda}{\lambda+s}.$$

According to (1.2) and (1.3), mixing with respect to λ leads to a distribution on $\mathbb{R}_+$ with density f and pLSt π of the form

$$f(x) = \int_{(0,\infty)} \lambda\,\mathrm{e}^{-\lambda x}\,\mathrm{d}G(\lambda), \quad \pi(s) = \int_{(0,\infty)} \frac{\lambda}{\lambda+s}\,\mathrm{d}G(\lambda), \tag{3.1}$$

with G a distribution function on $(0,\infty)$. Note that the resulting mixtures can be viewed as *scale mixtures*; putting $\lambda = 1/\theta$ one sees that π in (3.1) can be rewritten in the form (2.10):

$$\pi(s) = \int_{\mathbb{R}_+} \frac{1}{1+\theta s}\, \mathrm{d}H(\theta), \tag{3.2}$$

with H a distribution function on $(0,\infty)$. So, *mixtures of exponential densities* correspond to random variables of the form ZY with Z *positive* and Y exponential, and Z and Y independent. Random variables ZY with Z *nonnegative* have pLSt's π of the form (3.2) with H a distribution function on $\mathbb{R}_+$; this means that an additional mixing with the degenerate distribution at zero is allowed. Since this distribution can be viewed as an exponential distribution 'with $\lambda = \infty$', we extend π in (3.1) in this sense and agree that any distribution with pLSt π of the following form will be called a *mixture of exponential distributions*:

$$\pi(s) = \alpha + (1-\alpha) \int_{(0,\infty)} \frac{\lambda}{\lambda + s}\, \mathrm{d}G(\lambda), \tag{3.3}$$

with $\alpha \in [0,1]$ and G a distribution function on $(0,\infty)$. This has the advantage that the resulting class of distributions is *closed under weak convergence*.

Proposition 3.1. *If a sequence (π_n) of mixtures of exponential pLSt's converges (pointwise) to a pLSt π, then π is a mixture of exponential pLSt's.*

PROOF. Let π_n be of the form (3.2) with H replaced by a distribution function H_n on $\mathbb{R}_+$, and suppose that $\pi := \lim_{n\to\infty} \pi_n$ exists and is a pLSt. According to Helly's selection theorem (H_n) has a subsequence $(H_{n(k)})$ that is convergent in the sense that

$$\lim_{k\to\infty} H_{n(k)}(\theta) = H(\theta) \quad [\,\theta \in \mathbb{R} \text{ such that } H \text{ is continuous at } \theta\,],$$

where H is a right-continuous nondecreasing function with $H(\theta) = 0$ for $\theta < 0$ and $H(\theta) \le 1$ for $\theta \ge 0$. Now, for $s > 0$ the function $\theta \mapsto 1/(1+\theta s)$ is continuous and bounded on $\mathbb{R}_+$ and tends to 0 as $\theta \to \infty$. Hence, for $s > 0$ we have

$$\pi(s) = \lim_{k\to\infty} \int_{\mathbb{R}_+} \frac{1}{1+\theta s}\, \mathrm{d}H_{n(k)}(\theta) = \int_{\mathbb{R}_+} \frac{1}{1+\theta s}\, \mathrm{d}H(\theta).$$

Since π is a pLSt, by letting $s \downarrow 0$ we see that $\lim_{\theta \to \infty} H(\theta) = 1$, so H is a distribution function on $\mathbb{R}_+$. We conclude that π is of the form (3.2) and hence is a mixture of exponential pLSt's; the proposition is proved. Additionally, by using a uniqueness theorem for transforms of type (3.2), one sees that any other convergent subsequence of (H_n) has the same limit H; hence (H_n) converges weakly to H. □

From Bernstein's theorem it follows that the mixtures of exponential densities coincide with the *completely monotone* densities; see Proposition A.3.11. Therefore, Theorem III.10.7 immediately yields the infinite divisibility of such mixtures, and applying Proposition III.2.2 then accounts for the more general mixtures in (3.3); cf. the proof of Theorem 3.3 below. So we have the following result.

Theorem 3.2. *A mixture of exponential densities is log-convex, and hence infinitely divisible. The more general mixtures of exponential distributions are infinitely divisible as well.*

Though this may seem to be all one would want to know, it is of interest to consider the mixtures of exponential distributions from a somewhat different perspective. This will lead to a more direct proof of their infinite divisibility, to a representation formula for their pLSt's, to a characterization of them in terms of the Lévy canonical function, and to some generalizations, one of which is deferred to Section 4.

Before showing this we recall some basic facts from Section III.4. According to Theorem III.4.3 a function π on $\mathbb{R}_+$ is the pLSt of an infinitely divisible distribution on $\mathbb{R}_+$ if π has the form

$$\text{(3.4)} \qquad \pi(s) = \exp\Big[-\int_0^\infty (1 - \mathrm{e}^{-sx})\,\frac{1}{x}\,k(x)\,\mathrm{d}x\Big] \qquad [\,s \geq 0\,],$$

with k a *nonnegative* function on $(0, \infty)$; k is called the *canonical density* of π. Note that multiplying two functions of the form (3.4) corresponds to addition of their canonical densities; cf. Proposition III.4.5 (ii). Further, from Example III.4.8 it is known that for $\lambda > 0$:

$$\text{(3.5)} \qquad \pi(s) = \frac{\lambda}{\lambda + s} \implies \pi \text{ has the form (3.4) with } k(x) = \mathrm{e}^{-\lambda x}.$$

We first consider *finite* mixtures of exponential densities; for $n \in \mathbb{N}$ with $n \geq 2$ let the density f_n on $(0, \infty)$ and its Lt π_n be given by

$$(3.6) \qquad f_n(x) = \sum_{j=1}^{n} g_j \, \lambda_j \mathrm{e}^{-\lambda_j x}, \quad \pi_n(s) = \sum_{j=1}^{n} g_j \frac{\lambda_j}{\lambda_j + s},$$

where the g_j are positive with $\sum_{j=1}^{n} g_j = 1$ and $0 < \lambda_1 < \cdots < \lambda_n$. Then π_n can be written as $\pi_n = Q_{n-1}/P_n$, where Q_{n-1} and P_n are polynomials of degree $n-1$ and n, respectively; in fact, if we take $P_n(s) = \prod_{j=1}^{n}(\lambda_j + s)$, then the leading coefficient of Q_{n-1} is given by $\sum_{j=1}^{n} g_j \lambda_j$. Extending the domain of both polynomials to $\mathbb{R}$ we see that π_n has poles at $-\lambda_1, \ldots, -\lambda_n$ with

$$\lim_{s \uparrow -\lambda_j} \pi_n(s) = -\infty, \quad \lim_{s \downarrow -\lambda_j} \pi_n(s) = \infty \qquad [\, j = 1, \ldots, n \,].$$

Since π_n is continuous on $\mathbb{R} \setminus \{-\lambda_1, \ldots, -\lambda_n\}$, it follows that π_n, and hence Q_{n-1}, has $n-1$ zeroes $-\mu_1, \ldots, -\mu_{n-1}$ satisfying $\mu_j \in (\lambda_j, \lambda_{j+1})$ for all j. As $\pi_n(0) = 1$, we conclude that π_n can be written as

$$(3.7) \qquad \pi_n(s) = \frac{\left(\sum_{j=1}^{n} g_j \lambda_j\right) \prod_{j=1}^{n-1}(\mu_j + s)}{\prod_{j=1}^{n}(\lambda_j + s)} = \frac{\prod_{j=1}^{n} \lambda_j/(\lambda_j + s)}{\prod_{j=1}^{n-1} \mu_j/(\mu_j + s)}.$$

Now, using (3.5) and the remark preceding it, we see that π_n can be put in the form (3.4) with k replaced by k_n, where

$$(3.8) \qquad k_n(x) = \sum_{j=1}^{n-1} (\mathrm{e}^{-\lambda_j x} - \mathrm{e}^{-\mu_j x}) + \mathrm{e}^{-\lambda_n x} \qquad [\, x > 0 \,].$$

Since $\mu_j > \lambda_j$ for all j, k_n is a nonnegative function on $(0, \infty)$; hence π_n is *infinitely divisible* with canonical density k_n. Thus we are led to the following result.

Theorem 3.3. *A mixture of exponential distributions is infinitely divisible. Equivalently, a random variable X with $X \overset{\mathrm{d}}{=} ZY$ is infinitely divisible if Z and Y are independent, Z is nonnegative and Y is exponential.*

PROOF. Let π be of the form (3.3) with $\alpha \in [0, 1)$ and G a distribution function on $(0, \infty)$. As is well known, we can write G as the weak limit of a sequence (G_n) of distribution functions on $(0, \infty)$ with *finite* supports. Since $\lambda \mapsto \lambda/(\lambda + s)$ is a bounded continuous function on $(0, \infty)$, it follows that $\int_{(0,\infty)} \lambda/(\lambda + s) \, \mathrm{d}G(\lambda)$ is the pointwise limit of the same integral with G

replaced by G_n. Hence there exist probabilities $g_{n,j}$ and parameters $\lambda_{n,j}$ such that

$$\pi(s) = \lim_{\lambda\to\infty} \lim_{n\to\infty} \Big\{\sum_{j=1}^{n} (1-\alpha)\, g_{n,j} \frac{\lambda_{n,j}}{\lambda_{n,j}+s} + \alpha \frac{\lambda}{\lambda+s}\Big\}; \tag{3.9}$$

this means that π is the pointwise limit of pLSt's π_n of the form (3.6). As we saw above, these π_n are infinitely divisible; hence so is π, because of Proposition III.2.2. The final statement in terms of random variables now follows from the equivalence of (3.2) and (3.3). ◫

The method of proof above suggests both a *representation formula* for the pLSt's of mixtures of exponential distributions and a *characterization*, via the canonical density k, of these distributions among the infinitely divisible ones. To show this we start again by looking at *finite* mixtures.

Proposition 3.4. *Let $n \geq 2$ and $0 < \lambda_1 < \cdots < \lambda_n$. Then a function π on $\mathbb{R}_+$ is a mixture of the type*

$$\pi(s) = \sum_{j=1}^{n} g_j \frac{\lambda_j}{\lambda_j + s}, \tag{3.10}$$

where the g_j are positive with $\sum_{j=1}^{n} g_j = 1$, iff π is an infinitely divisible pLSt having a canonical density k such that $x \mapsto k(x)/x$ is completely monotone with Bernstein representation of the form

$$\frac{1}{x}\, k(x) = \int_0^\infty e^{-\lambda x}\, v(\lambda)\, d\lambda \qquad [x > 0], \tag{3.11}$$

where for some $\mu_1, \ldots, \mu_{n-1}$ satisfying $\lambda_j < \mu_j < \lambda_{j+1}$ for all j:

$$v(\lambda) = \begin{cases} 1 & , \text{ if } \lambda \in (\lambda_j, \mu_j) \text{ for some } j \text{ or } \lambda > \lambda_n, \\ 0 & , \text{ otherwise.} \end{cases} \tag{3.12}$$

Equivalently, a function π on $\mathbb{R}_+$ is a mixture of type (3.10) iff π has the form

$$\pi(s) = \exp\Big[-\int_0^\infty \Big(\frac{1}{\lambda} - \frac{1}{\lambda+s}\Big)\, v(\lambda)\, d\lambda\Big] \qquad [s \geq 0] \tag{3.13}$$

with v as in (3.12).

PROOF. As we saw above, a function π of type (3.10) is an infinitely divisible pLSt with canonical density k given by (3.8); clearly, k can be rewritten as in (3.11). Conversely, let π be an infinitely divisible pLSt with canonical density k as indicated. Then k can be written as in (3.8), from which by use of (3.4) and (3.5) it follows that π coincides with the function of s in the right-hand side of (3.7). Partial-fraction expansion of this function shows that π is of the form (3.10) with

$$g_j = \frac{1}{\lambda_j}\Big(\prod_{\ell\neq j}\frac{\lambda_\ell}{\lambda_\ell-\lambda_j}\Big)\Big(\prod_{\ell=1}^{n-1}\frac{\mu_\ell-\lambda_j}{\mu_\ell}\Big) \qquad [\,j=1,\ldots,n\,].$$

By taking $s=0$ we see that $\sum_{j=1}^n g_j = 1$; the positivity of the g_j follows from the inequalities on the μ_j. The final statement is now an immediate consequence; the representation formula (3.13) is obtained by inserting (3.11) in the canonical representation (3.4) for π and changing the order of integration. □

This result can be generalized to arbitrary mixtures of exponential distributions, by taking limits. In doing so it is easiest to start with the representation part.

Theorem 3.5 (Canonical representation). *A function π on $\mathbb{R}_+$ is the pLSt of a mixture of exponential distributions iff π has the form*

$$(3.14)\qquad \pi(s) = \exp\Big[-\int_0^\infty\Big(\frac{1}{\lambda}-\frac{1}{\lambda+s}\Big)\,v(\lambda)\,d\lambda\Big] \qquad [\,s\geq 0\,],$$

where v is a measurable function on $(0,\infty)$ satisfying $0\leq v\leq 1$ and, necessarily,

$$(3.15)\qquad \int_0^1 \frac{1}{\lambda}\,v(\lambda)\,d\lambda < \infty.$$

PROOF. Let π be the pLSt of a mixture of exponential distributions. Then from the proof of Theorem 3.3 it follows that $\pi = \lim_{n\to\infty}\pi_n$ with π_n of the form (3.10), and hence by Proposition 3.4 of the form (3.13) with v as in (3.12). One can show that this implies (3.14) with $0\leq v\leq 1$. We do not give the details; see Notes.

Conversely, let π be given by (3.14), where v satisfies $0\leq v\leq 1$. Then one can show (again we do not give the details) that $\pi = \lim_{n\to\infty}\pi_n$ with π_n of the form (3.13) with v as in (3.12). By Proposition 3.4 the π_n are mixtures of exponential pLSt's, and hence so is π because of Proposition 3.1.

Finally, condition (3.15) on v follows from the fact that the integral in (3.14) should be finite for all s; note that $\int_0^1 1/(\lambda+s)\,v(\lambda)\,\mathrm{d}\lambda < \infty$ for $s > 0$, since $0 \le v \le 1$. ◻

Sometimes we shall call the function v in the representation (3.14) for π the *second canonical density* of π, in order to distinguish it from the (first) canonical density k in the representation (3.4) for π. We state some consequences of Theorem 3.5. The first one is immediate.

Corollary 3.6. *If π is the pLSt of a mixture of exponential distributions, then so is π^a for any $a \in [0,1]$. Consequently, the convolution roots of a mixture of exponential distributions are mixtures of exponential distributions as well.*

It follows that if G is a distribution function on $(0,\infty)$ and if $a \in (0,1)$, then

$$(3.16)\qquad \Big(\int_{(0,\infty)} \frac{\lambda}{\lambda+s}\,\mathrm{d}G(\lambda)\Big)^a = \int_{(0,\infty)} \frac{\lambda}{\lambda+s}\,\mathrm{d}G_a(\lambda),$$

where G_a is a distribution function on $(0,\infty)$. This result does not seem obvious without the use of Theorem 3.5. The next consequence is easily obtained by letting $s \to \infty$ in (3.14) and using monotone convergence.

Corollary 3.7. *A mixture of exponential distributions has positive mass α, say, at zero iff its second canonical density v satisfies*

$$(3.17)\qquad \int_1^\infty \frac{1}{\lambda}\,v(\lambda)\,d\lambda < \infty,$$

in which case α is given by $\alpha = \exp\big[-\int_0^\infty (1/\lambda)\,v(\lambda)\,d\lambda\big]$.

A last consequence of Theorem 3.5 concerns the canonical function K of a pLSt π as in (3.14). As known from Section III.4, the LSt $\widehat{K}$ of K is given by the ρ-function of π for which

$$(3.18)\qquad \rho(s) := -\frac{\mathrm{d}}{\mathrm{d}s}\log \pi(s) = \int_0^\infty \frac{1}{(\lambda+s)^2}\,v(\lambda)\,\mathrm{d}\lambda.$$

Now, $s \mapsto 1/(\lambda+s)^2$ is the Lt of the function $x \mapsto x\,\mathrm{e}^{-\lambda x}$ on $(0,\infty)$. Using this in (3.18) and changing the order of integration, we see that $\widehat{K}$ is the Lt of the function k in (3.11). Thus we are led to the following generalization of the first part of Proposition 3.4; the converse statement again follows by inserting the expression for k in (3.4), and then using Theorem 3.5.

Theorem 3.8. *A probability distribution on $\mathbb{R}_+$ is a mixture of exponential distributions iff it is infinitely divisible having a canonical density k such that $x \mapsto k(x)/x$ is completely monotone with Bernstein representation of the form*

$$(3.19) \qquad \frac{1}{x}\, k(x) = \int_0^\infty e^{-\lambda x}\, v(\lambda)\, d\lambda \qquad [\, x > 0\,],$$

where v is a measurable function on $(0,\infty)$ satisfying $0 \le v \le 1$ and, necessarily, condition (3.15). In this case v is the second canonical density.

Condition (3.15) also follows from (3.19) and the fact that $\int_1^\infty (1/x)\, k(x)\, dx$ is finite. Note that the completely monotone function $x \mapsto k(x)/x$ is precisely the restriction of the *Lévy canonical density* m to $(0,\infty)$; cf. (IV.4.16). Finally we mention a consequence of (3.18), or (3.19), and (III.7.2).

Corollary 3.9. *Let the distribution of a random variable X be a mixture of exponential distributions with second canonical density v. Then*

$$(3.20) \qquad \mathbb{E}X = \int_0^\infty \frac{1}{\lambda^2}\, v(\lambda)\, d\lambda \qquad [\le \infty\,].$$

We return to Theorem 3.3; the method used there to prove infinite divisibility can be extended to distributions that are obtained from exponential distributions by a more general mixing procedure. First we will show that a restricted use of 'negative probabilities' in the mixing function is allowed. Consider n exponential densities with parameters $\lambda_1, \ldots, \lambda_n$ satisfying $0 < \lambda_1 < \cdots < \lambda_n$, and mix them as follows: let the function f_n on $(0,\infty)$ and its Lt π_n be given by

$$(3.21) \qquad f_n(x) = \sum_{j=1}^n g_j\, \lambda_j \mathrm{e}^{-\lambda_j x}, \quad \pi_n(s) = \sum_{j=1}^n g_j\, \frac{\lambda_j}{\lambda_j + s},$$

where now the g_j are non-zero, *possibly negative*, with $\sum_{j=1}^n g_j = 1$, so f_n integrates to one and $\pi_n(0) = 1$. Of course, we want f_n to be a density, i.e., $f_n(x) \ge 0$ for all x. By letting $x \to \infty$ and $x \downarrow 0$ we see that for this to be the case it is necessary that

$$(3.22) \qquad g_1 > 0, \quad \sum_{j=1}^n g_j \lambda_j \ge 0.$$

One easily verifies that this condition is also sufficient when $n = 2$, but not for larger n. If, however, as in the case $n = 2$, in the sequence $(g_1, \ldots, g_n)$

there is *not more than one change of sign*, then condition (3.22) is sufficient for f_n to be nonnegative, also when $n > 2$. Moreover, the density f_n turns out to be *infinitely divisible* in that case.

In order to show this we suppose that $\delta_n := \sum_{j=1}^n g_j \lambda_j \geq 0$ and that the g_j satisfy for some $m \in \{1, \ldots, n-1\}$:

$$(3.23) \qquad g_1 > 0, \ldots, g_m > 0,\; g_{m+1} < 0, \ldots, g_n < 0.$$

Now, observe that the mixture f_n can be written in the following form:

$$(3.24) \qquad f_n(x) = \sum_{j=1}^n g_j \lambda_j \left(\mathrm{e}^{-\lambda_j x} - \mathrm{e}^{-\lambda_m x}\right) + \delta_n\, \mathrm{e}^{-\lambda_m x} \qquad [\,x > 0\,].$$

Then it follows that $f_n(x) \geq 0$ for all x, so f_n is a probability density. For proving that f_n is infinitely divisible, one could proceed as before and look for the zeroes of the corresponding pLSt π_n; $n-2$ of them are easily (and similarly) found, whereas in case $\delta_n > 0$ an additional zero is obtained by observing that $\pi_n(s) = \{\delta_n + o(1)\}/\{s + O(1)\}$ as $s \to -\infty$. Thus we find that f_n is *infinitely divisible* with canonical density k_n given by

$$(3.25) \qquad k_n(x) = \sum_{j \neq m} \left(\mathrm{e}^{-\lambda_j x} - \mathrm{e}^{-\mu_j x}\right) + \mathrm{e}^{-\lambda_m x} \qquad [\,x > 0\,],$$

where $\mu_j \in (\lambda_j, \lambda_{j+1})$ for $j \leq n-1$ (and $j \neq m$), and $\mu_n > \lambda_n$ if $\delta_n > 0$, and $\mu_n := \infty$ (so $\mathrm{e}^{-\mu_n x} = 0$) if $\delta_n = 0$. This searching for zeroes of π_n can be avoided, however, by making use of the special representation of f_n in (3.24); taking Lt's there, one easily verifies that π_n can be written as

$$(3.26) \qquad \pi_n(s) = \Big\{\sum_{j=1}^n g_j\, \frac{\lambda_m - \lambda_j}{\lambda_m}\, \frac{\lambda_j}{\lambda_j + s} + \frac{\delta_n}{\lambda_m}\Big\}\, \frac{\lambda_m}{\lambda_m + s}.$$

So, π_n corresponds to the convolution of an exponential distribution and a finite mixture of exponential distributions (with nonnegative weights). Therefore, we can apply Theorem 3.3 and (3.8) to conclude again that π_n is *infinitely divisible* with canonical density k_n as given by (3.25).

This result, including the *factorization* in (3.26), can be extended to all generalized mixtures of exponential densities in the following way.

Theorem 3.10. *Let G be a right-continuous bounded function satisfying $G(\lambda) = 0$ for $\lambda \leq 0$ and such that for some $\lambda_0 > 0$*

$$(3.27) \qquad \begin{cases} G \text{ is nondecreasing on } [\,0, \lambda_0), \\ G \text{ is nonincreasing on } [\,\lambda_0, \infty) \text{ with } \lim_{\lambda \to \infty} G(\lambda) = 1. \end{cases}$$

With this G, let the function f on $(0,\infty)$ and its Lt π be defined by

$$(3.28)\qquad f(x) = \int_{(0,\infty)} \lambda\, \mathrm{e}^{-\lambda x}\, dG(\lambda), \quad \pi(s) = \int_{(0,\infty)} \frac{\lambda}{\lambda+s}\, dG(\lambda).$$

Then f can be viewed as a probability density iff $\delta := \int_{(0,\infty)} \lambda\, dG(\lambda) \ge 0$, in which case f is infinitely divisible; in fact, π can then be written as

$$(3.29)\qquad \pi(s) = \pi_1(s)\, \pi_2(s),$$

where π_1 is the pLSt of the exponential (λ_0) distribution and π_2 is the pLSt of a mixture of exponential distributions.

Proof. Note that $\int_0^\infty f(x)\, \mathrm{d}x = \int_{(0,\infty)} \mathrm{d}G(\lambda) = 1$, and that $f(0+) = \delta$ with $\delta \in [-\infty, \infty)$. So, if $\delta < 0$, then f is not a density. Let further $\delta \ge 0$. Then f can be written as

$$f(x) = \int_{(0,\infty)} \lambda\, (\mathrm{e}^{-\lambda x} - \mathrm{e}^{-\lambda_0 x})\, \mathrm{d}G(\lambda) + \delta\, \mathrm{e}^{-\lambda_0 x} \qquad [\,x > 0\,],$$

so $f(x) \ge 0$ for all x, and by taking Lt's we see that π can be put in the following form:

$$\pi(s) = \Big\{ \int_{(0,\infty)} \frac{\lambda}{\lambda+s}\, \frac{1}{\lambda_0}\, (\lambda_0 - \lambda)\, \mathrm{d}G(\lambda) + \frac{\delta}{\lambda_0} \Big\}\, \frac{\lambda_0}{\lambda_0+s}.$$

Hence π satisfies (3.29) with π_1 and π_2 as indicated. From Theorem 3.3 it now follows that π is infinitely divisible. ◻

By writing $\mathrm{e}^{-\lambda x}$ as $\int_\lambda^\infty x\, \mathrm{e}^{-tx}\, \mathrm{d}t$ and changing the order of integration we see that the function f in (3.28) can also be written as

$$(3.30)\qquad f(x) = x \int_0^\infty \mathrm{e}^{-\lambda x}\, A(\lambda)\, \mathrm{d}\lambda \qquad [\,x > 0\,],$$

where $A(\lambda) := \int_{(0,\lambda]} y\, \mathrm{d}G(y)$ for $\lambda > 0$. Note that A, like G, is unimodal; it satisfies (3.27), but with $\lim_{\lambda\to\infty} A(\lambda) = \delta$. Let further $\delta \ge 0$. Then A is *nonnegative*, so from (3.30) it is seen once more that $f(x) \ge 0$ for all x. Moreover, it follows that f is a *mixture of gamma (2) densities*, i.e., of gamma densities with shape parameter $r = 2$; the mixing function is rather special: it is absolutely continuous with density $\lambda \mapsto A(\lambda)/\lambda^2$ on $(0,\infty)$. The infinite divisibility of these special mixtures, which follows from Theorem 3.10, will be extended in Section 4 to *all* mixtures of gamma (2) distributions.

A second generalization concerns an extension to distributions on $\mathbb{R}$; we will show that in mixing exponential distributions also 'negative scales' are allowed. To do so we start with using a mixture of exponential distribution functions as a mixing function in the *power mixture* of (2.1) or (2.2). Applying Proposition 2.1 then leads to the following generalization of Theorem 3.3.

Theorem 3.11. *If ϕ_1 is an infinitely divisible characteristic function, then so is ϕ given by*

$$\phi(u) = \alpha + (1-\alpha) \int_{(0,\infty)} \frac{\lambda}{\lambda - \log\, \phi_1(u)}\, dG(\lambda), \tag{3.31}$$

where $\alpha \in [\,0,1\,]$ and G is a distribution function on $(0,\infty)$.

Now, take for ϕ_1 the characteristic function of a *normal* $(0,2)$ distribution, so take $\phi_1(u) = \exp\,[-u^2]$, and replace G by a distribution function of the form $\lambda \mapsto G(\sqrt{\lambda})$; then for ϕ in (3.31) we get

$$\phi(u) = \alpha + (1-\alpha) \int_{(0,\infty)} \frac{\lambda^2}{\lambda^2 + u^2}\, \mathrm{d}G(\lambda). \tag{3.32}$$

Since the integrand here is recognized as the characteristic function of the *Laplace* (λ) distribution, we are led to the result in Theorem 3.12 below. Essentially, it was already obtained as Theorem IV.10.1 with the same proof; but now the degenerate distribution at zero is also viewed as a Laplace distribution, 'with $\lambda = \infty$'. The final statement follows as before by rewriting (3.32) similar to (3.2).

Theorem 3.12. *A mixture of Laplace distributions is infinitely divisible. Equivalently, a random variable X with $X \stackrel{\mathrm{d}}{=} WV$ is infinitely divisible if W and V are independent, W is nonnegative and V has a Laplace distribution.*

This result can easily be translated in terms of *scale mixtures* of exponential distributions; cf. the remark following (2.10). To do so, we let A be a symmetric Bernoulli variable with values ± 1. Then a random variable V is *Laplace* iff $V \stackrel{\mathrm{d}}{=} A\,Y$ with A and Y independent and Y *exponential*. Moreover, a random variable Z is *symmetric* iff $Z \stackrel{\mathrm{d}}{=} A\,W$ with A and W independent and W *nonnegative*. Thus Theorem 3.12 implies that a random variable X with $X \stackrel{\mathrm{d}}{=} Z\,Y$ is *infinitely divisible* if Z and Y are independent, Z has a *symmetric* distribution and Y is exponential.

It can be shown, however, that symmetry of Z is not necessary here; any characteristic function ϕ of the following form is infinitely divisible:

$$\phi(u) = \int_{\mathbb{R}} \frac{1}{1 - iu\theta}\, \mathrm{d}H(\theta), \tag{3.33}$$

with H a distribution function on $\mathbb{R}$. In proving this we prefer to rewrite (3.33) as

$$\phi(u) = \alpha + \beta \int_{(0,\infty)} \frac{\lambda}{\lambda - iu}\, \mathrm{d}G_1(\lambda) + \gamma \int_{(0,\infty)} \frac{\lambda}{\lambda + iu}\, \mathrm{d}G_2(\lambda), \tag{3.34}$$

where $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$ and G_1 and G_2 are distribution functions on $(0,\infty)$. As before, ϕ can be written as the pointwise limit of finite mixtures ϕ_n of the form

$$\phi_n(u) = \sum_{j=1}^n g_j \frac{\lambda_j}{\lambda_j - iu} + \sum_{j=1}^n g_j' \frac{\lambda_j'}{\lambda_j' + iu}, \tag{3.35}$$

where the g_j and g_j' are positive with $\sum_{j=1}^n (g_j + g_j') = 1$, $0 < \lambda_1 < \cdots < \lambda_n$ and $0 < \lambda_1' < \cdots < \lambda_n'$. Hence by Proposition IV.2.3 it is sufficient to show that such a ϕ_n is infinitely divisible. To do so we proceed as before; we extend ϕ_n analytically and write for $s \notin \{-\lambda_1, \ldots, -\lambda_n, \lambda_1', \ldots, \lambda_n'\}$:

$$\phi_n(is) = \sum_{j=1}^n g_j \frac{\lambda_j}{\lambda_j + s} + \sum_{j=1}^n g_j' \frac{\lambda_j'}{\lambda_j' - s} = \frac{Q_{2n-1}(s)}{P_{2n}(s)}, \tag{3.36}$$

where P_{2n} is a polynomial of degree $2n$, and Q_{2n-1} is a polynomial of degree $2n-2$ if $\delta_n = 0$, and of degree $2n-1$ if $\delta_n \neq 0$; here we define $\delta_n := \sum_{j=1}^n (g_j\lambda_j - g_j'\lambda_j')$. As in the proof of Theorem 3.3 one sees that Q_{2n-1} has $2n-2$ zeroes $-\mu_1, \ldots, -\mu_{n-1}$ and $\mu_1', \ldots, \mu_{n-1}'$ satisfying $\mu_j \in (\lambda_j, \lambda_{j+1})$ and $\mu_j' \in (\lambda_j', \lambda_{j+1}')$ for all j. As in the first proof of Theorem 3.10, in case $\delta_n \neq 0$ we find a final zero $-\mu_n$ satisfying $\mu_n > \lambda_n$, or μ_n' satisfying $\mu_n' > \lambda_n'$, depending on the sign of δ_n. Put the quantities μ_n and/or μ_n' that are not yet defined, equal to ∞, and agree that $\mu/(\mu + c) := 1$ if $\mu = \infty$. Then we conclude that in all cases $\phi_n(is)$ can be written as

$$\phi_n(is) = \Bigl(\prod_{j=1}^n \frac{\lambda_j}{\lambda_j + s} \Big/ \frac{\mu_j}{\mu_j + s}\Bigr) \Bigl(\prod_{j=1}^n \frac{\lambda_j'}{\lambda_j' - s} \Big/ \frac{\mu_j'}{\mu_j' - s}\Bigr). \tag{3.37}$$

Now, use (3.5); since $\mu_j > \lambda_j$ and $\mu_j' > \lambda_j'$ for all j, it follows that ϕ_n is of the form

$$(3.38)\qquad \phi_n(u) = \phi_n^{(1)}(u)\,\phi_n^{(2)}(u),$$

with $\phi_n^{(1)}$ the characteristic function of an infinitely divisible distribution on $\mathbb{R}_+$ and $\phi_n^{(2)}$ the characteristic function of an infinitely divisible distribution on $\mathbb{R}_-$. Therefore, ϕ is *infinitely divisible*. We thus have proved the following result.

Theorem 3.13. *A mixture of the form (3.34) is infinitely divisible. Equivalently, a random variable X with $X \stackrel{d}{=} Z\,Y$ is infinitely divisible if Z and Y are independent and Y is exponential.*

Taking $\alpha = 0$ in (3.34) we get the characteristic function of an absolutely continuous distribution with density f given by

$$(3.39)\qquad f(x) = \begin{cases} \beta \int_{(0,\infty)} \lambda\,\mathrm{e}^{-\lambda x}\,\mathrm{d}G_1(\lambda) & \text{, if } x > 0, \\ (1-\beta) \int_{(0,\infty)} \lambda\,\mathrm{e}^{-\lambda |x|}\,\mathrm{d}G_2(\lambda) & \text{, if } x < 0. \end{cases}$$

Since by Bernstein's theorem a completely monotone density is a mixture of exponential densities (see Proposition A.3.11), for this case Theorem 3.13 can be reformulated as follows.

Corollary 3.14. *A probability density f on $\mathbb{R}$ with the property that both $x \mapsto f(x)$ and $x \mapsto f(-x)$ are completely monotone on $(0,\infty)$, is infinitely divisible.*

4. Mixtures of gamma distributions

The gamma distribution with parameters $r > 0$ and $\lambda > 0$ is a probability distribution on $\mathbb{R}_+$ with density $f_{r,\lambda}$ and pLSt $\pi_{r,\lambda}$ given by

$$f_{r,\lambda}(x) = \frac{\lambda^r}{\Gamma(r)}\,x^{r-1}\,\mathrm{e}^{-\lambda x}, \quad \pi_{r,\lambda}(s) = \Big(\frac{\lambda}{\lambda + s}\Big)^r.$$

We want to generalize the fact that mixtures of exponential distributions are infinitely divisible; cf. Theorem 3.3. Therefore, we first mix with respect to λ at a *fixed* value of r; the resulting *mixtures of gamma (r) densities* have the form

$$(4.1)\qquad f(x) = \int_{(0,\infty)} \frac{\lambda^r}{\Gamma(r)}\,x^{r-1}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}G(\lambda) \qquad [\,x > 0\,],$$

with G a distribution function on $(0,\infty)$, or, equivalently (by Bernstein's theorem),

$$f(x) = x^{r-1}\,\psi(x) \qquad [\,x>0\,], \tag{4.2}$$

with ψ *completely monotone*. Slightly more generally, as in Section 3 we allow an additional mixing with the degenerate distribution at zero; by a *mixture of gamma* (r) *distributions* we understand any distribution on $\mathbb{R}_+$ with pLSt π of the following form:

$$\pi(s) = \alpha + (1-\alpha)\int_{(0,\infty)} \Big(\frac{\lambda}{\lambda+s}\Big)^r \,\mathrm{d}G(\lambda), \tag{4.3}$$

with $\alpha \in [\,0,1\,]$ and G a distribution function on $(0,\infty)$. Of course, Proposition 3.1 and its proof generalize to the present situation; the class of mixtures of gamma (r) distributions is *closed under weak convergence.*

Proposition 4.1. *If a sequence* (π_n) *of mixtures of gamma* (r) *pLSt's converges (pointwise) to a pLSt* π*, then* π *is a mixture of gamma* (r) *pLSt's.*

Also, the mixtures (4.3) are *scale mixtures*; they correspond to random variables of the form ZY with Z *nonnegative* and Y gamma (r), and Z and Y independent. By giving Z an appropriate *beta* distribution we can thus obtain the gamma distributions with shape parameter smaller than r; in fact, as is well known, for all $\alpha \in (0,1)$ and $\lambda > 0$:

$$Z \text{ beta}\big(\alpha r, (1-\alpha)\, r\big),\;\; Y \text{ gamma}\,(r,\lambda) \implies ZY \text{ gamma}\,(\alpha r, \lambda). \tag{4.4}$$

Multiplying ZY here by a nonnegative random variable proves the following result; for densities it is also immediate from (4.2).

Proposition 4.2. *For* $\alpha \in (0,1)$*, mixtures of gamma* (αr) *distributions can be regarded as mixtures of gamma* (r) *distributions.*

It follows that for $r \le 1$ all mixtures of gamma (r) distributions are *infinitely divisible.* On the other hand one can show (see Section 12) that for $r > 2$ these mixtures are in general *not* infinitely divisible. Hence by Proposition III.2.2 there must exist r_0 with $1 \le r_0 \le 2$ such that all mixtures of gamma (r) distributions are infinitely divisible if $r \le r_0$, and not all such mixtures are infinitely divisible if $r > r_0$. We will show that $r_0 = 2$: *all mixtures of gamma (2) distributions are infinitely divisible.*

We may expect the proof in this boundary case to be harder than in the exponential case where $r = 1$. Nevertheless, we try to proceed as for Theorem 3.3, and consider first *finite* mixtures of gamma (2) densities; for $n \geq 2$ let the density f_n on $(0, \infty)$ and its Lt π_n be given by

$$f_n(x) = \sum_{j=1}^{n} g_j \, \lambda_j^2 \, x \, \mathrm{e}^{-\lambda_j x}, \quad \pi_n(s) = \sum_{j=1}^{n} g_j \Big(\frac{\lambda_j}{\lambda_j + s}\Big)^2, \tag{4.5}$$

where the g_j are positive satisfying $\sum_{j=1}^{n} g_j = 1$ and $0 < \lambda_1 < \cdots < \lambda_n$. Then π_n can be written as $\pi_n = Q_{2n-2}/P_{2n}$, where Q_{2n-2} and P_{2n} are polynomials of degree $2n - 2$ and $2n$, respectively. Extend the domain of both polynomials to $\mathbb{C}$; since $\pi_n(s) > 0$ for $s \in \mathbb{R} \setminus \{-\lambda_1, \ldots, -\lambda_n\}$, the $2n - 2$ zeroes of Q_{2n-2} occur in $n - 1$ pairs of complex conjugates $(-\sigma_1, -\overline{\sigma}_1)$, …,$(-\sigma_{n-1}, -\overline{\sigma}_{n-1})$, say. Hence π_n can be written as

$$\pi_n(s) = \Big\{\prod_{j=1}^{n} \Big(\frac{\lambda_j}{\lambda_j + s}\Big)^2\Big\} \Big/ \Big\{\prod_{j=1}^{n-1} \frac{\sigma_j}{\sigma_j + s} \, \frac{\overline{\sigma}_j}{\overline{\sigma}_j + s}\Big\}. \tag{4.6}$$

Using (3.5), including an extension to complex λ, we now see that π_n can be put in the form (3.4) with k replaced by k_n given by

$$\begin{aligned} k_n(x) &= 2 \sum_{j=1}^{n} \mathrm{e}^{-\lambda_j x} - \sum_{j=1}^{n-1} \{\mathrm{e}^{-\sigma_j x} + \mathrm{e}^{-\overline{\sigma}_j x}\} = \\ &= 2 \sum_{j=1}^{n-1} (\mathrm{e}^{-\lambda_j x} - \mathrm{e}^{-\mu_j x} \cos \nu_j x) + 2 \, \mathrm{e}^{-\lambda_n x} \qquad [x > 0], \end{aligned} \tag{4.7}$$

where we have written $\sigma_j = \mu_j + i\nu_j$ for all j; without restriction we suppose that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$. We conclude that the following condition is sufficient for infinite divisibility of π_n:

$$\sum_{j=1}^{n-1} \mathrm{e}^{-\lambda_j x} \geq \sum_{j=1}^{n-1} \mathrm{e}^{-\mu_j x} \qquad [x > 0]. \tag{4.8}$$

In proving this inequality one might hope that $\mu_j \geq \lambda_j$ for all j, as in the case $r = 1$. The situation is, however, not that simple. It is not difficult to show that $\mu_j \in (\lambda_1, \lambda_n)$ for all j, but this helps us only when $n = 2$. It turns out that (4.8) can be proved by using the following special case of *Karamata's inequality*; cf. Section A.5.

Lemma 4.3. *Let the function* $f : \mathbb{R} \to \mathbb{R}$ *be convex and nondecreasing, and let* $(a_1, \ldots, a_n)$ *and* $(b_1, \ldots, b_n)$ *be finite nondecreasing sequences in* $\mathbb{R}$ *satisfying* $\sum_{j=1}^{m} a_j \leq \sum_{j=1}^{m} b_j$ *for* $m = 1, \ldots, n$. *Then:*

$$\sum_{j=1}^{n} f(-a_j) \geq \sum_{j=1}^{n} f(-b_j).$$

By taking f in this lemma of the form $t \mapsto \mathrm{e}^{tx}$ with $x > 0$ and replacing n by $n-1$ we see that for obtaining (4.8) it is sufficient to have the following analytical result. Because of its crucial role we formulate it independently and in a slightly more general setting.

Lemma 4.4. *For* $n \in \mathbb{N}$ *with* $n \geq 2$ *let* R *be a function of the form*

$$R(z) = \sum_{j=1}^{n} \frac{A_j}{(\lambda_j + z)^2} \qquad \big[\, z \in \mathbb{C} \setminus \{-\lambda_1, \ldots, -\lambda_n\} \,\big],$$

where the A_j *are positive and* $0 < \lambda_1 < \cdots < \lambda_n$. *Denote the* $2n-2$ *zeroes of* R *by* $-\mu_j \pm i\,\nu_j$, $j = 1, \ldots, n-1$, *and order them in such a way that* $\mu_1 \leq \cdots \leq \mu_{n-1}$. *Then the following inequalities hold:*

$$\sum_{j=1}^{m} \lambda_j \leq \sum_{j=1}^{m} \mu_j \quad \text{for } m = 1, \ldots, n-1.$$

Unfortunately, this lemma is very hard to prove. Therefore, we do not give a proof; see Notes.

We conclude that the finite mixture π_n in (4.5) is infinitely divisible with canonical density k_n given by (4.7). In exactly the same way as in the proof of Theorem 3.3 we are led to the following generalization, the main result of this section.

Theorem 4.5. *A mixture of gamma (2) distributions is infinitely divisible. Equivalently, a random variable* X *with* $X \overset{\mathrm{d}}{=} Z\,Y$ *is infinitely divisible if* Z *and* Y *are independent,* Z *is nonnegative and* Y *is gamma (2).*

Because of (4.2) the theorem can be reformulated in the following appealing way; this also leads to a result on so-called *size-biased* densities.

Corollary 4.6. *A probability density* f *on* $(0, \infty)$ *is infinitely divisible if*

$$(4.9) \qquad f(x) = x\,\psi(x) \qquad [\, x > 0 \,]$$

with ψ completely monotone. Consequently, f is infinitely divisible if

$$(4.10)\qquad f(x) = \frac{1}{\mu}\, x\, g(x) \qquad [\, x > 0\,]$$

with g a mixture of exponential densities with finite mean μ.

As we saw above, the zeroes of a mixture of gamma (2) pLSt's are not so simply related to the poles as for mixtures of exponential pLSt's. Therefore, representation and characterization theorems and generalizations to 'negative probabilities' and 'negative scales' as in the preceding section seem hard to obtain for mixtures of gamma (2) distributions. We only note that a pLSt π that is a generalized mixture of *two* gamma (2) pLSt's:

$$(4.11)\qquad \pi(s) = \alpha\, \Bigl(\frac{\lambda_1}{\lambda_1 + s}\Bigr)^2 + (1-\alpha)\, \Bigl(\frac{\lambda_2}{\lambda_2 + s}\Bigr)^2,$$

with $0 < \lambda_1 < \lambda_2$ and $\alpha > 0$, is infinitely divisible for *all* α, also when $\alpha > 1$; see Section 12. On the other hand, we do have analogues of Theorems 3.11 and 3.12, of course.

Theorem 4.7. *If ϕ_1 is an infinitely divisible characteristic function, then so is ϕ given by*

$$(4.12)\qquad \phi(u) = \alpha + (1-\alpha) \int_{(0,\infty)} \Bigl(\frac{\lambda}{\lambda - \log\, \phi_1(u)}\Bigr)^2\, dG(\lambda),$$

where $\alpha \in [\,0,1\,]$ and G is a distribution function on $(0,\infty)$. In particular, ϕ is an infinitely divisible characteristic function if ϕ is of the form

$$(4.13)\qquad \phi(u) = \alpha + (1-\alpha) \int_{(0,\infty)} \Bigl(\frac{\lambda^2}{\lambda^2 + u^2}\Bigr)^2\, dG(\lambda),$$

so a random variable X with $X \stackrel{\rm d}{=} WV$ is infinitely divisible if W and V are independent, W is nonnegative and V has a sym-gamma (2) distribution.

Also the remark following Theorem 3.12 has an analogue. With (4.13) one might expect to have obtained the infinite divisibility of the random variables $X \stackrel{\rm d}{=} ZY$ with Z and Y independent, Z symmetric and Y gamma (2), but this is *not* so: If A is a symmetric Bernoulli variable with values ± 1 and independent of Y, then AY is *double-gamma* (2) and hence *not* infinitely divisible; cf. Example IV.11.15. The right analogue is obtained by observing that a random variable V is *sym-gamma* (2) iff $V \stackrel{\rm d}{=} AY$ with A and Y independent and Y having density f given by

$$(4.14)\qquad f(x) = \tfrac{1}{2}\,(1 + x)\, {\rm e}^{-x} \qquad [\, x > 0\,],$$

which is a mixture of an exponential and a gamma (2) density. Using again the fact that a random variable Z is *symmetric* iff $Z \stackrel{d}{=} AW$ with A and W independent and W *nonnegative*, one is then led to the following consequence of the last part of Theorem 4.7.

Corollary 4.8. *A random variable X with $X \stackrel{d}{=} ZY$ is infinitely divisible if Z and Y are independent, Z has a symmetric distribution and Y has density f given by (4.14).*

From Proposition 4.2 it follows that in Theorem 4.7 the exponent 2 may be replaced by any positive $r < 2$. Moreover, we may mix with respect to such an r. We show this by generalizing Theorem 4.5 rather than Theorem 4.7.

Theorem 4.9. *A pLSt π of the form*

$$\pi(s) = \alpha + (1-\alpha) \int_{(0,2]\times(0,\infty)} \Big(\frac{\lambda}{\lambda+s}\Big)^r \, dG(r,\lambda), \tag{4.15}$$

with $\alpha \in [0,1]$ and G a distribution function on $(0,2] \times (0,\infty)$, is infinitely divisible.

PROOF. According to Proposition 4.2, or (4.4), a gamma (r,λ) distribution with $r \leq 2$ can be represented as a mixture of gamma (2) distributions:

$$\Big(\frac{\lambda}{\lambda+s}\Big)^r = \int_{(0,\infty)} \Big(\frac{\mu}{\mu+s}\Big)^2 \, \mathrm{d}H_{r,\lambda}(\mu);$$

in fact, $H_{r,\lambda}$ is the distribution function of λ/Z with Z beta $(r, 2-r)$ distributed. Inserting this in (4.15) shows that π can be put in the form

$$\pi(s) = \alpha + (1-\alpha) \int_{(0,\infty)} \Big(\frac{\mu}{\mu+s}\Big)^2 \, \mathrm{d}H(\mu),$$

where $H(\mu) := \int_{(0,2]\times(0,\infty)} H_{r,\lambda}(\mu) \, \mathrm{d}G(r,\lambda)$ for $\mu > 0$. From Theorem 4.5 we conclude that π is infinitely divisible. ◻

It follows that for $\lambda > 0$ and a distribution function G on $(0,\infty)$ the *power mixture*

$$\pi(s) = \int_{(0,\infty)} \Big(\frac{\lambda}{\lambda+s}\Big)^r \, \mathrm{d}G(r) \tag{4.16}$$

is infinitely divisible if the support $S(G)$ of G is restricted to $(0,2]$. Of course, another sufficient condition is infinite divisibility of G; cf. Proposition 2.1.

5. Generalized gamma convolutions

In this section we consider a class of infinitely divisible distributions that is closely related to the mixtures of gamma distributions discussed in Sections 3 and 4. About this class of so-called *generalized gamma convolutions* a whole book has been written; see Notes. Therefore, we want to be rather brief, and it will be impossible to do justice to this very rich, but rather technical subject. We shall present some highlights, often without giving detailed proofs. Emphasis will be on results providing the infinite divisibility of distributions, not on specific properties of or relations between generalized gamma convolutions.

A distribution on $\mathbb{R}_+$ is said to be a *generalized gamma convolution* if it is the weak limit of finite convolutions of gamma distributions. Equivalently, as has been done in the book mentioned above, the generalized gamma convolutions can be defined by the specific form of their pLSt's. This form is suggested by rewriting the pLSt π_n of a convolution of m_n gamma distributions in the following way:

$$(5.1)\qquad \pi_n(s) = \prod_{j=1}^{m_n} \Big(\frac{\lambda_{n,j}}{\lambda_{n,j}+s}\Big)^{r_{n,j}} = \exp\Big[\sum_{j=1}^{m_n} r_{n,j}\,\log\frac{\lambda_{n,j}}{\lambda_{n,j}+s}\Big],$$

and then letting $n \to \infty$. Also note that the degenerate distribution at $a > 0$ can be obtained as the weak limit of gamma $(n, n/a)$ distributions as $n \to \infty$. Thus one is led to the following *representation* result.

Theorem 5.1 (Canonical representation). *A function π on $\mathbb{R}_+$ is the pLSt of a generalized gamma convolution iff it has the form*

$$(5.2)\qquad \pi(s) = \exp\Big[-as + \int_{(0,\infty)} \log\frac{\lambda}{\lambda+s}\,dU(\lambda)\Big] \qquad [\,s \ge 0\,],$$

where $a \ge 0$ and U is an LSt-able function with $U(0) = 0$ and, necessarily,

$$(5.3)\qquad \int_{(0,1]} \log\frac{1}{\lambda}\,dU(\lambda) < \infty, \quad \int_{(1,\infty)} \frac{1}{\lambda}\,dU(\lambda) < \infty.$$

Here condition (5.3) is equivalent to the integral in (5.2) being finite. The proof of Theorem 5.1 is related to the not surprising fact that the class of generalized gamma convolutions is *closed under weak convergence.*

Proposition 5.2. *If a sequence (π_n) of LSt's of generalized gamma convolutions converges (pointwise) to a pLSt π, then π is the LSt of a generalized gamma convolution.*

The proof of this result is similar to that of Proposition 3.1 for mixtures of exponential distributions. Clearly, the class of generalized gamma convolutions is also closed with respect to convolutions and convolution roots. In particular, a generalized gamma convolution is *infinitely divisible*; this also follows from its definition and Section III.2, of course. As is known from Section III.4, the LSt $\widehat{K}$ of the canonical function K of π as in (5.2) is given by the ρ-function of π for which

$$(5.4)\qquad \rho(s) := -\frac{\mathrm{d}}{\mathrm{d}s}\log \pi(s) = a + \int_{(0,\infty)} \frac{1}{\lambda+s}\,\mathrm{d}U(\lambda).$$

It follows that $K(0) = a$, so a is the *left extremity* of the distribution corresponding to π, and for many purposes it is sufficient to consider the case where $a = 0$. By using Fubini's theorem in (5.4) one easily shows that $K - K(0)$ is absolutely continuous with density $k = \widehat{U}$, so with k *completely monotone*. We conclude that the canonical pair (a, U) of π is uniquely determined by π; the function U is called the *Thorin function* of π (and of the corresponding distribution); cf. Notes. We have thus proved the direct part of the following *characterization* result; the converse part is similarly obtained from Theorem 5.1 by using Bernstein's theorem and Theorem III.4.2 or III.4.3.

Theorem 5.3. *A distribution with left extremity zero is a generalized gamma convolution iff it is infinitely divisible with an absolutely continuous canonical function K having a completely monotone density k. In this case $k = \widehat{U}$ with U the Thorin function of the distribution.*

We apply this theorem to obtain a sufficient condition for infinite divisibility of a so-called *size-biased* density; see Corollary 4.6 for another sufficient condition.

Proposition 5.4. *A probability density f on $(0,\infty)$ is infinitely divisible if it is of the form*

$$(5.5)\qquad f(x) = \frac{1}{\mu}\, x\, g(x) \qquad [\,x > 0\,],$$

with g a density of a generalized gamma convolution with left extremity zero and finite mean μ.

PROOF. Let π_1 be the Lt of g and let ρ_1 be the ρ-function of π_1; according to Theorem 5.3 ρ_1 is the Lt of a completely monotone function. Now, note that the Lt π of f can be written as

$$\pi(s) = -\frac{1}{\mu}\,\pi_1'(s) = \frac{1}{\mu}\,\pi_1(s)\,\rho_1(s) = \pi_1(s)\,\pi_2(s),$$

where $\pi_2 := \rho_1/\mu$ is the Lt of a probability density that is completely monotone and hence infinitely divisible. It follows that $\pi = \pi_1\pi_2$ is also infinitely divisible. ◻◻

Theorem 5.3 can also be used to relate the generalized gamma convolutions to the *self-decomposable* and *stable* distributions on $\mathbb{R}_+$. Since by Theorem V.2.11 an infinitely divisible distribution on $\mathbb{R}_+$ having a *nonincreasing* canonical density k is self-decomposable, it is immediately clear that a generalized gamma convolution is self-decomposable; this also follows from its definition, of course, because the class of self-decomposable distributions contains the gamma distributions and is closed with respect to convolutions and weak limits. In view of Theorems V.2.16, V.2.17 and Corollary V.2.18 we may conclude the following.

Proposition 5.5. *A generalized gamma convolution is self-decomposable. When its left extremity is zero, it is unimodal and absolutely continuous with a density f that is positive and continuous on $(0,\infty)$; moreover, f is nonincreasing or, equivalently, $f(0+) > 0$ (possibly ∞) iff the canonical density k satisfies $k(0+) \le 1$.*

So, the generalized-gamma-convolution property is much stronger than self-decomposability. This is also apparent from Theorem V.2.6 and the following result, which can be derived with some difficulty (by induction) from Theorem 5.3.

Theorem 5.6. *A pLSt π is the LSt of a generalized gamma convolution iff its ρ-function has the following property:*

$$s \mapsto \frac{d^m}{ds^m}\big[s^m \rho(s)\big] \text{ is completely monotone for all } m \in \mathbb{Z}_+.$$

As is known from Section V.3, the class of self-decomposable distributions on $\mathbb{R}_+$ contains the stable distributions on $\mathbb{R}_+$, with LSt's of the form

(5.6) $\qquad \pi(s) = \exp\,[-\lambda\, s^\gamma],$

where $\lambda > 0$ and $\gamma \in (0,1\,]$. Moreover, if $\gamma < 1$, then π in (5.6) has canonical density k given by $k(x) = \lambda\gamma\, x^{-\gamma}/\Gamma(1-\gamma)$ for $x > 0$. Theorem 5.3 now immediately implies the following result (see Section A.5 for the gamma function); alternatively, one can use Theorem 5.6 and (5.4).

Proposition 5.7. *A stable distribution on $\mathbb{R}_+$ is a generalized gamma convolution. Moreover, if $\gamma < 1$, then π in (5.6) has an absolutely continuous Thorin function U with density u on $(0,\infty)$ given by*

$$u(x) = \frac{\lambda\gamma}{\Gamma(\gamma)\,\Gamma(1-\gamma)}\, x^{-(1-\gamma)} = \lambda\gamma\, \frac{\sin\gamma\pi}{\pi}\, x^{-(1-\gamma)}.$$

Before identifying a more interesting subclass of the class of generalized gamma convolutions, we establish a connection with *mixtures of gamma distributions*. First, recall from Theorem 3.8 that the mixtures of *exponential* distributions correspond to the infinitely divisible distributions on $\mathbb{R}_+$ that have a canonical density k such that $x \mapsto k(x)/x$ is the Lt of a function v satisfying $0 \le v \le 1$; the function v is called the second canonical density of the mixture. Now, by using Fubini's theorem one easily verifies that if $k = \widehat{U}$ then $x \mapsto k(x)/x$ is the Lt of the function U:

$$(5.7) \qquad \frac{1}{x}\, k(x) = \int_0^\infty e^{-\lambda x}\, U(\lambda)\, d\lambda \qquad [\,x > 0\,].$$

From Theorem 5.3 we conclude that the generalized gamma convolutions that can be viewed as mixtures of exponential distributions, can be characterized as follows.

Proposition 5.8. *A generalized gamma convolution with canonical pair (a, U) is a mixture of exponential distributions iff $a = 0$ and U satisfies $\lim_{\lambda\to\infty} U(\lambda) \le 1$. In this case for the second canonical density v of the mixture one may take $v = U$.*

In view of Proposition 5.5 and because $\lim_{\lambda\to\infty} U(\lambda) = k(0+)$, Proposition 5.8 can be reformulated in the following way.

Corollary 5.9. *The continuous density f of a generalized gamma convolution with left extremity zero is completely monotone iff it is monotone. Equivalently, f is completely monotone iff $f(0+) > 0$ (possibly ∞).*

The computation leading to (5.7) can be reversed; so if (5.7) holds with U an LSt-able function, then $k = \widehat{U}$. Together with Theorem 5.3 and Proposition 5.8 this shows that also the mixtures of exponential distributions that can be viewed as generalized gamma convolutions, can be identified.

Corollary 5.10. *A mixture of exponential distributions with second canonical density v is a generalized gamma convolution iff v can be chosen to be nondecreasing.*

By using the canonical representation of Theorem 5.1 one easily sees that the main part of Proposition 5.8 can be generalized as follows.

Corollary 5.11. *For $r > 0$ a generalized gamma convolution with canonical pair (a, U) is the r-fold convolution of a mixture of exponential distributions iff $a = 0$ and U satisfies $\lim_{\lambda\to\infty} U(\lambda) \le r$.*

The generalized gamma convolutions in this corollary, at a fixed r, can also be viewed as *mixtures of gamma (r) distributions*. To show this we start with considering the convolution of a gamma (s, λ) and a gamma (t, μ) distribution, and set $r := s + t$. When $\mu = \lambda$ we get a gamma (r, λ) distribution, and if $\mu \neq \lambda$, then the convolution has density f on $(0, \infty)$ with

$$f(x) = \int_0^x \frac{\lambda^s}{\Gamma(s)}\,(x-y)^{s-1}\,\mathrm{e}^{-\lambda(x-y)}\,\frac{\mu^t}{\Gamma(t)}\,y^{t-1}\,\mathrm{e}^{-\mu y}\,\mathrm{d}y =$$
$$= x^{r-1}\,\frac{\lambda^s\mu^t}{\Gamma(s)\,\Gamma(t)}\int_0^1 \mathrm{e}^{-\{\lambda v+\mu(1-v)\}x}\,v^{s-1}(1-v)^{t-1}\,\mathrm{d}v,$$

so f has the form

(5.8) $\qquad f(x) = x^{r-1}\,\psi(x),\;$ with ψ completely monotone.

Convolving f in (5.8) with a third gamma density similarly yields a density of the form (5.8) with r now given by the sum of the three shape parameters. Iterating this procedure we find that π_n in (5.1) corresponds to a density of the form (5.8) with $r := \sum_{j=1}^{m_n} r_{n,j}$, so by (4.2) π_n is the LSt of a mixture of gamma (r) distributions. Now, by comparing (5.1) and (5.2) one sees that $r = \lim_{\lambda\to\infty} U_n(\lambda)$, with U_n the Thorin function of π_n. Since an arbitrary generalized gamma convolution with $\lim_{\lambda\to\infty} U(\lambda) = r$ finite can be obtained as the weak limit of generalized gamma convolutions as in (5.1) with $\sum_{j=1}^{m_n} r_{n,j} = r$, from Propositions 4.1 and 4.2 we conclude that the claim above does indeed hold. As in Proposition 5.8 there is a converse. To show this we first observe that the Thorin function U of a generalized gamma convolution with left extremity zero satisfies

$$\lim_{\lambda\to\infty} U(\lambda) = \lim_{s\to\infty} s\,\rho(s), \tag{5.9}$$

with ρ the corresponding ρ-function as given by (5.4) $\big($or by (5.11) below$\big)$. Next, one easily verifies that the ρ-function of a mixture of gamma$\,(r)$ distributions satisfies the inequality $s\,\rho(s) \le r$ for all s. Combining these two results immediately yields the desired converse. It can also be obtained by noting that a density f of the form (5.8) satisfies $f(x)/x^{r-1} \not\to 0$ as $x \downarrow 0$, and by using the following alternative for (5.9) (which can be proved by a well-known Tauberian theorem):

$$\lim_{\lambda\to\infty} U(\lambda) = \sup\big\{t > 0 : f(x)/x^{t-1} \to 0 \text{ as } x \downarrow 0\big\}, \tag{5.10}$$

where f is the continuous density of the generalized gamma convolution. Thus we have proved the following generalization of the main part of Proposition 5.8.

Theorem 5.12. *For $r > 0$ a generalized gamma convolution with canonical pair (a, U) is a mixture of gamma$\,(r)$ distributions iff $a = 0$ and U satisfies* $\lim_{\lambda\to\infty} U(\lambda) \le r$.

Combining this with Corollary 5.11 yields the following curious result.

Corollary 5.13. *For $r > 0$ a generalized gamma convolution is a mixture of gamma$\,(r)$ distributions iff it is the r-fold convolution of a mixture of exponential distributions.*

We stress that by no means all densities f of the form (5.8) correspond to generalized gamma convolutions. In fact, for $r > 2$ not all of them are infinitely divisible, and for $r = 1$ we have Corollary 5.10.

The densities f in (5.8) do correspond to generalized gamma convolutions if the complete monotonicity of ψ is replaced by so-called *hyperbolic complete monotonicity* of ψ. For proving this very useful result the *real* characterizations of the generalized gamma convolutions in Theorems 5.1, 5.3 and 5.6 will not suffice; we need a special *complex* characterization.

Return to the expression (5.4) for the *ρ-function* of a generalized gamma convolution with canonical pair (a, U):

$$\rho(s) = a + \int_{(0,\infty)} \frac{1}{\lambda + s}\, \mathrm{d}U(\lambda) \qquad [\, s > 0\,]; \tag{5.11}$$

this function of $s > 0$, which is a Stieltjes transform, has an analytic continuation to $\mathbb{C} \setminus (-\infty, 0\,]$ satisfying, as is easily verified, $\operatorname{Im} \rho(z) \geq 0$ if $\operatorname{Im} z < 0$. Now, there is a converse; this is a consequence of the following general result (see Notes).

Lemma 5.14. *A function* $\psi : (0, \infty) \to \mathbb{R}$ *has an analytic continuation to* $\mathbb{C} \setminus (-\infty, 0\,]$ *satisfying Im* $\psi(z) \geq 0$ *for Im* $z < 0$ *iff it has the form*

$$\psi(s) = -bs + c + \int_{\mathbb{R}_+} \Big(\frac{1}{\lambda + s} - \frac{\lambda}{1 + \lambda^2} \Big)\, dU(\lambda) \qquad [\, s > 0\,], \tag{5.12}$$

where $b \geq 0$, $c \in \mathbb{R}$, *and* U *is an LSt-able function with the property that* $\int_{\mathbb{R}_+} 1/(1 + \lambda^2)\, dU(\lambda) < \infty$.

Theorem 5.15. *A pLSt* π *is the LSt of a generalized gamma convolution iff its ρ-function has an analytic continuation to* $\mathbb{C} \setminus (-\infty, 0\,]$ *for which*

$$\text{Im}\ \rho(z) \geq 0 \ \text{ if Im}\ z < 0. \tag{5.13}$$

Equivalently, π *is the LSt of a generalized gamma convolution iff it has an analytic continuation to* $\mathbb{C} \setminus (-\infty, 0\,]$ *for which*

$$\text{Im}\ \pi'(z)\, \overline{\pi(z)} \leq 0 \ \text{ if Im}\ z < 0. \tag{5.14}$$

Proof. We only need to show the 'if' part. So, let the ρ-function of π satisfy (5.13). By the lemma it then has the form (5.12) with b, c and U as indicated. Now, from the fact that $\rho(s) \geq 0$ for all $s > 0$, it follows that $b = 0$, $d := \int_{\mathbb{R}_+} \lambda/(1 + \lambda^2)\, \mathrm{d}U(\lambda) < \infty$ and $a := c - d \geq 0$. So, the expression (5.12) for ρ reduces to (5.11) with $(0, \infty)$ replaced by $\mathbb{R}_+$. Since $-\log \pi(s) = \int_0^s \rho(u)\, \mathrm{d}u$ for $s > 0$, we see that necessarily $U(0) = 0$ and hence that π can be represented as in (5.2). So π is the LSt of a generalized gamma convolution.

The equivalent formulation is obtained by noting that ρ can be written as $\rho(z) = -\pi'(z)\, \overline{\pi(z)} / |\pi(z)|^2$ and showing that π has no zeroes in $\mathbb{C} \setminus (-\infty, 0\,]$ if it satisfies (5.14). ◻

In most cases it is not easy to apply this theorem directly; the case of a stable distribution on $\mathbb{R}_+$ is an exception (cf. Proposition 5.7). As announced above, the theorem can be used, however, for identifying a very important class of generalized gamma convolutions. To this end the following curious definition is needed: A function $\psi : (0,\infty) \to \mathbb{R}_+$ is said to be *hyperbolically completely monotone* if for every $u > 0$ the function

$$(5.15) \qquad v \mapsto \psi(uv)\,\psi(u/v) \qquad [\,v > 0\,]$$

is completely monotone $\big($on $(\,2,\infty)\big)$ as a function of $w := v + 1/v$; one easily verifies that (5.15) is indeed a function of w. Before showing the relation with generalized gamma convolutions, we give some examples and elementary properties of hyperbolically completely monotone functions, and note in advance that *not* all these functions are completely monotone.

Proposition 5.16. *In each of the following five cases the function ψ on $(0,\infty)$ is hyperbolically completely monotone: (i) $\psi(s) = s^\alpha$ with $\alpha \in \mathbb{R}$; (ii) $\psi(s) = e^{-s}$; (iii) $\psi(s) = e^{-1/s}$; (iv) $\psi(s) = (1+s)^{-\gamma}$ with $\gamma > 0$; (v) $\psi(s) = \exp\,[-s^\alpha]$ with $|\alpha| \le 1$.*

PROOF. We only consider case (v); the others are easily handled. So, let $|\alpha| \le 1$ and take $u > 0$. Then for ψ in (v) we have

$$\psi(uv)\,\psi(u/v) = \exp\big[-u^\alpha\,\{v^\alpha + v^{-\alpha}\}\big] \qquad [\,v > 0\,],$$

so we are ready if g with $g(v) := v^\alpha + v^{-\alpha}$, as a function of $w := v + 1/v$, has a completely monotone derivative; cf. Proposition A.3.7. Of course, in proving this we may restrict ourselves to $\alpha \in (0,1)$; then

$$v^{-\alpha} = c_\alpha \int_0^\infty \frac{1}{x+v}\, x^{-\alpha}\,\mathrm{d}x \qquad [\,v > 0\,]$$

for some $c_\alpha > 0$. This simply follows by substituting $x = vt$ in the integral; in fact, by using (5.4) in the special case of Proposition 5.7 with $\gamma = 1 - \alpha$ one can show that $c_\alpha = (\sin\alpha\pi)\big/\pi$. It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}w} g(v) &= \frac{g'(v)}{w'(v)} = \frac{g'(v)}{1 - v^{-2}} = \alpha\,\frac{v^\alpha - v^{-\alpha}}{v - v^{-1}} = \\ &= \alpha\, c_\alpha\, \frac{1}{v - v^{-1}} \int_0^\infty \Big(\frac{1}{x + 1/v} - \frac{1}{x+v}\Big)\, x^{-\alpha}\,\mathrm{d}x = \\ &= \alpha\, c_\alpha \int_0^\infty \frac{1}{1 + x^2 + x\,w}\, x^{-\alpha}\,\mathrm{d}x,\end{aligned}$$

which, as a function of w, is a mixture of completely monotone functions and hence itself is completely monotone. We conclude that ψ in (v) is hyperbolically completely monotone. ◻◻

The result we just proved, can be generalized to a useful *closure property* as follows. Let ψ be hyperbolically completely monotone, so by Bernstein's theorem for every $u > 0$ there exists an LSt-able function L_u such that

$$\psi(uv)\,\psi(u/v) = \int_{\mathbb{R}_+} \exp\big[-\lambda u\,(v + 1/v)\big]\,\mathrm{d}L_u(\lambda) \qquad [v > 0], \tag{5.16}$$

where the factor u in the exponent is inserted for convenience. We note in passing that from (5.16) it easily follows that ψ is *positive* unless $\psi \equiv 0$. Now, let $|\alpha| \le 1$ and define $\chi(s) := \psi(s^\alpha)$. Then for $u > 0$

$$\chi(uv)\,\chi(u/v) = \int_{\mathbb{R}_+} \exp\big[-\lambda u^\alpha (v^\alpha + v^{-\alpha})\big]\,\mathrm{d}L_{u^\alpha}(\lambda) \qquad [v > 0],$$

which by (the proof of) Proposition 5.16 (v), as a function of $w := v+1/v$, is a mixture of completely monotone functions and hence itself is completely monotone. We conclude that χ is hyperbolically completely monotone. We formally state this result together with some other closure properties, which are easily verified.

Proposition 5.17. *The set of hyperbolically completely monotone functions is closed under each of the following operations: (i) scale transformation; (ii) pointwise multiplication; (iii) pointwise limit; (iv) composition with the function $s \mapsto s^\alpha$, where $|\alpha| \le 1$.*

We now come to the main theorem of this section, and consider *probability densities* f that are hyperbolically completely monotone; as we shall see later, *pLSt's* π and *distribution functions* F with this property are also of interest. First, note that from parts (i) and (ii) of Propositions 5.16 and 5.17 it follows that the *gamma densities* are all hyperbolically completely monotone. On the other hand, by looking at the convolution of two different exponential distributions, for instance, one sees that *not* all generalized gamma convolutions have hyperbolically completely monotone densities. In fact, it goes the other way around, as is shown in the following theorem; we only give a sketch of a proof, which also contains a serious restriction.

Theorem 5.18. *A probability distribution on $\mathbb{R}_+$ that has a hyperbolically completely monotone density, is a generalized gamma convolution, and is therefore self-decomposable and infinitely divisible.*

PROOF. Let f be a probability density that is hyperbolically completely monotone, and let π be the Lt of f. We want to apply Theorem 5.15 and therefore consider

$$J(z) := -\pi'(z)\,\overline{\pi(z)} = \int_0^\infty\!\!\int_0^\infty x\,\exp\left[-zx - \overline{z}y\right] f(x)\,f(y)\,\mathrm{d}x\,\mathrm{d}y;$$

we have to show that $\operatorname{Im} J(z) \geq 0$ if $\operatorname{Im} z < 0$. *We will do so only for* $\operatorname{Re} z > 0$; we only know that on this area J is a well-defined analytic function. Now, make the *hyperbolic substitution* $x = uv$, $y = u/v$, with Jacobian $-2u/v$; then it follows that

$$J(z) = \int_0^\infty\!\!\int_0^\infty 2u^2\,\exp\left[-zuv - \overline{z}u/v\right] f(uv)\,f(u/v)\,\mathrm{d}u\,\mathrm{d}v.$$

Since here $f(uv)\,f(u/v)$ can be represented as in (5.16), we get, changing the order of integration,

$$J(z) = \int_0^\infty 2u^2\left(\int_{\mathbb{R}_+} I(z;\lambda,u)\,\mathrm{d}L_u(\lambda)\right)\mathrm{d}u.$$

Here the integral I is defined and next rewritten, by setting $v = u\,(\lambda+\overline{z})\,\theta$, in the following way:

$$I(z;\lambda,u) := \int_0^\infty \exp\Big[-(\lambda+z)\,uv - (\lambda+\overline{z})\,u/v\Big]\,\mathrm{d}v =$$

$$= u\,(\lambda+\overline{z})\int_{\ell(z,\lambda)} \exp\Big[-|\lambda+z|^2\,u^2\,\theta - 1/\theta\Big]\,\mathrm{d}\theta,$$

where $\ell(z,\lambda)$ is the ray $\big\{c\,(\lambda+z) : c > 0\big\}$, directed away from the origin. Now, it can be proved that integrating over this ray may be replaced by integrating over the positive half-line. Then obviously $\operatorname{Im} I(z;\lambda,u) \geq 0$ if $\operatorname{Im} z < 0$, and hence $\operatorname{Im} J(z) \geq 0$ if $\operatorname{Im} z < 0$. Thus π is the LSt of a generalized gamma convolution. ◻

The distribution of a random variable X is a generalized gamma convolution if for every $n \in \mathbb{N}$ the (better behaved!) distribution of $X\,Y_n$ is a generalized gamma convolution, where Y_n is gamma (n,n) and independent of X; just note that $X\,Y_n \xrightarrow{\mathrm{d}} X$ as $n \to \infty$ and use Proposition 5.2. This observation is one of the means by which the proof of Theorem 5.18

can be made rigorous, because if X has a hyperbolically completely monotone density, then so has $X\,Y_n$ for every $n \in \mathbb{N}$. The last implication is an immediate consequence of the second part of the following useful result.

Proposition 5.19.

(i) *If X has a hyperbolically completely monotone density, then so has the q-th power X^q of X for all $q \in \mathbb{R}$ with $|q| \geq 1$. So, X has a hyperbolically completely monotone density iff $1/X$ has.*

(ii) *If X and Y are independent and both have a hyperbolically completely monotone density, then so have $X\,Y$ and X/Y.*

PROOF. Part (i) follows from Propositions 5.16 (i) and 5.17 (ii), (iv) by noting that $Y := X^q$ has density f_Y given by $f_Y(y) = |\alpha|\, y^{\alpha-1} f_X(y^\alpha)$ for $y > 0$, where $\alpha := 1/q$ satisfies $|\alpha| \leq 1$ if $|q| \geq 1$.
Because of (i), in part (ii) it suffices to consider X/Y; it has a density h that can be written as

$$(5.17) \qquad h(a) = \int_0^\infty f(ay)\, g(y)\, \mathrm{d}y \qquad [\,a > 0\,],$$

where $f := f_X$ and g with $g(y) := y\, f_Y(y)$ for $y > 0$ are both hyperbolically completely monotone. In order to show that also h is hyperbolically completely monotone, we take $s > 0$ and want to prove that the following function of $t > 0$ is completely monotone in $w := t + 1/t$:

$$h(st)\, h(s/t) = \int_0^\infty \int_0^\infty f(stx)\, f(sy/t)\, g(x)\, g(y)\, \mathrm{d}x\, \mathrm{d}y.$$

Now, proceed as in the proof of Theorem 5.18. Using the same hyperbolic substitution and applying representation (5.16) to both f and g, one easily shows that

$$h(st)\, h(s/t) = \int_0^\infty 2u \Big(\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} I(t; \lambda, \widetilde{\lambda}, u)\, \mathrm{d}L_{su}^{(1)}(\lambda)\, \mathrm{d}L_u^{(2)}(\widetilde{\lambda}) \Big)\, \mathrm{d}u,$$

where

$$I(t; \lambda, \widetilde{\lambda}, u) := \int_0^\infty \exp\big[-\lambda s u\, \big(tv + 1/(tv)\big) - \widetilde{\lambda} u\, (v + 1/v)\big]\, \frac{\mathrm{d}v}{v} =$$

$$= \int_0^\infty \exp\big[-(\lambda^2 s^2 + \widetilde{\lambda}^2 + \lambda\widetilde{\lambda} s\, w)\, u^2 \theta - 1/\theta\big]\, \frac{\mathrm{d}\theta}{\theta};$$

here we have set $v = u\, (\lambda s/t + \widetilde{\lambda})\, \theta$. We conclude that the desired complete monotonicity in w does indeed hold. ◻

By using the preceding theory on hyperbolically completely monotone densities, one can simply solve several classical problems in infinite divisibility that were open for a long time; see Notes. We show this in two special cases. The first result, on products and powers of *gamma* variables, is immediate from Proposition 5.19 and Theorem 5.18.

Theorem 5.20. *If $X_1, X_2, \ldots, X_n$ are independent gamma random variables and $q_1, q_2, \ldots, q_n$ are real numbers satisfying $|q_j| \ge 1$ for all j, then*

$$X_1^{q_1}\, X_2^{q_2} \cdots X_n^{q_n}$$

is self-decomposable and hence infinitely divisible.

The second result concerns the *log-normal* distribution, which is of interest in financial mathematics. Its infinite divisibility was very hard to prove before the theory on hyperbolic complete monotonicity was developed. Now it is easy.

Theorem 5.21. *The log-normal distribution with density f given by*

$$f(x) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{x}\,\exp\big[-\tfrac{1}{2}\,(\log x)^2\big] \qquad [x > 0],$$

is self-decomposable and hence infinitely divisible.

Proof. We apply Theorem 5.18; it is then sufficient to show that the function ψ with $\psi(s) := \exp\big[-\tfrac{1}{2}\,(\log s)^2\big]$ for $s > 0$ is hyperbolically completely monotone. To do so we take $u > 0$, and consider

$$\psi(uv)\,\psi(u/v) = \exp\big[-(\log u)^2 - (\log v)^2\big] \qquad [v > 0].$$

As in the proof of Proposition 5.16 we are ready if g with $g(v) := (\log v)^2$, as a function of $w := v + 1/v$, has a completely monotone derivative. Now, since

$$\log v = \int_0^\infty \Big(\frac{1}{x+1} - \frac{1}{x+v}\Big)\, \mathrm{d}x \qquad [v > 0],$$

we see that

$$\frac{\mathrm{d}}{\mathrm{d}w}g(v) = \frac{g'(v)}{1 - v^{-2}} = \frac{\log v - \log 1/v}{v - 1/v} =$$
$$= \frac{1}{v - 1/v}\int_0^\infty \Big(\frac{1}{x + 1/v} - \frac{1}{x+v}\Big)\, \mathrm{d}x = \int_0^\infty \frac{1}{1 + x^2 + x\,w}\, \mathrm{d}x,$$

which, as a function of w, is a mixture of completely monotone functions and hence itself is completely monotone. We conclude that ψ, and therefore f, is hyperbolically completely monotone. ◻◻

We state a more general result. Starting from the log-normal density and the densities generated by Proposition 5.16, and applying the operations of Proposition 5.17 (i), (ii) and (iv), we find a large class of concrete infinitely divisible densities several of which are well known; see also the examples in Section 12. Note that in the following theorem *not* every combination of the parameters yields a probability density; for instance, if $n = 0$, $d = 0$, $m \neq 0$ and the α_j are positive, then necessarily $0 < r < \sum_{j=1}^{m} \alpha_j \gamma_j$.

Theorem 5.22. *Any probability density f on $(0,\infty)$ of the following form is hyperbolically completely monotone and hence self-decomposable and infinitely divisible:*

$$f(x) = c\, x^{r-1} \prod_{j=1}^{m} \Big(\frac{1}{1 + a_j x^{\alpha_j}}\Big)^{\gamma_j} \exp\Big[-\sum_{j=1}^{n} b_j x^{\beta_j} - d\,(\log x)^2\Big], \tag{5.18}$$

where $r \in \mathbb{R}$, $m, n \in \mathbb{Z}_+$, $d \geq 0$, and for all j: $a_j > 0$, $b_j > 0$, $\alpha_j \neq 0$ with $-1 \leq \alpha_j \leq 1$, $\beta_j \neq 0$ with $-1 \leq \beta_j \leq 1$, $\gamma_j > 0$, and c is a norming constant.

To conclude the study of hyperbolically completely monotone densities we state the following immediate consequence of Theorem 5.18 and Corollary 5.9.

Proposition 5.23. *Let f be a probability density on $(0,\infty)$ that is hyperbolically completely monotone and continuous. Then f is completely monotone iff it is monotone. Equivalently, f is completely monotone iff $f(0+) > 0$ (possibly ∞).*

We next turn to *pLSt's* that are hyperbolically completely monotone; we will show that the class of distributions with such LSt's *coincides* with the class of generalized gamma convolutions. Clearly, a pLSt π_n as in (5.1) is a hyperbolically completely monotone function, and hence so is any limit of it as $n \to \infty$; cf. Propositions 5.16 (iv) and 5.17. So, the LSt π of a generalized gamma convolution is hyperbolically completely monotone. To show that the converse holds, is more difficult.

Theorem 5.24. *A continuous function π on $(0,\infty)$ with $\pi(0+) = 1$ is hyperbolically completely monotone iff it is the LSt of a generalized gamma convolution. So, a probability distribution on $\mathbb{R}_+$ with a hyperbolically completely monotone LSt is self-decomposable and hence infinitely divisible.*

PROOF. Let π be hyperbolically completely monotone; as $\pi(0+) = 1$, we have $\pi \not\equiv 0$, so π is positive. First, assume that $x \mapsto x^{n-1}\,\pi(x/n)$ is integrable over $(0,\infty)$ for all $n \in \mathbb{N}$. Taking $n = 1$ we see that π is proportional to a continuous probability density, so from Proposition 5.23 it follows that π is completely monotone and hence is a pLSt. To show that π corresponds to a generalized gamma convolution, we let X be a random variable with pLSt π and take Y_n independent of X and gamma (n,n) distributed; then $X\,Y_n \xrightarrow{\mathrm{d}} X$ as $n \to \infty$. From the continuity theorem it then follows that π is the pointwise limit as $n \to \infty$ of the pLSt $\pi^{(n)}$ with

$$\pi^{(n)}(s) := \mathbb{E}\exp[-s\,X\,Y_n] = \int_0^\infty \pi(sy)\,\frac{n^n}{\Gamma(n)}\,y^{n-1}\,\mathrm{e}^{-ny}\,\mathrm{d}y =$$

$$= s^{-n}\int_0^\infty \mathrm{e}^{-(1/s)\,x}\,\frac{1}{\Gamma(n)}\,x^{n-1}\,\pi(x/n)\,\mathrm{d}x = c_n\,s^{-n}\,\pi_n(1/s),$$

where $c_n := \int_0^\infty x^{n-1}\,\pi(x/n)\,\mathrm{d}x/\Gamma(n)$ and π_n is the Lt of the probability density f_n given by

$$f_n(x) = \frac{1}{c_n\,\Gamma(n)}\,x^{n-1}\,\pi(x/n) \qquad [\,x > 0\,].$$

Now, π is hyperbolically completely monotone, hence so is f_n. From Theorem 5.18 it follows that π_n is the LSt of a generalized gamma convolution. Moreover, because of (5.10) its Thorin function U_n has the property that $\lim_{\lambda\to\infty} U_n(\lambda) = n$. Using the canonical representation of Theorem 5.1 for π_n, one then sees that $\pi^{(n)}$ satisfies

$$\log\,\pi^{(n)}(s) = \log\,c_n + \int_{(0,\infty)} \log\,\frac{\lambda}{\lambda s + 1}\,\mathrm{d}U_n(\lambda).$$

As letting $s \downarrow 0$ shows that $\log\,c_n = -\int_{(0,\infty)} \log\lambda\,\mathrm{d}U_n(\lambda)$, we conclude that

$$\log\,\pi^{(n)}(s) = \int_{(0,\infty)} \log\,\frac{\lambda}{\lambda + s}\,\mathrm{d}U_n(1/\lambda),$$

so by Theorem 5.1 $\pi^{(n)}$ is the LSt of a generalized gamma convolution. Hence so is π; this follows from Proposition 5.2 by letting $n \to \infty$.

Next, consider the case where π does not satisfy the integrability conditions above. Then for any $\varepsilon > 0$ π_ε does, where $\pi_\varepsilon(s) := \mathrm{e}^{-\varepsilon s}\,\pi(s)$. Moreover, π_ε is continuous with $\pi_\varepsilon(0+) = 1$ and is also hyperbolically completely monotone, of course. From the first part of the proof it now follows that, for any $\varepsilon > 0$, π_ε is the LSt of a generalized gamma convolution. Hence so is $\pi = \lim_{\varepsilon \downarrow 0} \pi_\varepsilon$, because of Proposition 5.2. ◻

This theorem has several nice implications. For instance, together with Proposition 5.17 (iv) it immediately yields the following *closure property* of the class of generalized gamma convolutions; cf. Propositions III.6.4 and V.2.14 (iii).

Proposition 5.25. *If π is the LSt of a generalized gamma convolution, then so is $s \mapsto \pi(s^\gamma)$ for every $\gamma \in (0,1]$.*

Another closure property that is often useful in dealing with concrete examples, is the following; it easily follows from Theorems 5.1 or 5.2.

Proposition 5.26. *If π is the pLSt of a generalized gamma convolution, then so is the function $s \mapsto \pi(a+s)/\pi(a)$ for every $a > 0$.*

It follows that if f is a density of a generalized gamma convolution, then so is any density g on $(0,\infty)$ of the form $g(x) = c\,\mathrm{e}^{-ax}\,f(x)$ with $a > 0$, where $c > 0$ is a norming constant.

From Theorem 5.24 one can also easily obtain the following counterpart to Proposition 5.19 (ii).

Proposition 5.27. *Let X and Y be independent, let the distribution of X be a generalized gamma convolution, and let Y have a hyperbolically completely monotone density. Then the distributions of XY and X/Y are generalized gamma convolutions.*

Proof. By Theorem 5.24 the pLSt π_X of X is hyperbolically completely monotone, as is the density f_Y of Y. Now, the pLSt π of XY can be written as

$$\pi(s) = \int_0^\infty \pi_X(sy)\,f_Y(y)\,\mathrm{d}y, \tag{5.19}$$

similar to the function h in (5.17). Hence we can proceed as in the proof of Proposition 5.19 (ii) to verify that π is hyperbolically completely monotone. Applying Theorem 5.24 once more shows that π corresponds to a generalized gamma convolution. The assertion on X/Y now immediately follows from Proposition 5.19 (i). ◻◻

An important special case is obtained by taking here Y gamma (r). Then considering XY yields a result, stated in the first corollary below, that supplements Theorem 4.5 when $r > 2$; and by calculating a density of X/Y with $r = n \in \mathbb{N}$ one easily obtains the *closure property* formulated in the second corollary.

Corollary 5.28. *For $r > 0$ a scale mixture of gamma (r) distributions is a generalized gamma convolution, and is hence infinitely divisible, if the mixing function is a generalized gamma convolution.*

Corollary 5.29. *If π is the LSt of a generalized gamma convolution and if $n \in \mathbb{N}$, then the following function f is a density of a generalized gamma convolution:*

$$(5.20)\qquad f(x) = \frac{(-1)^n}{(n-1)!}\,(1/x)^{n+1}\,\pi^{(n)}(1/x) \qquad [\,x > 0\,].$$

Not all densities f in (5.20) are hyperbolically completely monotone; see Section 12. In view of this it is important to note that if π is the Lt of a hyperbolically completely monotone density g, say, then writing

$$(5.21)\qquad (-1)^n\,\pi^{(n)}(s) = \int_0^\infty e^{-sx}\,x^n\,g(x)\,dx$$

and comparing this with (5.19) we see that $(-1)^n\,\pi^{(n)}$ is hyperbolically completely monotone for all n, and hence so is f in (5.20). Actually, one can show that the class of Lt's π of hyperbolically completely monotone densities g is *characterized* by the condition that $(-1)^n\,\pi^{(n)}$ is hyperbolically completely monotone for all $n \in \mathbb{Z}_+$ (and π does not correspond to a degenerate distribution).

Finally, we pay some attention to *distribution functions* that are hyperbolically completely monotone; they turn out to correspond to generalized gamma convolutions as well. Return to X/Y in Proposition 5.27 with Y exponential (or to Corollary 5.29 with $n = 1$). One easily verifies that the distribution function F of X/Y is then given by

$$(5.22)\qquad F(x) = \pi(1/x) \qquad [\,x > 0\,],$$

where $\pi := \pi_X$. Now, if π is the LSt of a generalized gamma convolution, then F not only corresponds to a generalized gamma convolution, but F is also hyperbolically completely monotone; cf. Theorem 5.24 and Proposition 5.17 (iv). There is a converse. To see this, let F be a continuous distribution function on $\mathbb{R}_+$ and suppose that F is hyperbolically completely monotone. Then the function π with $\pi(s) := F(1/s)$ for $s > 0$ is continuous with $\pi(0+) = 1$ and is also hyperbolically completely monotone, of course. From Theorem 5.24 we then conclude that π is the LSt of a generalized gamma convolution. Since F satisfies (5.22), it follows that F is the distribution function of X/Y with X and Y as above, so F corresponds to a generalized gamma convolution. Thus we have proved the following result.

Theorem 5.30. *A continuous distribution function F on $\mathbb{R}_+$ that is hyperbolically completely monotone, corresponds to a generalized gamma convolution, and is therefore self-decomposable and infinitely divisible. In fact, F is hyperbolically completely monotone iff it is the distribution function of X/Y, where X and Y are independent, the distribution of X is a generalized gamma convolution, and Y is exponential.*

In a similar way the generalized gamma convolutions in Corollary 5.29 with $n \geq 2$ give also rise to useful criteria in terms of hyperbolic complete monotonicity. Taking $n = 2$, for instance, one can show that a continuous distribution function F on $\mathbb{R}_+$ has the property that $x \mapsto \int_0^x F(y)\,dy$ is hyperbolically completely monotone, iff it is the distribution function of X/Y, where X and Y are independent, the distribution of X is a generalized gamma convolution, and Y is gamma (2). We do not pursue this, and conclude with an obvious counterpart to Proposition 5.19.

Proposition 5.31.

(i) *If X has a hyperbolically completely monotone distribution function, then so has X^q for all $q \geq 1$.*

(ii) *If X and Y are independent and both have a hyperbolically completely monotone distribution function, then so has* $\max\{X, Y\}$.

There is a simple dual result to Theorem 5.30. Consider, instead of X/Y, the random variable Y/X with X and Y independent and Y exponential. From Section 3 we know that, for every $X > 0$, Y/X then has a *completely monotone* density, and hence is infinitely divisible. One easily verifies that the distribution function F of Y/X is given by

$$(5.23) \qquad F(x) = 1 - \pi(x) \qquad [x > 0],$$

where $\pi := \pi_X$. Now, if π is the LSt of a generalized gamma convolution, then by Theorem 5.24 the *tail function* $\overline{F}$ with $\overline{F}(x) := 1 - F(x)$ is hyperbolically completely monotone. Conversely, if a continuous distribution function F on $\mathbb{R}_+$ has a hyperbolically completely monotone tail function $\overline{F}$, then by Theorem 5.24 $\overline{F}$ is the LSt of a generalized gamma convolution; so, because of (5.23) F is the distribution function of Y/X with X and Y as above. We summarize, and state an obvious counterpart to Proposition 5.31.

Theorem 5.32. *A continuous distribution function F on $\mathbb{R}_+$ for which the tail function $\overline{F}$ is hyperbolically completely monotone, has a completely monotone density, and is hence infinitely divisible. In fact, F has a hyperbolically completely monotone tail function $\overline{F}$ iff it is the distribution function of Y/X, where X and Y are independent, the distribution of X is a generalized gamma convolution, and Y is exponential.*

Proposition 5.33.

(i) *If X has a hyperbolically completely monotone tail function, then so has X^q for all $q \geq 1$.*
(ii) *If X and Y are independent and both have a hyperbolically completely monotone tail function, then so has $\min\{X, Y\}$.*

Note that a random variable X has a hyperbolically completely monotone distribution function iff $1/X$ has a hyperbolically completely monotone tail function. Using this together with parts (i) of Propositions 5.19, 5.31 and 5.33 yields the following result on infinite divisibility of *powers* of random variables.

Corollary 5.34. *If X is a random variable for which a density, the distribution function or the tail function is hyperbolically completely monotone, then X^q is infinitely divisible for all $q \in \mathbb{R}$ with $|q| \geq 1$.*

One might wonder whether hyperbolic complete monotonicity of the LSt of X is also sufficient for X^q being infinitely divisible for all $|q| \geq 1$. When

$q \geq 1$, there is much support for the stronger assertion that if X corresponds to a generalized gamma convolution, then so does X^q; cf. Theorem 5.24. In case $q \leq -1$, however, if one takes X with $\ell_X > 0$, then X^q is bounded and hence not infinitely divisible. For a counter-example with $\ell_X = 0$ we refer to Section 12.

Summarizing the main results of this section, i.e., Theorems 5.18, 5.24 and 5.30, we can say that a sufficient condition for *self-decomposability*, and hence for *infinite divisibility*, of a probability distribution on $\mathbb{R}_+$ is given by the *hyperbolic complete monotonicity* of a *density*, the *LSt* or the *distribution function* of the distribution. Illustrative examples of the use of this condition are given in Section 12.

6. Mixtures of Poisson distributions

Consider the Poisson distribution; it has a positive parameter, which here we denote by t. When mixing with respect to t we allow t to be zero; the corresponding 'Poisson' distribution is degenerate at zero. Thus by a *mixture of Poisson distributions* we understand a distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ of the form

$$(6.1) \qquad p_k = \int_{\mathbb{R}_+} \frac{t^k}{k!}\, \mathrm{e}^{-t}\, \mathrm{d}G(t) \qquad [\, k \in \mathbb{Z}_+ \,],$$

with G a distribution function on $\mathbb{R}_+$. The corresponding pgf P can be written in terms of the LSt π of G as follows:

$$(6.2) \qquad P(z) = \int_{\mathbb{R}_+} \exp\big[-t\,(1-z)\big]\, \mathrm{d}G(t) = \pi(1-z).$$

As noted in Section 2, a mixture of Poisson distributions can be viewed as a *power mixture* and as a *scale mixture*. This means that a random variable X with distribution (6.1) and pgf (6.2) can be represented as

$$(6.3) \qquad X \stackrel{\mathrm{d}}{=} N(T) \quad \text{and} \quad X \stackrel{\mathrm{d}}{=} T \odot N,$$

where $N(\cdot)$ is a Poisson process of rate one, $N := N(1)$ and T is an $\mathbb{R}_+$-valued random variable that is independent of $N(\cdot)$ with $F_T = G$. Now, let Z be nonnegative and independent of $\big(N(\cdot), T\big)$; then considering $Z \odot X$ with X as in (6.3) and using (A.4.11) one sees that

$$(6.4) \qquad Z \odot N(T) \stackrel{\mathrm{d}}{=} Z \odot (T \odot N) \stackrel{\mathrm{d}}{=} (ZT) \odot N \stackrel{\mathrm{d}}{=} N(ZT).$$

In the sequel most results on mixtures of Poisson distributions will be formulated, without further comment, in terms of the first representation in (6.3). From the well-known fact that an LSt is determined by its values on a finite interval, it follows that the distributions of T and $N(T)$ determine each other, so

$$(6.5)\qquad N(T_1) \stackrel{\mathrm{d}}{=} N(T_2) \iff T_1 \stackrel{\mathrm{d}}{=} T_2.$$

Moreover, similar to the proof of Proposition 3.1 one shows that the class of mixtures of Poisson distributions is *closed under weak convergence.*

Proposition 6.1. *If a sequence (P_n) of mixtures of Poisson pgf's converges (pointwise) to a pgf P, then P is a mixture of Poisson pgf's.*

We want to know what mixtures of Poisson distributions are *infinitely divisible*. By looking for complex zeroes of the pgf P or by considering the tail behaviour of the distribution (p_k) one can show that a *finite mixture* of (different) Poisson distributions can*not* be infinitely divisible; cf. Example II.11.2. In a similar way one obtains the following more general result.

Proposition 6.2. *The random variable $N(T)$ is not infinitely divisible if T is non-degenerate and bounded.*

PROOF. Let T be bounded by some $a > 0$. Consider the tail probability $\mathbb{P}\big(N(T) > k\big)$ for $k \in \mathbb{Z}_+$; then using (6.1) and the fact that $\mathbb{P}\big(N(t) > k\big)$ is nondecreasing in t, we see that

$$\mathbb{P}\big(N(T) > k\big) \le \mathbb{P}\big(N(a) > k\big) \qquad [\,k \in \mathbb{Z}_+\,].$$

Now, suppose that $N(T)$ is infinitely divisible, and apply Corollary II.9.5. Then it follows that $N(T)$ is Poisson distributed and hence by (6.5) that T is degenerate. ◻◻

On the other hand, from Proposition 2.1 it follows that $N(T)$ is infinitely divisible if T is infinitely divisible or, equivalently (cf. Theorem III.4.1), if the *ρ-function* of (the pLSt π of) T, i.e., $\rho(s) := -(\mathrm{d}/\mathrm{d}s)\log \pi(s)$, is *completely monotone*: ρ has alternating derivatives on the whole interval $(0,\infty)$. Now, this sufficient condition can be weakened in such a way that it becomes also necessary, as follows.

Theorem 6.3. *The random variable $N(T)$ is infinitely divisible iff the ρ-function of T is completely monotone on the interval* $(0,1\,]$.

PROOF. According to Theorem II.4.3 $N(T)$ is infinitely divisible iff the R-function of (the pgf P of) $N(T)$, i.e., $R(z) := (\mathrm{d}/\mathrm{d}z)\log P(z)$, is *absolutely monotone*: R has nonnegative derivatives on the interval $[\,0,1)$. Now, by (6.2) the R-function of P and the ρ-function of T are related by

$$R(z) = \rho(1-z) \;\text{ for } z \in [\,0,1), \qquad \rho(s) = R(1-s) \;\text{ for } s \in (0,1\,].$$

From these relations the theorem immediately follows. ◻◻

In Section 12 we shall give an example of a random variable T that itself is *not* infinitely divisible, but does make $N(T)$ infinitely divisible.

Theorem 6.3 suggests a *characterization* of infinite divisibility on $\mathbb{R}_+$ by infinite divisibility of certain mixtures of Poisson distributions. Consider, for $\theta > 0$, the random variable $N(\theta T)$; since the ρ-function ρ_θ of θT is related to the ρ-function ρ of T by $\rho_\theta(s) = \theta\,\rho(\theta s)$, we see that $N(\theta T)$ is infinitely divisible iff ρ is completely monotone on the interval $(0,\theta\,]$. Thus we are led to the following result.

Theorem 6.4. *An $\mathbb{R}_+$-valued random variable T is infinitely divisible iff $N(\theta T)$ is infinitely divisible for all $\theta > 0$.*

This theorem can also be obtained by (only) using the fact that the class of infinitely divisible distributions on $\mathbb{R}_+$ is closed under weak convergence; cf. Proposition III.2.2. This will be clear from the following general result which shows two things: The property of being a pLSt can be *characterized* in a 'discrete' way related to mixtures of Poisson pgf's, and the pLSt can then be *constructed* from these mixtures.

Proposition 6.5. *A function π on $\mathbb{R}_+$ is the pLSt of an $\mathbb{R}_+$-valued random variable T iff the function P_θ defined by*

(6.6) $$P_\theta(z) = \pi\big(\theta\{1-z\}\big) \qquad [\,0 \le z \le 1\,],$$

is the pgf of a $\mathbb{Z}_+$-valued random variable X_θ for all $\theta > 0$. In this case $X_\theta \overset{\mathrm{d}}{=} N(\theta T)$ and, conversely, T can be obtained from the X_θ by

(6.7) $$\frac{1}{\theta}\,X_\theta \xrightarrow{\;\mathrm{d}\;} T \;\text{ as }\; \theta \to \infty.$$

PROOF. The 'only-if' part is trivial; cf. (6.2). So, let the function P_θ given by (6.6) be the pgf of a random variable X_θ for all $\theta > 0$. Then, since a pgf is absolutely monotone on $[0,1)$ and $\pi(s) = P_\theta(1 - s/\theta)$ for $s \in [0,\theta]$, the function π is completely monotone on $(0,\theta]$ for all $\theta > 0$, so π is completely monotone $\big($on $(0,\infty)\big)$. By Bernstein's theorem it follows that π is an LSt, and since $\pi(0) = 1$, π is the pLSt of a random variable T. Next, by (6.2) and (6.3) the random variable $N(\theta T)$ has pgf P_θ, so $X_\theta \stackrel{d}{=} N(\theta T)$. Finally, the pLSt π_θ of X_θ/θ satisfies

$$\pi_\theta(s) = P_\theta(e^{-s/\theta}) = \pi\big(\theta\,\{1 - e^{-s/\theta}\}\big) \longrightarrow \pi(s) \text{ as } \theta \to \infty,$$

so by the continuity theorem we have $X_\theta/\theta \xrightarrow{d} T$ as $\theta \to \infty$. ◻

We further note that for $\theta > 0$ the distribution $\big(p_k(\theta)\big)_{k\in\mathbb{Z}_+}$ of $N(\theta T)$ can be obtained by repeated differentiation of the pLSt π_T of T:

$$(6.8) \qquad p_k(\theta) = \int_{\mathbb{R}_+} \frac{(\theta t)^k}{k!}\, e^{-\theta t}\, dF_T(t) = \frac{(-\theta)^k}{k!}\, \pi_T^{(k)}(\theta) \qquad [\,k \in \mathbb{Z}_+\,].$$

In a way similar to Theorem 6.4 the mixtures of Poisson distributions provide a link between several other classes of distributions on $\mathbb{R}_+$ and their counterparts on $\mathbb{Z}_+$. We will show this for the *self-decomposable* distributions on $\mathbb{R}_+$, for the *stable* distributions on $\mathbb{R}_+$ with fixed exponent γ, for the *gamma* (r) distributions with r fixed and hence for the *exponential* distributions, for the *compound-exponential* distributions on $\mathbb{R}_+$, for the *monotone* densities on $(0,\infty)$, for the *log-convex* densities on $(0,\infty)$, for the *completely monotone* densities on $(0,\infty)$ and, more general, for the *mixtures of gamma* (r) *distributions* with r fixed, and, finally, for the *generalized gamma convolutions*. The resulting relations can be used to simply obtain results on $\mathbb{R}_+$, such as canonical representations, from their counterparts on $\mathbb{Z}_+$; in some cases one can also go the other way around. We will do this only to obtain discrete analogues of some of the results of Sections 3, 4 and 5; see Sections 7 and 8.

We start with the *self-decomposable* distributions; this case can be handled in a way similar to the first proof of Theorem 6.4 (including the proof of Theorem 6.3).

Theorem 6.6. *An $\mathbb{R}_+$-valued random variable T is self-decomposable iff $N(\theta T)$ is (discrete) self-decomposable for all $\theta > 0$.*

Proof. Take $\theta > 0$, and let ρ be the ρ-function of (the pLSt π of) T and let R be the R-function of (the pgf P of) $N(\theta T)$. Recall from Theorem V.2.6 that T is self-decomposable iff ρ_0 with $\rho_0(s) := (\mathrm{d}/\mathrm{d}s)\,[s\,\rho(s)]$ is completely monotone. Similarly, from Theorem V.4.8 we know that $N(\theta T)$ is self-decomposable iff R_0 with $R_0(z) := -(\mathrm{d}/\mathrm{d}z)\,[(1-z)\,R(z)]$ is absolutely monotone. Now, since by (6.6) P and π are related by $P(z) = \pi\big(\theta\{1-z\}\big)$, one easily verifies that

$$R(z) = \theta\,\rho\big(\theta\{1-z\}\big), \quad \text{and hence} \quad R_0(z) = \theta\,\rho_0\big(\theta\{1-z\}\big). \tag{6.9}$$

It follows that R_0 is absolutely monotone iff ρ_0 is completely monotone on $(0,\theta\,]$. The assertion of the theorem now immediately follows. ◻◻

From the proof just given it will be clear that the mixtures of Poisson distributions that are self-decomposable, can be characterized in a way similar to Theorem 6.3: *$N(T)$ is self-decomposable iff the ρ_0-function of T is completely monotone on* $(0,1\,]$. To prove Theorem 6.6 one might also use just the definitions of self-decomposability on $\mathbb{R}_+$ and on $\mathbb{Z}_+$ together with Proposition 6.5. The resulting proof is, however, somewhat more delicate than the second proof of Theorem 6.4 because X_θ/θ, having a discrete distribution, is *not* self-decomposable on $\mathbb{R}_+$ if X_θ is self-decomposable on $\mathbb{Z}_+$. On the other hand, the alternative proof can be adapted so as to obtain the following more general result, which is an extension of (V.8.27): *A pLSt π is self-decomposable iff the pgf P_θ with*

$$P_\theta(z) := \pi\big(\theta\,A(z)\big) = \pi\big(\theta\,\{1-B(z)\}\big)$$

is $\mathcal{F}$-self-decomposable for all $\theta > 0$; here $\mathcal{F} = (F_t)_{t\geq 0}$ is a continuous composition semigroup of pgf's, and A and B are determined by $\mathcal{F}$ as in (V.8.11). In a similar way one can characterize the distributions on $\mathbb{R}_+$ that are self-decomposable with respect to a continuous composition semigroup $\mathcal{C} = (C_t)_{t\geq 0}$ of cumulant generating functions; see the end of Section V.8. We don't go into this, and only note that similar generalizations can be obtained of the following result on the subclasses of *stable* distributions.

Theorem 6.7. *For $\gamma \in (0,1\,]$ an $\mathbb{R}_+$-valued random variable T is stable with exponent γ iff $N(\theta T)$ is (discrete) stable with exponent γ for some, and then all, $\theta > 0$.*

PROOF. We use the canonical representations for the pLSt π and the pgf P of a stable distribution with exponent γ on $\mathbb{R}_+$ and on $\mathbb{Z}_+$, respectively; according to Theorems V.3.5 and V.5.5 they are given by

$$\pi(s) = \exp\big[-\lambda\, s^\gamma\big], \quad P(z) = \exp\big[-\lambda\,(1-z)^\gamma\big], \tag{6.10}$$

where $\lambda > 0$. Since the pgf of $N(\theta T)$ is given by (6.6) with $\pi = \pi_T$, the 'only-if' part of the theorem is now immediately clear. For the converse we may take $\theta = 1$. So, let $N(T)$ be stable with exponent γ; then it has pgf P as in (6.10) with some $\lambda > 0$. Hence $N(T) \stackrel{d}{=} N(\widetilde{T})$, where $\widetilde{T}$ is an $\mathbb{R}_+$-valued random variable with pLSt π as in (6.10) with the same λ. From (6.5) it follows that $T \stackrel{d}{=} \widetilde{T}$, so T is stable with exponent γ. □

From (6.10) it is clear that any stable distribution on $\mathbb{Z}_+$ is a mixture of Poisson distributions. Therefore we can reverse matters and obtain the following characterization of stability on $\mathbb{Z}_+$.

Corollary 6.8. *For $\gamma \in (0,1\,]$ a $\mathbb{Z}_+$-valued random variable X is stable with exponent γ iff $X \stackrel{d}{=} N(T)$ where T is $\mathbb{R}_+$-valued and stable with exponent γ.*

Similarly one shows that Theorem 6.7 and its corollary have close analogues for the *gamma* and *negative-binomial* distributions with shape parameter r, whose pLSt's π and pgf's P, respectively, are of the form

$$\pi(s) = \Big(\frac{\lambda}{\lambda+s}\Big)^r, \quad P(z) = \Big(\frac{1-p}{1-pz}\Big)^r, \tag{6.11}$$

where $\lambda > 0$ and $p \in (0,1)$. In fact, T has pLSt π as in (6.11) iff $N(\theta T)$ has pgf P as in (6.11), if $p = \theta/(\lambda+\theta)$ or $\lambda = (1/p-1)\,\theta$. We state the results.

Theorem 6.9. *For $r>0$ an $\mathbb{R}_+$-valued random variable T has a gamma (r) distribution iff $N(\theta T)$ has a negative-binomial (r) distribution for some, and then all, $\theta > 0$.*

Corollary 6.10. *For $r>0$ a $\mathbb{Z}_+$-valued random variable X has a negative-binomial (r) distribution iff $X \stackrel{d}{=} N(T)$ with T gamma (r) distributed.*

Taking $r = 1$ in Theorem 6.9 we see that T has an *exponential* distribution iff $N(\theta T)$ has a *geometric* distribution for some, and then all, $\theta > 0$. The 'all-θ' version of this result can be extended to the *compound-exponential* distributions on $\mathbb{R}_+$ and on $\mathbb{Z}_+$; their pLSt's π and pgf's P are of the form

$$\pi(s) = \frac{1}{1 - \log\, \pi_0(s)}, \quad P(z) = \frac{1}{1 - \log\, P_0(z)}, \tag{6.12}$$

where π_0 is an infinitely divisible pLSt and P_0 is an infinitely divisible pgf. Recall from Theorem II.3.6 that the class of compound-exponential distributions on $\mathbb{Z}_+$ coincides with the class of *compound-geometric* distributions on $\mathbb{Z}_+$.

Theorem 6.11. *An $\mathbb{R}_+$-valued random variable T is compound-exponential iff $N(\theta T)$ is compound-exponential or, equivalently, compound-geometric for all $\theta > 0$.*

PROOF. The direct part of the theorem is easily obtained from (6.6), (6.12) and Theorem 6.4. The converse part is an immediate consequence of Theorem III.3.8 and Proposition 6.5; note that X_θ/θ is compound-geometric on $\mathbb{R}_+$ if X_θ is compound-geometric on $\mathbb{Z}_+$.
Alternatively, one can use the characterizations given in Theorems III.5.1 and II.5.6; the ρ_0-function of $\pi := \pi_T$ with $\rho_0(s) := (\mathrm{d}/\mathrm{d}s)\big(1/\pi(s)\big)$ and the R_0-function of $P := P_\theta$ with $R_0(z) := -(\mathrm{d}/\mathrm{d}z)\big(1/P(z)\big)$ satisfy (6.9) again, because of (6.6). ◻◻

We now turn to the classes of *monotone* densities f on $(0,\infty)$ and *monotone* distributions $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$.

Theorem 6.12. *A $(0,\infty)$-valued random variable T has a monotone density iff $N(\theta T)$ has a monotone distribution for all $\theta > 0$.*

PROOF. Let U be a random variable with a uniform distribution on $(0,1)$. First, suppose that T has a monotone density. Then by (the proof of) Theorem A.3.13 there exists a positive random variable Z, independent of U, such that $T \overset{\mathrm{d}}{=} UZ$. Taking $\theta = 1$ without restriction, we now see that by (6.4)

$$N(T) \overset{\mathrm{d}}{=} N(UZ) \overset{\mathrm{d}}{=} U \odot N(Z),$$

so $N(T)$ has a monotone distribution because of Theorem A.4.11. In view of this theorem, for the converse we may suppose that for every $\theta > 0$

there exists a $\mathbb{Z}_+$-valued random variable Z_θ, independent of U, such that $N(\theta T) \stackrel{d}{=} U \odot Z_\theta$. Now, use (A.4.17); then by Proposition 6.5 the random variable T can be obtained as

$$\frac{1}{\theta}\big\lfloor U(Z_\theta+1)\big\rfloor \xrightarrow{d} T, \quad \text{and hence } U\,\frac{1}{\theta}\,(Z_\theta+1) \xrightarrow{d} T,$$

as $\theta \to \infty$; the fractional part of $U(Z_\theta+1)$ is $o(\theta)$, of course. By Theorem A.3.13 it follows that the distribution function F of T is the weak limit of concave distribution functions, so F itself is concave as well, and hence has a monotone density; note that $F(0)=0$. ◻◻

We next consider the subclasses of *log-convex* densities f on $(0,\infty)$ and *log-convex* distributions $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$, for which

$$\text{(6.13)}\qquad \begin{cases} \big\{f\big(\tfrac{1}{2}(x+y)\big)\big\}^2 \le f(x)\,f(y) & \text{for } x>0,\ y>0,\\ p_k^2 \le p_{k-1}\,p_{k+1} & \text{for } k\in\mathbb{N}; \end{cases}$$

recall that such f and (p_k) are infinitely divisible by Theorems II.10.1 and III.10.2. To relate these two types of log-convexity via mixtures of Poisson distributions takes more effort than in the previous cases; we will not give all details of the proof.

Theorem 6.13. *A $(0,\infty)$-valued random variable T has a log-convex density iff $N(\theta T)$ has a log-convex distribution for all $\theta>0$.*

PROOF. Denote the distribution function of T by G and its pLSt by π; note that $G(0)=0$. First, we consider the direct part of the theorem; clearly, we may then restrict ourselves to the case $\theta=1$. Let G have a log-convex density g, and put $h(t) := \mathrm{e}^{-t}\,g(t)$ for $t>0$; h is log-convex as well. By (6.1) we have to prove that (p_k) is log-convex where

$$p_k = \int_0^\infty \frac{t^k}{k!}\,h(t)\,\mathrm{d}t \qquad [\,k\in\mathbb{Z}_+\,].$$

To do so we further suppose that g, and hence h, has a continuous second derivative on $(0,\infty)$. This is no essential restriction as one can see by writing an arbitrary g as the limit of a nondecreasing sequence of functions g_n with this property; cf. the proof of Theorem V.2.17. From Proposition A.3.9 it follows that the log-convexity of h can be represented by

$$\big\{h'(t)\big\}^2 \le h''(t)\,h(t) \qquad [\,t>0\,].$$

Also, since g and h are convex, the functions g, $-g'$, h and $-h'$ are both nonnegative and nonincreasing; hence, as is easily verified, $\lim_{t\downarrow 0} t\,h(t) = \lim_{t\downarrow 0} t^2 \{-h'(t)\} = 0$, and also $\lim_{t\to\infty} t^n\,h(t) = \lim_{t\to\infty} t^n \{-h'(t)\} = 0$ for all $n \in \mathbb{N}$. By partial integration it follows that (p_k) can be written in terms of h' and of h'' as follows:

$$p_k = \int_0^\infty \frac{t^{k+1}}{(k+1)!} \{-h'(t)\}\,\mathrm{d}t = \int_0^\infty \frac{t^{k+2}}{(k+2)!}\, h''(t)\,\mathrm{d}t \qquad [\,k \in \mathbb{Z}_+].$$

Using this, the inequality on h and Schwarz's inequality we conclude that for $k \in \mathbb{N}$

$$\begin{aligned} p_k^2 &= \Big(\int_0^\infty \frac{t^{k+1}}{(k+1)!}\,\big|h'(t)\big|\,\mathrm{d}t\Big)^2 \le \\ &\le \Big(\int_0^\infty \frac{t^{k+1}}{(k+1)!}\,\sqrt{h''(t)}\,\sqrt{h(t)}\,\mathrm{d}t\Big)^2 \le \\ &\le \Big(\int_0^\infty \frac{t^{k+1}}{(k+1)!}\,h''(t)\,\mathrm{d}t\Big)\Big(\int_0^\infty \frac{t^{k+1}}{(k+1)!}\,h(t)\,\mathrm{d}t\Big) = p_{k-1}\,p_{k+1}. \end{aligned}$$

Turning to the converse we suppose that the distribution $\big(p_k(\theta)\big)$ of $N(\theta T)$ is log-convex for all $\theta > 0$. From (6.8) we know that for $\theta > 0$

$$p_k(\theta) = \int_{(0,\infty)} \frac{(\theta x)^k}{k!}\,\mathrm{e}^{-\theta x}\,\mathrm{d}G(x) = \frac{(-\theta)^k}{k!}\,\pi^{(k)}(\theta) \qquad [\,k \in \mathbb{Z}_+\,],$$

so for $p_k'(\theta)$ we have

$$\theta\,p_k'(\theta) = k\,p_k(\theta) - (k+1)\,p_{k+1}(\theta) \qquad [\,k \in \mathbb{Z}_+\,].$$

In order to show that G has a log-convex density, we use the fact that $T/Y_k \xrightarrow{\mathrm{d}} T$ as $k \to \infty$, where Y_k is independent of T and gamma (k,k) distributed. This means (see Notes) that the normalized version G^* of G, with $G^*(x) := \frac{1}{2}\{G(x-) + G(x)\}$, can be obtained as

$$G^*(x) = \lim_{k\to\infty} G_k(x) \qquad [\,x > 0\,],$$

where G_k is the distribution function of T/Y_k, so with density g_k given by

$$g_k(t) = \int_{(0,\infty)} (x/t^2)\,f_{Y_k}(x/t)\,\mathrm{d}G(x) = (k/t)\,p_k(k/t) \qquad [\,t > 0\,].$$

Now, g_k turns out to be log-convex. In fact, using the expression for $p_k'(\theta)$ above one shows that for $t > 0$

$$t^2 g_k'(t) = k(k+1)\,\big\{p_{k+1}(k/t) - p_k(k/t)\big\},$$

$$t^3 g_k''(t) = k(k+1)(k+2)\,\big\{p_{k+2}(k/t) - 2p_{k+1}(k/t) + p_k(k/t)\big\},$$

and hence

$$\frac{t^4}{k^2(k+1)}\Big[g_k''(t)\,g_k(t)-\{g_k'(t)\}^2\Big]=\{p_{k+1}(k/t)-p_k(k/t)\}^2+$$
$$+(k+2)\,\{p_k(k/t)\,p_{k+2}(k/t)-p_{k+1}^2(k/t)\},$$

which is nonnegative because of the log-convexity of $\big(p_k(\theta)\big)$ for all θ. Being convex, g_k is nonincreasing; therefore, G_k is concave on $(0,\infty)$, and hence so is G^*. It follows (see Notes) that G^*, and hence G, is absolutely continuous with a density g^*, say, that is nonincreasing and for which

$$g^*(t)=\lim_{k\to\infty} g_k(t)\qquad [\,t>0 \text{ such that } g^* \text{ is continuous at } t\,].$$

Now, observe that g^* has countably many discontinuities, and that the sequence $\big(g_k(t)\big)$ is bounded for every t; in fact, putting $f_k(y):=y\,f_{Y_k}(y)$ for $y>0$, we can estimate as follows:

$$g_k(t)=\int_0^\infty f_k(y)\,g^*(ty)\,\mathrm{d}y\le$$
$$\le f_k(\tfrac12)\int_0^{\frac12} g^*(ty)\,\mathrm{d}y+g^*(\tfrac12 t)\int_{\frac12}^\infty f_k(y)\,\mathrm{d}y,$$

where $\int_{\frac12}^\infty f_k(y)\,\mathrm{d}y\le \mathbb{E}\,Y_k=1$ for all $k\in\mathbb{N}$ and $f_k(\frac12)\sim c\sqrt{k}\,(\frac12\sqrt{\mathrm{e}})^k$ as $k\to\infty$ for some $c>0$. Therefore, we can use Cantor's diagonal procedure to conclude that for some (sub-) sequence $\big(k(n)\big)$ in $\mathbb{N}$ the limit $g(t):=\lim_{n\to\infty} g_{k(n)}(t)$ exists for *all* $t>0$. Of course, the resulting function g is a density of G, and g, as a limit of log-convex functions, is log-convex as well. ◻

Somewhat more generally, in Theorem 6.13 one may start, as in previous cases, from an $\mathbb{R}_+$-valued random variable T. By adapting the proof of the theorem and using the fact that the class of log-convex distributions on $\mathbb{Z}_+$ is closed under mixing (see Proposition II.10.6), one can show that log-convexity of the distribution of $N(\theta T)$ for all θ is necessary and sufficient for T to have a distribution that is a mixture of a degenerate distribution at zero and a distribution with a log-convex density. A similar generalization is possible of Theorem 6.12 and of the following result on the subclasses of *completely monotone* densities on $(0,\infty)$ and *completely monotone* distributions on $\mathbb{Z}_+$; this is due to the fact that also the class of (completely) monotone distributions on $\mathbb{Z}_+$ is closed under mixing.

Theorem 6.14. *A $(0,\infty)$-valued random variable T has a completely monotone density iff $N(\theta T)$ has a completely monotone distribution for some, and then all, $\theta > 0$.*

This theorem, including its generalization mentioned above, is proved by using the facts (see Propositions A.3.11 and A.4.10) that the completely monotone densities f on $(0,\infty)$ and the completely monotone distributions (p_k) on $\mathbb{Z}_+$ coincide with the *mixtures of exponential densities* and the *mixtures of geometric distributions*, respectively:

$$(6.14) \qquad f(x) = \int_{(0,\infty)} \lambda\, \mathrm{e}^{-\lambda x}\, \mathrm{d}G(\lambda), \quad p_k = \int_{[0,1)} (1-p)\, p^k\, \mathrm{d}H(p),$$

where G is a distribution function on $(0,\infty)$ and H is one on $[0,1)$. Note that in the latter case mixing with the degenerate distribution at zero is included; such mixing in the former case is accounted for by the definition of *mixtures of exponential distributions* in Section 3.

More generally, for $r > 0$ we will relate the *mixtures of gamma (r) distributions* and the *mixtures of negative-binomial (r) distributions* with pLSt's π and pgf's P of the form

$$(6.15) \qquad \begin{cases} \pi(s) = \alpha + (1-\alpha) \displaystyle\int_{(0,\infty)} \Big(\frac{\lambda}{\lambda + s}\Big)^r \mathrm{d}G(\lambda), \\ P(z) = \displaystyle\int_{[0,1)} \Big(\frac{1-p}{1-pz}\Big)^r \mathrm{d}H(p), \end{cases}$$

where $\alpha \in [0,1]$ and G and H are as above. In fact, from the relation between π and P in (6.11) one easily obtains the following generalization of Theorem 6.9 and its corollary, and hence also (the generalized version of) Theorem 6.14.

Theorem 6.15. *For $r > 0$ the distribution of an $\mathbb{R}_+$-valued random variable T is a mixture of gamma (r) distributions iff the distribution of $N(\theta T)$ is a mixture of negative-binomial (r) distributions for some, and then all, $\theta > 0$.*

Corollary 6.16. *For $r > 0$ the distribution of a $\mathbb{Z}_+$-valued random variable X is a mixture of negative-binomial (r) distributions iff $X \stackrel{\mathrm{d}}{=} N(T)$ where the distribution of T is a mixture of gamma (r) distributions.*

The relation between π and P in (6.11) also immediately yields analogous results for the *generalized gamma convolutions* and the (similarly defined) *generalized negative-binomial convolutions* with pLSt's π and pgf's P that can be obtained as

$$(6.16)\qquad \pi(s) = \lim_{n\to\infty} \prod_{j=1}^{m_n} \Big(\frac{\lambda_{n,j}}{\lambda_{n,j}+s}\Big)^{r_{n,j}}, \quad P(z) = \lim_{n\to\infty} \prod_{j=1}^{m_n} \Big(\frac{1-p_{n,j}}{1-p_{n,j}\,z}\Big)^{r_{n,j}},$$

with obvious restrictions on the parameters.

Theorem 6.17. *The distribution of an $\mathbb{R}_+$-valued random variable T is a generalized gamma convolution iff the distribution of $N(\theta T)$ is a generalized negative-binomial convolution for some, and then all, $\theta > 0$.*

Corollary 6.18. *The distribution of a $\mathbb{Z}_+$-valued random variable X is a generalized negative-binomial convolution iff $X \stackrel{d}{=} N(T)$ where the distribution of T is a generalized gamma convolution.*

In the next two sections we will use Corollaries 6.16 and 6.18 to translate results from Sections 3, 4 and 5 into similar results for the mixtures of negative-binomial (r) distributions and the generalized negative-binomial convolutions.

7. Mixtures of negative-binomial distributions

We will mostly consider the negative-binomial (r,p) distribution for a fixed value of r. By a *mixture of negative-binomial (r) distributions* we understand any probability distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ of the form

$$(7.1)\qquad p_k = \int_{[0,1)} \binom{k+r-1}{k} p^k\,(1-p)^r\,\mathrm{d}H(p) = \binom{k+r-1}{k}\,\alpha_k,$$

with H a distribution function on $[\,0,1)$ and $(\alpha_k)_{k\in\mathbb{Z}_+}$ *completely monotone*; cf. Hausdorff's theorem. The pgf of (p_k) in (7.1) is given by P in (6.15). It can be represented as in the following reformulation of Corollary 6.16; recall that $N(\cdot)$ is a Poisson process of rate one, independent of T.

Proposition 7.1. *A function P on $[\,0,1\,]$ is the pgf of a mixture of negative-binomial (r) distributions iff P has the form*

$$P(z) = \pi(1-z) \qquad [\,0 \le z \le 1\,],$$

where π is the pLSt of a mixture of gamma (r) distributions.

From Corollary 6.16 it also follows that a $\mathbb{Z}_+$-valued random variable X has a distribution (p_k) of the form (7.1) iff $X \stackrel{d}{=} N(ZS)$ where $Z \geq 0$, S is standard gamma (r), and Z and S are independent. Now, by (6.4) we have $N(ZS) \stackrel{d}{=} Z \odot N(S)$. Moreover, $N(S)$ has a negative-binomial $(r, \frac{1}{2})$ distribution; see (6.11). Thus, the mixtures of negative-binomial (r) distributions can be viewed as *scale mixtures*; they correspond to random variables X of the form

$$X \stackrel{d}{=} Z \odot Y, \quad \text{with } Z \geq 0 \text{ and } Y \text{ negative-binomial}\,(r, \tfrac{1}{2}), \tag{7.2}$$

and, of course, Z and Y independent. More specifically, if X has a negative-binomial (r, W) distribution, i.e., X has distribution (7.1) with $H = F_W$, then X is of the form (7.2) with Z given by $Z \stackrel{d}{=} W/(1-W)$.

The *representation* result of Proposition 7.1 will be used repeatedly in this section. It enables us to easily translate results for mixtures of gamma (r) distributions into results in the present context. We start with translating Propositions 4.1 and 4.2, and the main result on infinite divisibility in Theorem 4.5.

Proposition 7.2. *If a sequence (P_n) of mixtures of negative-binomial (r) pgf's converges (pointwise) to a pgf P, then P is a mixture of negative-binomial (r) pgf's.*

Proof. Write $P_n(z) = \pi_n(1-z)$ with π_n a mixture of gamma (r) pLSt's. Then $\lim_{n\to\infty} \pi_n(s) = P(1-s)$ for $s \in [0,1]$. On the other hand, by Helly's theorem and the continuity theorem there is a subsequence (n_k) such that $\lim_{k\to\infty} \pi_{n_k}(s) = \pi(s)$ for $s > 0$, where π is a completely monotone function on $(0,\infty)$. It follows that $\pi(s) = P(1-s)$ for $s \in (0,1]$; hence, as P is a pgf, we have $\pi(0+) = 1$, so π is a pLSt. Now, apply Proposition 4.1: π is a mixture of gamma (r) pLSt's. Since $P(z) = \pi(1-z)$, we conclude that P is a mixture of negative-binomial (r) pgf's. □

Proposition 7.3. *For $\alpha \in (0,1)$, mixtures of negative-binomial (αr) distributions can be regarded as mixtures of negative-binomial (r) distributions.*

Proof. This is an immediate consequence of Proposition 4.2 and the representation result in Proposition 7.1 as stated and with r replaced by αr. □

Theorem 7.4. *A mixture of negative-binomial (2) distributions is infinitely divisible. Equivalently, a probability distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ is infinitely divisible if*

$$(7.3)\qquad p_k = (k+1)\,\alpha_k \qquad [\,k\in\mathbb{Z}_+\,],$$

with (α_k) completely monotone.

PROOF. Let (p_k) be a mixture of negative-binomial (2) distributions. Then its pgf P can be represented as $P(z) = \pi(1-z)$ with π the pLSt of a mixture of gamma (2) distributions; according to Theorem 4.5 π is infinitely divisible. From (the easy part of) Theorem 6.4 it follows that P is infinitely divisible as well. □

By combining this theorem with Proposition 7.3 we see that for $r \le 2$ all mixtures of negative-binomial (r) distributions are *infinitely divisible.* Now, use the pgf of such a mixture for P in the *power mixture* of (2.5), and apply Proposition 2.2. Then one obtains the following generalization of Theorem 7.4; cf. the first part of Theorem 4.7.

Theorem 7.5. *A function ϕ of the following form is an infinitely divisible characteristic function:*

$$(7.4)\qquad \phi(u) = \int_{[0,1)} \Big(\frac{1-p}{1-p\,\phi_1(u)}\Big)^r\, dH(p),$$

where $r\in(0,2\,]$, ϕ_1 is a characteristic function, and H is a distribution function on $[\,0,1)$.

We take $r=1$ in (7.1) and thus consider *mixtures of geometric distributions* or, equivalently (cf. Proposition A.4.10), *completely monotone* distributions (p_k) on $\mathbb{Z}_+$; from Theorem II.10.4 we know that they are log-convex, and hence *infinitely divisible.* A more direct proof of their infinite divisibility is obtained by combining Proposition 7.1 and Theorem 3.3; cf. the proof of Theorem 7.4. We now want to characterize the mixtures of geometric distributions among the infinitely divisible distributions; this was done in Theorem II.10.5 without proof. To this end we first derive a *canonical representation* from Theorem 3.5.

Theorem 7.6 (Canonical representation). *A function P on $[0,1]$ is the pgf of a mixture of geometric distributions iff P has the form*

$$(7.5)\qquad P(z) = \exp\Big[-\int_0^1 \Big(\frac{1}{1-p} - \frac{1}{1-p\,z}\Big)\frac{1}{p}\,w(p)\,dp\Big] \qquad [0 \le z \le 1],$$

where w is a measurable function on $(0,1)$ satisfying $0 \le w \le 1$ and, necessarily,

$$(7.6)\qquad \int_{\frac{1}{2}}^1 \frac{1}{1-p}\,w(p)\,dp < \infty.$$

PROOF. From Proposition 7.1 and Theorem 3.5 it follows that P is the pgf of a mixture of geometric distributions iff P has the form

$$P(z) = \exp\Big[-\int_0^\infty \Big(\frac{1}{\lambda} - \frac{1}{\lambda+1-z}\Big)\,v(\lambda)\,\mathrm{d}\lambda\Big] \qquad [0 \le z \le 1],$$

where v satisfies $0 \le v \le 1$ and (3.15). As in (6.11) we put $1/(\lambda+1) = p$ or $\lambda = 1/p - 1$; this yields (7.5) and (7.6) with $w(p) := v(1/p - 1)$. ◻◻

Corollary 7.7. *If P is the pgf of a mixture of geometric distributions, then so is P^a for any $a \in [0,1]$. Consequently, the convolution roots of a mixture of geometric distributions are mixtures of geometric distributions as well.*

As known from Section II.4, the canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ of the infinitely divisible pgf P in (7.5) has gf R given by

$$(7.7)\qquad R(z) := \frac{\mathrm{d}}{\mathrm{d}z}\log P(z) = \int_0^1 \Big(\frac{1}{1-p\,z}\Big)^2 w(p)\,\mathrm{d}p.$$

Now, $z \mapsto 1/(1-p\,z)^2$ is the gf of the sequence $\big((k+1)\,p^k\big)_{k\in\mathbb{Z}_+}$. Using this in (7.7) one is led to the direct part of the following *characterization* result; the converse part is easily obtained by using the canonical representations of Theorems II.4.1 and 7.6.

Theorem 7.8. *A probability distribution (p_k) on $\mathbb{Z}_+$ is a mixture of geometric distributions iff it is infinitely divisible having a canonical sequence (r_k) such that $\big(r_k/(k+1)\big)$ is completely monotone with Hausdorff representation of the form*

$$(7.8)\qquad \frac{1}{k+1}\,r_k = \int_0^1 p^k\,w(p)\,dp \qquad [k \in \mathbb{Z}_+],$$

where w is a measurable function on $(0,1)$ satisfying $0 \le w \le 1$ and (7.6).

Note that condition (7.6) on w also follows from (7.8) and the fact that the r_k necessarily satisfy $\sum_{k=0}^{\infty} r_k/(k+1) < \infty$; cf. (II.4.4).

We now turn to a discrete counterpart of Theorem 3.10 on mixing with 'negative probabilities'. As in the $\mathbb{R}_+$-case we start with considering *finite* mixtures; for $n \geq 2$ fixed let the sequence $(p_k)_{k \in \mathbb{Z}_+}$ and its gf P be given by

$$(7.9) \qquad p_k = \sum_{j=1}^{n} h_j\,(1-a_j)\,a_j^k, \quad P(z) = \sum_{j=1}^{n} h_j\,\frac{1-a_j}{1-a_j\,z},$$

where $0 < a_1 < \cdots < a_n < 1$ and $h_1 < 0,\,..,\,h_m < 0,\, h_{m+1} > 0,\,..,\,h_n > 0$ for some $m \in \{1,\ldots,n-1\}$ with $\sum_{j=1}^{n} h_j = 1$. Put $\varepsilon := \sum_{j=1}^{n} h_j\,(1-a_j)$, so $\varepsilon = p_0$, and rewrite p_k as

$$(7.10) \qquad p_k = \sum_{j=1}^{n} h_j\,(1-a_j)\,(a_j^k - a_m^k) + \varepsilon\, a_m^k \qquad [\,k \in \mathbb{Z}_+\,].$$

Then one sees that (p_k) can be viewed as a probability distribution on $\mathbb{Z}_+$ with $p_0 > 0$ iff $\varepsilon > 0$. In order to determine whether (p_k) is infinitely divisible in this case, one is tempted to write P as a mixture of Poisson pgf's and to use the $\mathbb{R}_+$-result in Theorem 3.10. Indeed, as in Proposition 7.1 the function P in (7.9) can be represented as

$$(7.11) \qquad P(z) = \pi(1-z), \;\text{ with } \pi(s) := \sum_{j=1}^{n} h_j\,\frac{\lambda_j}{\lambda_j + s}, \;\; \lambda_j := \frac{1-a_j}{a_j};$$

note that the λ_j are arranged in decreasing order. Now, according to the discussion preceding Theorem 3.10 the function π is a pLSt iff $\delta := \sum_{j=1}^{n} h_j\,(1-a_j)/a_j \geq 0$. It follows that $\delta \geq 0$ implies $\varepsilon > 0$; this can be verified directly by observing that $h_j\,a_m/a_j < h_j$ for all $j \neq m$, so $a_m\,\delta < \varepsilon$. Unfortunately, the inverse implication does not hold; *not* all pgf's P as in (7.9) can be viewed as mixtures of Poisson pgf's. Therefore, rather than using Theorem 3.10 one might try to imitate its proof and take gf's in (7.10), with $\varepsilon > 0$, to obtain

$$(7.12) \qquad P(z) = \Big\{ z \sum_{j=1}^{n} h_j\,\frac{a_j - a_m}{1-a_m}\,\frac{1-a_j}{1-a_j\,z} + \frac{\varepsilon}{1-a_m} \Big\}\,\frac{1-a_m}{1-a_m\,z}.$$

Thus P is of the form

$$(7.13) \qquad P(z) = P_1(z)\,\big\{\gamma + (1-\gamma)\,z\,P_2(z)\big\},$$

where $\gamma \in (0,1)$, P_1 is the pgf of a geometric distribution and P_2 is the pgf of a mixture of geometric distributions (with nonnegative weights), so P_2 is infinitely divisible. Nevertheless, the pgf $z \mapsto \gamma + (1-\gamma)\, z\, P_2(z)$ need not be infinitely divisible, so nothing can be said about P. Actually, one can show that *not* all pgf's P as in (7.9) are infinitely divisible. We therefore restrict ourselves to pgf's satisfying $\delta \geq 0$; their infinite divisibility is easily proved and extended to non-finite generalized mixtures as in the following theorem. For illustrative examples we refer to Section 12.

Theorem 7.9. *Let H be a right-continuous bounded function satisfying $H(p) = 0$ for $p \leq 0$, $H(p) = 1$ for $p \geq 1$ and such that for some $\alpha \in (0,1)$*

$$(7.14) \qquad \begin{cases} H \text{ is nonincreasing on } [\,0, \alpha), \\ H \text{ is nondecreasing on } [\,\alpha, 1\,] \text{ with } H(1-) = 1. \end{cases}$$

With this H, let the sequence $(p_k)_{k \in \mathbb{Z}_+}$ and its gf P be defined by

$$(7.15) \qquad p_k = \int_{(0,1)} p^k\,(1-p)\,dH(p), \quad P(z) = \int_{(0,1)} \frac{1-p}{1-p\,z}\,dH(p).$$

Then (p_k) can be viewed as a probability distribution on $\mathbb{Z}_+$ with $p_0 > 0$ iff $\varepsilon := \int_{(0,1)} (1-p)\,dH(p) > 0$. Moreover, if $\delta := \int_{(0,1)} (1-p)/p\,dH(p) \geq 0$, then $\varepsilon > 0$ and (p_k) is infinitely divisible; in fact, P can then be written as

$$(7.16) \qquad P(z) = P_1(z)\,P_2(z),$$

where P_1 is the pgf of the geometric (α) distribution and P_2 is the pgf of a mixture of geometric distributions.

Proof. Note that $\sum_{k=0}^{\infty} p_k = \int_{(0,1)} \mathrm{d}H(p) = 1$ and that $p_0 = \varepsilon$. Rewriting p_k as

$$p_k = \int_{(0,1)} (p^k - \alpha^k)\,(1-p)\,\mathrm{d}H(p) + \varepsilon\,\alpha^k \qquad [\,k \in \mathbb{Z}_+\,],$$

one sees that (p_k) is a probability distribution on $\mathbb{Z}_+$ with $p_0 > 0$ iff $\varepsilon > 0$. As in Proposition 7.1 the function P in (7.15) can be represented as

$$P(z) = \pi(1-z), \quad \text{with } \pi(s) := \int_{(0,\infty)} \frac{\lambda}{\lambda + s}\,\mathrm{d}G(\lambda),$$

where $G(\lambda) := 1 - H\big(1/(\lambda+1)\big)$ for $\lambda \geq 0$; note that G satisfies condition (3.27) with $\lambda_0 := (1-\alpha)/\alpha$. Let further $\delta \geq 0$. Then $\int_{(0,\infty)} \lambda\,\mathrm{d}G(\lambda) \geq 0$, so according to Theorem 3.10 the function π is an infinitely divisible pLSt.

By Theorem 6.4 it follows that ($\varepsilon > 0$ and) P is an infinitely divisible pgf. Moreover, the factorization (3.29) for π yields (7.16) with $P_i(z) := \pi_i(1-z)$ for $i = 1, 2$; P_1 and P_2 are pgf's of types as indicated in the theorem because of Theorems 6.9 and 6.15. ◻◻

By writing $p^k = \int_0^p (k+1)\, r^k \, \mathrm{d}r/p$ and changing the order of integration we see that the sequence (p_k) in (7.15) can also be written as

$$(7.17) \qquad p_k = (k+1) \int_0^1 p^k \, B(p) \, \mathrm{d}p \qquad [\, k \in \mathbb{Z}_+ \,],$$

where B is defined by $B(p) := \int_{(p,1)} (1-t)/t \, \mathrm{d}H(t)$ for $p \in (0,1)$. If $\delta \geq 0$, then B is *nonnegative*, so (p_k) is a *mixture of negative-binomial (2) distributions*; the infinite divisibility of (p_k) now also follows from Theorem 7.4.

We next want to present a discrete counterpart of Theorem 3.13, or rather Corollary 3.14, on *two-sided complete monotonicity*. Let $(p_k)_{k \in \mathbb{Z}}$ be a probability distribution on $\mathbb{Z}$ such that both the sequences $(p_k)_{k \in \mathbb{Z}_+}$ and $(p_{-k})_{k \in \mathbb{Z}_+}$ are completely monotone. Then (p_k) can be (uniquely) represented as

$$(7.18) \qquad p_k = \begin{cases} c_+ \, u_k & , \text{ if } k \in \mathbb{Z}_+, \\ c_- \, v_{-k} & , \text{ if } k \in \mathbb{Z}_-, \end{cases}$$

where $(u_k)_{k \in \mathbb{Z}_+}$ and $(v_k)_{k \in \mathbb{Z}_+}$ are completely monotone probability distributions on $\mathbb{Z}_+$, and c_+ and c_- are numbers in $(0, 1\,]$ determined by the fact that $-p_0 + c_+ + c_- = 1$ and $p_0 = c_+ \, u_0 = c_- \, v_0$, so

$$(7.19) \qquad c_+ = \frac{v_0}{u_0 + v_0 - u_0 v_0}, \qquad c_- = \frac{u_0}{u_0 + v_0 - u_0 v_0}.$$

When (p_k) is *symmetric*, (7.18) reduces to (IV.10.5), and because of Theorem IV.10.2 (p_k) is *infinitely divisible*; this is essentially due to Theorem 7.5 with $r = 1$ and $\phi_1(u) = \cos u$. In order to show that (p_k) is infinitely divisible also in the *asymmetric* case, we rewrite (p_k) as a mixture: Let $(\delta_{0,k})_{k \in \mathbb{Z}}$ be the distribution on $\mathbb{Z}$ that is degenerate at zero, and extend (u_k) and (v_k) to distributions on $\mathbb{Z}$ by $u_k := v_k := 0$ for $k < 0$; then putting $c_0 := -p_0 < 0$, we see that (p_k) can be written as

$$(7.20) \qquad p_k = c_0 \, \delta_{0,k} + c_+ \, u_k + c_- \, v_{-k} \qquad [\, k \in \mathbb{Z} \,],$$

where the weights c_0, c_+, c_- satisfy $c_+ \geq 0$, $c_- \geq 0$ and $c_0 + c_+ + c_- = 1$, and moreover $c_0 = -c_+ \, u_0 = -c_- \, v_0$. For the approximation below it is convenient to know that the final condition here can be relaxed somewhat; we will further suppose that only

$$c_0 \geq -c_+ \, u_0, \quad c_0 \geq -c_- \, v_0. \tag{7.21}$$

In fact, one easily verifies that also in this seemingly more general case (p_k) is a probability distribution on $\mathbb{Z}$ that is two-sided completely monotone; just use the fact that an additional mixing with the degenerate distribution at zero does not destroy the two-sided complete monotonicity. Now, by Hausdorff's theorem (u_k) and (v_k) are mixtures of geometric distributions and, as before, in proving infinite divisibility of (p_k) we may suppose that these mixtures are *finite* mixtures of the same order n, say; cf. Proposition IV.2.3. Thus we assume that (p_k) is of the form

$$\begin{aligned} p_k = c_0 \, \delta_{0,k} + c_+ \sum_{j=1}^{n} h_j \, (1 - a_j) \, a_j^k \, 1_{\mathbb{Z}_+}(k) \, + \\ + c_- \sum_{j=1}^{n} h_j' \, (1 - a_j') \, (a_j')^{-k} \, 1_{\mathbb{Z}_-}(k), \end{aligned} \tag{7.22}$$

where the weights h_j and h_j' are positive with $\sum_{j=1}^n h_j = \sum_{j=1}^n h_j' = 1$ and the parameters a_j and a_j' are in $(0,1)$ with $a_1 < \cdots < a_n$ and $a_1' < \cdots < a_n'$. Consider the characteristic function ϕ of (p_k); it is of the form $\phi(u) = \psi(e^{iu})$ for some $\mathbb{C}$-valued function ψ on the unit circle in $\mathbb{C}$. Extend ψ analytically; then for $z \notin \{1/a_1, \ldots, 1/a_n, a_1', \ldots, a_n'\}$ we can write

$$\psi(z) = c_0 + c_+ \sum_{j=1}^{n} h_j \, \frac{1 - a_j}{1 - a_j z} + c_- \sum_{j=1}^{n} h_j' \, \frac{1 - a_j'}{1 - a_j'/z} = \frac{Q_{2n}(z)}{P_{2n}(z)}, \tag{7.23}$$

where P_{2n} is a polynomial of degree $2n$ and Q_{2n} is a polynomial of degree $2n$ if $c_0 > -c_- \, v_0$ and of degree at most $2n-1$ if $c_0 = -c_- \, v_0$; in fact, if the leading coefficient in P_{2n} is taken to be one, then the coefficient of z^{2n} in Q_{2n} is given by

$$c_0 + c_- \sum_{j=1}^{n} h_j' \, (1 - a_j') = c_0 + c_- \, v_0,$$

which is nonnegative by (7.21). Proceed as in the proof of Theorem 3.3. Considering the fact that $1/(1 - a_j z) \to \pm\infty$ as $z \to 1/a_j \mp 0$ and the fact that $1/(1 - a_j'/z) \to \pm\infty$ as $z \to a_j' \pm 0$, one sees that ψ, and hence Q_{2n}, has $2n-2$ zeroes $1/b_2, \ldots, 1/b_n$ and $b_2', \ldots, b_n'$ satisfying $b_j \in (a_{j-1}, a_j)$ and $b_j' \in (a_{j-1}', a_j')$ for all j. Since by (7.21)

$$\lim_{z\to 0} \psi(z) = c_0 + c_+ \sum_{j=1}^{n} h_j\,(1 - a_j) = c_0 + c_+\, u_0 \geq 0,$$

we find an additional zero b_1' satisfying $b_1' \in [\,0, a_1')$. In case $c_0 > -c_-\, v_0$ we find a final zero $1/b_1$ with $b_1 \in (0, a_1)$ by noting that

$$\lim_{z\to\infty} \psi(z) = c_0 + c_- \sum_{j=1}^{n} h_j'\,(1 - a_j') = c_0 + c_-\, v_0 > 0.$$

Putting $b_1 := 0$ if $c_0 = -c_-\, v_0$, and using the fact that $\psi(1) = 1$, we conclude that in all cases ψ can be written as

$$(7.24)\qquad \psi(z) = \Big(\prod_{j=1}^{n} \frac{1 - a_j}{1 - a_j z}\Big/\frac{1 - b_j}{1 - b_j z}\Big)\Big(\prod_{j=1}^{n} \frac{1 - a_j'}{1 - a_j'/z}\Big/\frac{1 - b_j'}{1 - b_j'/z}\Big).$$

Now, use the fact (cf. Example II.11.10) that for $a, b \in (0, 1)$:

$$z \mapsto \frac{1 - a}{1 - a\,z}\Big/\frac{1 - b}{1 - b\,z} \ \text{ is an infinitely divisible pgf } \iff \ b \leq a.$$

Since $b_j < a_j$ and $b_j' < a_j'$ for all j, it follows that the characteristic function ϕ of (p_k) is of the form

$$(7.25)\qquad \phi(u) = \phi_1(u)\,\phi_2(u),$$

with ϕ_1 the characteristic function of an infinitely divisible distribution on $\mathbb{Z}_+$ and ϕ_2 the characteristic function of an infinitely divisible distribution on $\mathbb{Z}_-$. Therefore, (p_k) is infinitely divisible. We summarize.

Theorem 7.10. *A probability distribution $(p_k)_{k\in\mathbb{Z}}$ on $\mathbb{Z}$ with the property that both the sequences $(p_k)_{k\in\mathbb{Z}_+}$ and $(p_{-k})_{k\in\mathbb{Z}_+}$ are completely monotone, is infinitely divisible.*

Finally, we briefly return to the negative-binomial distribution with general shape parameter $r > 0$, and also mix with respect to r. The following result is an immediate consequence of Theorem 4.9; cf. the proof of Theorem 7.4.

Theorem 7.11. *A pgf P of the form*

$$(7.26)\qquad P(z) = \int_{(0,2]\times[0,1)} \Big(\frac{1 - p}{1 - p\,z}\Big)^{r}\, dH(r, p),$$

with H a distribution function on $(0, 2\,] \times [\,0, 1)$, is infinitely divisible.

It follows that for $p \in (0,1)$ and a distribution function H on $(0,\infty)$ the *power mixture*

$$(7.27) \qquad P(z) = \int_{(0,\infty)} \Big(\frac{1-p}{1-p\,z}\Big)^r \,\mathrm{d}H(r)$$

is infinitely divisible if the support $S(H)$ of H is restricted to $(0,2]$. Of course, another sufficient condition is infinite divisibility of H; cf. Proposition 2.1.

8. Generalized negative-binomial convolutions

A probability distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ is said to be a *generalized negative-binomial convolution* if it is the weak limit of finite convolutions of negative-binomial distributions; the pgf P of such a (p_k) can be obtained as in (6.16). We reformulate Corollary 6.18 in terms of transforms.

Proposition 8.1. *A function P on $[0,1]$ is the pgf of a generalized negative-binomial convolution iff P has the form*

$$(8.1) \qquad P(z) = \pi(1-z) \qquad [0 \le z \le 1],$$

where π is the pLSt of a generalized gamma convolution.

This *representation* result will be used to translate results for generalized gamma convolutions into results in the present context. We start with doing so for Theorem 5.1 and Proposition 5.2; the proofs of the next two results are similar to those of Theorem 7.6 and Proposition 7.2, respectively.

Theorem 8.2 (Canonical representation). *A function P on $[0,1]$ is the pgf of a generalized negative-binomial convolution iff P has the form*

$$(8.2) \qquad P(z) = \exp\Big[-a\,(1-z) + \int_{(0,1)} \log\frac{1-p}{1-p\,z}\,dV(p)\Big] \qquad [0 \le z \le 1],$$

where $a \ge 0$ and V is a right-continuous nondecreasing function on $(0,1)$ satisfying $V(1-) = 0$ and, necessarily,

$$(8.3) \qquad \int_{(0,\frac12)} p\,dV(p) < \infty, \quad \int_{(\frac12,1)} \log\frac{1}{1-p}\,dV(p) < \infty.$$

In proving this theorem one observes that if P satisfies (8.1) with π the pLSt of a generalized gamma convolution with canonical pair (a, U), so with π of the form (5.2), then P satisfies (8.2) with the same a and with V given by

$$V(p) = -U(1/p - 1) \qquad \big[\text{V continuous at } p \in (0,1)\big]. \tag{8.4}$$

Condition (8.3) is equivalent to the integral in (8.2) being finite; in particular, it guarantees that the pgf P in (8.2) has $P(0)$ positive. If $a > 0$, then P has a *Poisson* factor; we will often restrict ourselves to the case where $a = 0$.

Proposition 8.3. *If a sequence (P_n) of pgf's of generalized negative-binomial convolutions converges (pointwise) to a pgf P, then P is the pgf of a generalized negative-binomial convolution.*

By its definition a generalized negative-binomial convolution is *infinitely divisible*; cf. Section II.2. The canonical sequence $(r_k)_{k \in \mathbb{Z}_+}$ of the pgf P in (8.2) has gf R given by

$$R(z) := \frac{\mathrm{d}}{\mathrm{d}z} \log P(z) = a + \int_{(0,1)} \frac{p}{1 - p\,z} \,\mathrm{d}V(p), \tag{8.5}$$

so the r_k themselves can be written as

$$r_k = a\,\delta_{0,k} + \int_{(0,1)} p^{k+1}\,\mathrm{d}V(p) = \int_{[0,1)} p^k\,\mathrm{d}V_1(p) \qquad [\,k \in \mathbb{Z}_+\,], \tag{8.6}$$

where $V_1(p) := 0$ for $p < 0$ and $V_1(p) := a + \int_{(0,p]} t\,\mathrm{d}V(t)$ for $p \in [0,1)$, which is finite because of (8.3). By the uniqueness part of Hausdorff's theorem it follows that the canonical pair (a, V) of P is uniquely determined by P; the function V is called the *Thorin function* of P (and of the corresponding distribution). Hausdorff's theorem also implies the direct part of the following *characterization* result; the converse part is similarly obtained by using the canonical representations of Theorems II.4.1 and 8.2.

Theorem 8.4. *A probability distribution (p_k) on $\mathbb{Z}_+$ is a generalized negative-binomial convolution iff it is infinitely divisible having a canonical sequence (r_k) that is completely monotone. In this case the Hausdorff representation of (r_k) can be obtained from the canonical pair (a, V) as in (8.6).*

This theorem can be used to obtain a sufficient condition for infinite divisibility as in the following proposition; compare with the sufficient condition given in Theorem 7.4.

Proposition 8.5. *A probability distribution* $(p_k)_{k\in\mathbb{Z}_+}$ *on* $\mathbb{Z}_+$ *is infinitely divisible if it is of the form*

$$p_k = \frac{1}{\mu}\,(k+1)\,q_{k+1} \qquad [\,k\in\mathbb{Z}_+\,] \tag{8.7}$$

with $(q_k)_{k\in\mathbb{Z}_+}$ *a generalized negative-binomial convolution that has finite mean* μ.

PROOF. Similar to that of Proposition 5.4. If P_1 is the pgf of (q_k) and R_1 is the R-function of P_1, then the pgf P of (p_k) can be written as

$$P(z) = \frac{1}{\mu}\,P_1'(z) = \frac{1}{\mu}\,P_1(z)\,R_1(z) = P_1(z)\,P_2(z),$$

where $P_2 := R_1/\mu$ is the gf of a probability distribution that is completely monotone and hence infinitely divisible. ◻

Theorem 8.4 can also be used to show that the generalized negative-binomial convolutions interpolate between the *stable* and the *self-decomposable* distributions on $\mathbb{Z}_+$; just apply Theorems V.4.13 and V.5.6, and note that, for $\gamma\in(0,1)$ and $k\in\mathbb{Z}_+$, $\binom{k-\gamma}{k}$ is the k-th moment of the beta $(1-\gamma,\gamma)$ distribution (cf. Section B.3). Alternatively, one might use Proposition 8.1 together with Propositions 5.5 and 5.7, Theorem 6.6 and Corollary 6.8. Thus the following two propositions hold; the final statement in the first one is a consequence of Theorem V.4.20.

Proposition 8.6. *A generalized negative-binomial convolution* $(p_k)_{k\in\mathbb{Z}_+}$ *is self-decomposable. Moreover,* (p_k) *is unimodal; it is nonincreasing iff it starts nonincreasing in the sense that* $p_1 \le p_0$.

Proposition 8.7. *A stable distribution on* $\mathbb{Z}_+$, *with pgf* P *of the form*

$$P(z) = \exp\big[-\lambda\,(1-z)^\gamma\big], \tag{8.8}$$

where $\lambda > 0$ *and* $\gamma\in(0,1\,]$, *is a generalized negative-binomial convolution.*

Next we establish a connection with the *mixtures of negative-binomial distributions* from the preceding section. First, recall from Theorem 7.8

that the mixtures of *geometric* distributions correspond to the infinitely divisible distributions on $\mathbb{Z}_+$ that have a canonical sequence (r_k) such that

$$(8.9) \qquad \frac{1}{k+1}\, r_k = \int_0^1 p^k\, w(p)\, \mathrm{d}p \qquad [\, k \in \mathbb{Z}_+ \,]$$

for some function w satisfying $0 \le w \le 1$; let us call w the *canonical density* of the mixture. Now, if (r_k) is as in (8.6), then by using Fubini's theorem one sees that

$$(8.10) \qquad \frac{1}{k+1}\, r_k = a\, \delta_{0,k} + \int_0^1 p^k\, \{-V(p)\}\, \mathrm{d}p \qquad [\, k \in \mathbb{Z}_+ \,].$$

By Theorem 8.4 and the uniqueness part of Hausdorff's theorem we thus obtain the following result.

Proposition 8.8. *A generalized negative-binomial convolution with canonical pair (a, V) is a mixture of geometric distributions iff $a = 0$ and V satisfies $-V(0+) \le 1$. In this case for the canonical density w of the mixture one may take $w = -V$.*

We make a few comments on this result. The first one concerns a generalized negative-binomial convolution (p_k) with canonical pair $(0, V)$. According to Proposition 8.6 and (8.6) (note that $r_0 = p_1/p_0$) such a (p_k) is *monotone* iff $\int_{(0,1)} p\, \mathrm{d}V(p) \le 1$. Since this condition is less restrictive than $-V(0+) = \int_{(0,1)} \mathrm{d}V(p) \le 1$, it follows from Proposition 8.8 that monotonicity of (p_k) is *not* equivalent to complete monotonicity of (p_k). This is contrary to the $\mathbb{R}_+$-case, where we do have such an equivalence for densities of generalized gamma convolutions; cf. Corollary 5.9. In view of Theorem 6.14 we conclude that there exist pgf's P of the form (8.1) that correspond to monotone distributions but for which π corresponds to a density that is *not* monotone; we refer to Section 12 for an example. The second comment concerns a converse; if (r_k) satisfies (8.10) with V nondecreasing, then (r_k) can be written as in (8.6). Together with Theorem 8.4 and Proposition 8.8 this shows that the mixtures of geometric distributions that can be viewed as generalized negative-binomial convolutions, can be identified as follows.

Corollary 8.9. *A mixture of geometric distributions with canonical density w is a generalized negative-binomial convolution iff w can be chosen to be nonincreasing.*

The final comment on Proposition 8.8 concerns the following generalization of its main part; it is easily obtained from the canonical representation in Theorem 8.2.

Corollary 8.10. *For $r > 0$ a generalized negative-binomial convolution with canonical pair (a, V) is the r-fold convolution of a mixture of geometric distributions iff $a = 0$ and V satisfies $-V(0+) \leq r$.*

The generalized negative-binomial convolutions in this corollary, with r fixed, can also be viewed as *mixtures of negative-binomial (r) distributions*, so as distributions $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ of the form

$$(8.11) \qquad p_k = \binom{k+r-1}{k}\,\alpha_k, \quad \text{with } (\alpha_k) \text{ completely monotone;}$$

cf. (7.1). This easily follows by translating Theorem 5.12 in the usual way: Apply Corollary 6.16, use Proposition 8.1 and observe that because of (8.4) the Thorin functions U of π and V of P satisfy the relation $\lim_{\lambda\to\infty} U(\lambda) = -V(0+)$. This leads to the following result; its corollary is a consequence of Corollary 8.10.

Theorem 8.11. *For $r > 0$ a generalized negative-binomial convolution with canonical pair (a, V) is a mixture of negative-binomial (r) distributions iff $a = 0$ and V satisfies $-V(0+) \leq r$.*

Corollary 8.12. *For $r > 0$ a generalized negative-binomial convolution is a mixture of negative-binomial (r) distributions iff it is the r-fold convolution of a mixture of geometric distributions.*

As in the $\mathbb{R}_+$-case, we will look for sufficient conditions for (p_k) of the form (8.11) to be a generalized negative-binomial convolution. More generally, we are interested in determining classes of explicit distributions $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ that are generalized negative-binomial convolutions, and hence are self-decomposable and infinitely divisible. Of course, translating Proposition 8.1 in terms of distributions yields the following result.

Proposition 8.13. *A probability distribution $(p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ of the form*

$$(8.12) \qquad p_k = \frac{1}{k!}\int_0^\infty x^k\, e^{-x}\, g(x)\, dx \qquad [\, k \in \mathbb{Z}_+ \,]$$

is a generalized negative-binomial convolution if (and only if) g is a density of a generalized gamma convolution.

The main results of Section 5, i.e., Theorems 5.18, 5.24 and 5.30, now give important criteria in terms of *hyperbolic complete monotonicity*: If the density g itself, or the Lt π of g, or the distribution function G corresponding to g, is hyperbolically completely monotone, then (p_k) in (8.12) is a generalized negative-binomial convolution. Such a distribution (p_k) can, however, seldom be obtained explicitly. Therefore, it would be of great interest to have a discrete analogue of Theorem 5.18, i.e., we would like to have an appropriate concept of *discrete* hyperbolic complete monotonicity for sequences, with the property that (p_k) is hyperbolically completely monotone iff (p_k) is of the form (8.12) with g hyperbolically completely monotone; then it would follow that any hyperbolically completely monotone distribution (p_k) is a generalized negative-binomial convolution, and is hence self-decomposable and infinitely divisible. Unfortunately, such a concept has not been found yet. If it exists, it will *not* satisfy the discrete counterparts of the product properties of Propositions 5.19 (ii) and 5.27, because the product XY of two independent geometrically distributed random variables X and Y is not even infinitely divisible; see Section 12. Using the $\odot$-product as considered in Section 2, however, we can state the following result; as in Section 6, $N(T)$ is the Poisson process of rate one, stopped at an independent random time T.

Proposition 8.14. *Let Z and Y be independent random variables, Z non-negative and Y $\mathbb{Z}_+$-valued. Then $Z \odot Y$ is well defined and its distribution is a generalized negative-binomial convolution in each of the following cases:*

(i) *Z has a hyperbolically completely monotone density and the distribution of Y is a generalized negative-binomial convolution;*

(ii) *The distribution of Z is a generalized gamma convolution and Y satisfies $Y \stackrel{\mathrm{d}}{=} N(T)$ with T a random variable with a hyperbolically completely monotone density.*

Proof. By Proposition 8.1 (or rather Corollary 6.18) in case (i) we have $Y \stackrel{\mathrm{d}}{=} N(T)$ where the distribution of T is a generalized gamma convolution. Hence in both cases

$$Z \odot Y \stackrel{\mathrm{d}}{=} N(ZT), \quad \text{with } Z \text{ and } T \text{ independent},$$

because by (6.4) $Z \odot N(T) \stackrel{\mathrm{d}}{=} N(ZT)$. Now apply Proposition 5.27, and use the fact that the distribution of $N(ZT)$ is a generalized negative-binomial convolution if (and only if) the distribution of ZT is a generalized gamma convolution. ◻◻

Let us look at some implications of this proposition in terms of distributions. If Z has an absolutely continuous distribution with density g, then $Z \odot Y$ has distribution (p_k) given by

$$(8.13) \qquad p_k = \int_0^\infty \mathbb{P}(x \odot Y = k)\, g(x)\, \mathrm{d}x \qquad [\, k \in \mathbb{Z}_+ \,].$$

So, by taking Y in part (i) of the proposition Poisson (1) and observing that $x \odot Y$ is Poisson (x) for $x > 0$, we get the special case of Proposition 8.13 with g hyperbolically completely monotone. Further, by taking Y in part (ii) negative-binomial $(r, \frac{1}{2})$ or, equivalently, T standard gamma (r) and observing that the distribution of $x \odot Y$ is negative-binomial $\big(r, x/(1+x)\big)$ for $x > 0$, we obtain the following counterpart to Proposition 8.13.

Proposition 8.15. *For $r > 0$ a probability distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ of the form*

$$(8.14) \qquad p_k = \binom{k+r-1}{k} \int_0^\infty \frac{x^k}{(1+x)^{k+r}}\, g(x)\, dx \qquad [\, k \in \mathbb{Z}_+ \,]$$

is a generalized negative-binomial convolution if g is the density of a generalized gamma convolution.

Note that (8.14) is of the form (8.11), so Proposition 8.15 gives a sufficient condition for a mixture of negative-binomial (r) distributions to be a generalized negative-binomial convolution; see also (7.2). In fact, the proposition can be reformulated as follows: A negative-binomial (r, W) distribution (so with random success parameter W) is a generalized negative-binomial convolution if the distribution of $Z := W/(1-W)$ is a generalized gamma convolution.

In some special cases Propositions 8.13 and 8.15 can be used to determine explicit distributions (p_k) that are generalized negative-binomial convolutions; for illustrating examples we refer to Section 12. In dealing with concrete examples it is sometimes convenient to make use of the following *closure properties*; the first one easily follows from Theorems 8.2 or 8.4, and the second one is obtained by combining Propositions 5.25 and 8.1.

Proposition 8.16. *If* $(p_k)_{k\in\mathbb{Z}_+}$ *having pgf* P *is a generalized negative-binomial convolution, then so is* $(p_k^{(\alpha)})_{k\in\mathbb{Z}_+}$, *with* $p_k^{(\alpha)} := \alpha^k p_k / P(\alpha)$, *for every* $\alpha \in (0,1)$.

Proposition 8.17.

(i) *If* π *is the LSt of a generalized gamma convolution, then the function* $z \mapsto \pi\big((1-z)^\gamma\big)$ *is the pgf of a generalized negative-binomial convolution for every* $\gamma \in (0,1\,]$.

(ii) *If* P *is the pgf of a generalized negative-binomial convolution, then so is the function* $z \mapsto P\big(1-(1-z)^\gamma\big)$ *for every* $\gamma \in (0,1\,]$.

Part (ii) of this proposition shows, for instance, that the generalized negative-binomial convolutions that are *stable*, can be obtained in a simple way from the Poisson distribution.

9. Mixtures of zero-mean normal distributions

We turn to probability distributions on $\mathbb{R}$, and consider *variance mixtures of normal distributions with mean zero.* In doing so it will appear more convenient to mix with respect to *half* the variance, so we consider normal $(0, 2T)$ distributions with positive random T. Their densities are called *mixtures of zero-mean normal densities* and are given by

$$(9.1) \qquad f(x) = \int_{(0,\infty)} \frac{1}{2\sqrt{\pi t}}\, \exp\big[-x^2/(4t)\big]\, \mathrm{d}G(t) \qquad [\,x \neq 0\,],$$

with $G = F_T$. Sometimes we allow T to be zero; the corresponding 'normal' distribution is then degenerate at zero. The characteristic functions ϕ of the resulting *mixtures of zero-mean normal distributions* can be written as

$$(9.2) \qquad \phi(u) = \int_{\mathbb{R}_+} \exp\,[-tu^2]\, \mathrm{d}G(t) = \pi(u^2),$$

with π the LSt of G. As noted in Section 2, these mixtures can be viewed as *power mixtures* and as *scale mixtures*; a random variable X with characteristic function ϕ given by (9.2) can be represented as

$$(9.3) \qquad X \overset{\mathrm{d}}{=} B(T) \quad \text{and} \quad X \overset{\mathrm{d}}{=} \sqrt{T}\, B,$$

where $B(\cdot)$ is Brownian motion with $B := B(1)$ normal $(0,2)$ and T is independent of $B(\cdot)$. We will mostly use, without further comment, the first representation in (9.3). Of course, the distributions of T and $B(T)$ determine each other, so

$$(9.4) \qquad B(T_1) \stackrel{\mathrm{d}}{=} B(T_2) \iff T_1 \stackrel{\mathrm{d}}{=} T_2.$$

Moreover, similar to the proof of Proposition 3.1 one shows that the class of mixtures of zero-mean normal distributions is *closed under weak convergence.*

Proposition 9.1. *If a sequence* (ϕ_n) *of mixtures of zero-mean normal characteristic functions converges (pointwise) to a characteristic function* ϕ*, then* ϕ *is a mixture of zero-mean normal characteristic functions.*

The mapping $T \mapsto B(T)$ transforms distributions on $\mathbb{R}_+$ into *symmetric* distributions on $\mathbb{R}$. It preserves *infinite divisibility*, i.e., if T is infinitely divisible, then so is $B(T)$; this immediately follows from Proposition 2.1. The converse is not true; see Section 12. But for $B(T)$ to be infinitely divisible the random variable T has to have several properties in common with infinitely divisible random variables. For instance, if T is non-degenerate, then T cannot have an arbitrarily thin tail; cf. Theorem III.9.1. We will not prove this rather technical result (see Notes), and only show that T must be unbounded.

Proposition 9.2. *The random variable* $B(T)$ *is infinitely divisible if* T *is infinitely divisible. On the other hand,* $B(T)$ *is not infinitely divisible if* T *is non-degenerate and bounded.*

Proof. We only have to prove the second assertion, so let T be bounded by some $a > 0$. Consider the tail probability $\mathbb{P}\big(|B(T)| > x\big)$ for $x > 0$; because of (9.1) it satisfies

$$x\,\mathbb{P}\big(|B(T)| > x\big) \le \int_{(0,a]} \Big(\int_x^\infty \frac{y}{\sqrt{\pi t}}\, \exp\big[-y^2/(4t)\big]\,\mathrm{d}y\Big)\,\mathrm{d}G(t) =$$

$$= \int_{(0,a]} \frac{2\sqrt{t}}{\sqrt{\pi}}\, \exp\big[-x^2/(4t)\big]\,\mathrm{d}G(t) \;\le\; 2\sqrt{a/\pi}\, \exp\big[-x^2/(4a)\big].$$

Now, suppose that $B(T)$ is infinitely divisible, and apply Corollary IV.9.9. Then from the estimation above it follows that $B(T)$ is normal and hence by (9.4) that T is degenerate. □

It is not known whether, as for $N(T)$ in Theorem 6.3, there are simple necessary and sufficient conditions on T for $B(T)$ to be infinitely divisible. Also, the analogue of Theorem 6.4 does not hold, i.e., infinite divisibility of T is not characterized by the infinite divisibility of $B(\theta T)$ for all $\theta > 0$; this follows from the fact that by (9.3) $B(\theta T) \stackrel{d}{=} \sqrt{\theta}\, B(T)$. Similar remarks can be made with respect to *self-decomposability*; using just the definitions one easily shows that

$$(9.5) \qquad T \text{ self-decomposable} \implies B(T) \text{ self-decomposable},$$

but there seems to be no converse. The mapping $T \mapsto B(T)$ also preserves *stability*; in this case there does exist a converse.

Theorem 9.3. *For $\gamma \in (0,1]$ an $\mathbb{R}_+$-valued random variable T is stable with exponent γ iff $B(T)$ is stable with exponent 2γ.*

PROOF. Use the canonical representation for the pLSt π of a stable distribution on $\mathbb{R}_+$ with exponent $\gamma \in (0,1]$ and for the characteristic function ϕ of a *symmetric* stable distribution on $\mathbb{R}$ with exponent $\gamma \in (0,2]$; according to Theorems V.3.5 and V.7.6 they are given by

$$(9.6) \qquad \pi(s) = \exp\left[-\lambda\, s^{\gamma}\right], \quad \phi(u) = \exp\left[-\lambda\, |u|^{\gamma}\right],$$

where $\lambda > 0$. The assertion of the theorem now immediately follows from (9.2) and (9.4). □□

Corollary 9.4. *For $\gamma \in (0,2]$ a random variable X has a symmetric stable distribution with exponent γ iff $X \stackrel{d}{=} B(T)$ where T is $\mathbb{R}_+$-valued and stable with exponent $\frac{1}{2}\gamma$.*

Similarly one shows that Theorem 9.3 and its corollary have close analogues for the *gamma* and *sym-gamma* distributions with shape parameter r, whose pLSt's π and characteristic functions ϕ, respectively, are of the form

$$(9.7) \qquad \pi(s) = \Big(\frac{\lambda}{\lambda + s}\Big)^{r}, \quad \phi(u) = \Big(\frac{\lambda^2}{\lambda^2 + u^2}\Big)^{r},$$

where $\lambda > 0$. We state the results.

Theorem 9.5. *For $r > 0$ an $\mathbb{R}_+$-valued random variable T has a gamma (r) distribution iff $B(T)$ has a sym-gamma (r) distribution.*

Corollary 9.6. *For $r > 0$ a random variable X has a sym-gamma (r) distribution iff $X \stackrel{d}{=} B(T)$ with T gamma (r) distributed.*

Taking $r = 1$ we see that T has an *exponential* distribution iff $B(T)$ has a *Laplace* distribution. Using the first part of Proposition 9.2 one easily shows that for the more general *compound-exponential* distributions on $\mathbb{R}_+$ and on $\mathbb{R}$ we have

$$\text{(9.8)} \qquad T \text{ compound-exponential} \implies B(T) \text{ compound-exponential;}$$

as for (9.5) there seems to be no converse.

We turn to *log-convex* and, more specially, *completely monotone* densities. Since log-convex densities on $\mathbb{R}$ do not exist and there seems to be no obvious concept of complete monotonicity for functions on $\mathbb{R}$, Theorems 6.13 and 6.14 have no (partial) analogues. On the other hand, completely monotone densities on $(0, \infty)$ coincide with *mixtures of exponential densities*, and these mixtures do have an obvious counterpart on $\mathbb{R}$, viz. *mixtures of Laplace densities*, which are of the form

$$\text{(9.9)} \qquad f(x) = \int_{(0,\infty)} \tfrac{1}{2}\lambda\, \mathrm{e}^{-\lambda|x|}\, \mathrm{d}H(\lambda) \qquad [\,x \neq 0\,]$$

with H a distribution function on $(0, \infty)$.

More generally, for $r > 0$ we will relate the *mixtures of gamma (r) distributions* and the *mixtures of sym-gamma (r) distributions* with pLSt's π and characteristic functions ϕ of the form

$$\text{(9.10)} \qquad \begin{cases} \pi(s) = \alpha + (1-\alpha) \displaystyle\int_{(0,\infty)} \Big(\frac{\lambda}{\lambda + s}\Big)^r \mathrm{d}G(\lambda), \\ \phi(u) = \alpha + (1-\alpha) \displaystyle\int_{(0,\infty)} \Big(\frac{\lambda^2}{\lambda^2 + u^2}\Big)^r \mathrm{d}H(\lambda), \end{cases}$$

where $\alpha \in [\,0, 1\,]$ and G and H are distribution functions on $(0, \infty)$. In fact, proceeding as for Theorem 9.5 one obtains the following result.

Theorem 9.7. *For $r > 0$ the distribution of an $\mathbb{R}_+$-valued random variable T is a mixture of gamma (r) distributions iff the distribution of $B(T)$ is a mixture of sym-gamma (r) distributions.*

Corollary 9.8. *For $r > 0$ the distribution of a random variable X is a mixture of sym-gamma (r) distributions iff $X \stackrel{d}{=} B(T)$ where the distribution of T is a mixture of gamma (r) distributions.*

The relation between π and ϕ in (9.7) also immediately yields an analogous relation between the *generalized gamma convolutions* and the (similarly defined) *generalized sym-gamma convolutions* with pLSt's π and characteristic functions ϕ that can be obtained as

$$(9.11) \qquad \pi(s) = \lim_{n\to\infty} \prod_{j=1}^{m_n} \Big(\frac{\lambda_{n,j}}{\lambda_{n,j}+s}\Big)^{r_{n,j}}, \quad \phi(u) = \lim_{n\to\infty} \prod_{j=1}^{m_n} \Big(\frac{\lambda_{n,j}^2}{\lambda_{n,j}^2+u^2}\Big)^{r_{n,j}},$$

with obvious restrictions on the parameters.

Theorem 9.9. *The distribution of an $\mathbb{R}_+$-valued random variable T is a generalized gamma convolution iff the distribution of $B(T)$ is a generalized sym-gamma convolution.*

Corollary 9.10. *The distribution of a random variable X is a generalized sym-gamma convolution iff $X \overset{\mathrm{d}}{=} B(T)$ where the distribution of T is a generalized gamma convolution.*

In the next two sections we will use Corollaries 9.8 and 9.10 to translate results from Sections 3, 4 and 5 into similar results for the mixtures of sym-gamma distributions and the generalized sym-gamma convolutions.

10. Mixtures of sym-gamma distributions

We will mostly consider the sym-gamma (r, λ) distribution for a fixed value of the shape parameter $r > 0$. As stated in (9.10), the characteristic function ϕ of a *mixture of sym-gamma (r) distributions* is of the form

$$(10.1) \qquad \phi(u) = \alpha + (1-\alpha) \int_{(0,\infty)} \Big(\frac{\lambda^2}{\lambda^2+u^2}\Big)^r \,\mathrm{d}H(\lambda),$$

where $\alpha \in [0,1]$ and H is a distribution function on $(0,\infty)$. These characteristic functions can be represented as in the following reformulation of Corollary 9.8.

Proposition 10.1. *A function ϕ on $\mathbb{R}$ is the characteristic function of a mixture of sym-gamma (r) distributions iff ϕ has the form*

$$\phi(u) = \pi(u^2) \qquad [u \in \mathbb{R}],$$

where π is the pLSt of a mixture of gamma (r) distributions.

Clearly, the mixtures in (10.1) are *scale mixtures*; they correspond to random variables of the form ZY with Z *nonnegative* and Y sym-gamma (r), and Z and Y independent. Use of Proposition 10.1 translates Propositions 4.1 and 4.2 into the following properties; cf. Propositions 7.2 and 7.3.

Proposition 10.2. *If a sequence (ϕ_n) of mixtures of sym-gamma (r) characteristic functions converges (pointwise) to a characteristic function ϕ, then ϕ is a mixture of sym-gamma (r) characteristic functions.*

Proposition 10.3. *For $\alpha \in (0,1)$, mixtures of sym-gamma (αr) distributions can be regarded as mixtures of sym-gamma (r) distributions.*

Similarly, Theorem 4.5 and Proposition 9.2 imply a result that was already obtained in the second part of Theorem 4.7 and can be reformulated as follows; see f_2 in Example IV.2.9.

Theorem 10.4. *A mixture of sym-gamma (2) distributions is infinitely divisible. Consequently, a probability density f on $\mathbb{R}$ is infinitely divisible if f is of the form*

$$(10.2) \qquad f(x) = \int_{(0,\infty)} \tfrac{1}{4}\lambda \left(1 + \lambda|x|\right) e^{-\lambda|x|}\, dH(\lambda) \qquad [\, x \neq 0\,]$$

with H a distribution function on $(0,\infty)$.

Note that f in (10.2) can be written as the following two-term mixture:

$$f(x) = \tfrac{1}{2} \int_{(0,\infty)} \tfrac{1}{2}\lambda\, \mathrm{e}^{-\lambda|x|}\, \mathrm{d}H(\lambda) + \tfrac{1}{2} \int_{(0,\infty)} \tfrac{1}{2}\lambda^2\, |x|\, \mathrm{e}^{-\lambda|x|}\, \mathrm{d}H(\lambda).$$

Now by Proposition 4.2 the first term here can be put in the same form as the second term, so f can be viewed as a *mixture of double-gamma (2) distributions*:

$$(10.3) \qquad f(x) = \int_{(0,\infty)} \tfrac{1}{2}\lambda^2\, |x|\, \mathrm{e}^{-\lambda|x|}\, \mathrm{d}G(\lambda) = |x|\, \psi\big(|x|\big) \qquad [\, x \neq 0\,],$$

where G is a distribution function on $(0,\infty)$ and ψ is *completely monotone*. Not every density f of the form (10.3) is, however, a mixture of sym-gamma (2) densities. In fact, not every f of the form (10.3) is infinitely divisible; cf. Example IV.11.15.

Thus, for $r \le 2$ all mixtures of sym-gamma (r) distributions are *infinitely divisible.* Moreover, Theorem 4.9 implies that mixing with respect to such values of r preserves infinite divisibility.

Theorem 10.5. *A characteristic function ϕ of the form*

$$\phi(u) = \alpha + (1-\alpha) \int_{(0,2]\times(0,\infty)} \Big(\frac{\lambda^2}{\lambda^2+u^2}\Big)^r \, dH(r,\lambda), \tag{10.4}$$

with $\alpha \in [0,1]$ and H a distribution function on $(0,2]\times(0,\infty)$, is infinitely divisible.

It follows that for $\lambda > 0$ and a distribution function H on $(0,\infty)$ the *power mixture*

$$\phi(u) = \int_{(0,\infty)} \Big(\frac{\lambda^2}{\lambda^2+u^2}\Big)^r \, \mathrm{d}H(r) \tag{10.5}$$

is infinitely divisible if the support $S(H)$ of H is restricted to $(0,2]$. Of course, another sufficient condition is infinite divisibility of H; cf. Proposition 2.1.

We take $r = 1$ in (10.1) and thus consider *mixtures of Laplace distributions.* As we saw above, these mixtures are infinitely divisible (this also immediately follows from Theorem 3.3), and we now want to characterize them among the infinitely divisible distributions. This can be done by using the following *canonical representation*, which by Proposition 10.1 easily follows from Theorem 3.5; cf. the proof of Theorem 7.6.

Theorem 10.6 (Canonical representation). *A function ϕ on $\mathbb{R}$ is the characteristic function of a mixture of Laplace distributions iff ϕ has the form*

$$\phi(u) = \exp\Big[-\int_0^\infty \Big(\frac{1}{\lambda^2} - \frac{1}{\lambda^2+u^2}\Big)\, w(\lambda)\, 2\lambda \, d\lambda\Big] \qquad [\,u \in \mathbb{R}\,], \tag{10.6}$$

where w is a measurable function on $(0,\infty)$ satisfying $0 \le w \le 1$ and, necessarily,

$$\int_0^1 \frac{1}{\lambda}\, w(\lambda)\, d\lambda < \infty. \tag{10.7}$$

Corollary 10.7. *If ϕ is the characteristic function of a mixture of Laplace distributions, then so is ϕ^a for any $a \in [0,1]$. Consequently, the convolution roots of a mixture of Laplace distributions are mixtures of Laplace distributions as well.*

We now come to the announced *characterization*, and recall that the infinitely divisible characteristic functions ϕ are uniquely determined by the canonical triple (a, σ^2, M) in the Lévy representation in Theorem IV.4.4.

Theorem 10.8. *A probability distribution on $\mathbb{R}$ is a mixture of Laplace distributions iff it is infinitely divisible having a Lévy canonical triple $(0, 0, M)$ such that M has a density m of the form*

$$m(x) = \int_0^\infty e^{-\lambda|x|}\, w(\lambda)\, d\lambda \qquad [\, x \neq 0\,], \tag{10.8}$$

where w is a measurable function on $(0, \infty)$ satisfying $0 \le w \le 1$ and, necessarily, condition (10.7).

Proof. Use Theorem 10.6. First, let ϕ be a characteristic function of the form (10.6) with w as indicated. Then using the fact that $u \mapsto \lambda^2/(\lambda^2 + u^2)$ is the characteristic function of the Laplace (λ) distribution, by Fubini's theorem we see that ϕ can be written as

$$\phi(u) = \exp\Big[\int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{iux} - 1)\, m(x)\, \mathrm{d}x\Big] \tag{10.9}$$

with m given by (10.8). Now observe that because of condition (10.7) and the fact that $0 \le w \le 1$, the function m satisfies

$$\int_x^\infty m(t)\, \mathrm{d}t < \infty \ \text{ for } x > 0, \qquad \int_0^\infty \frac{x}{1 + x^2}\, m(x)\, \mathrm{d}x < \infty. \tag{10.10}$$

From Theorem IV.4.4 it follows that ϕ has canonical triple (a, σ^2, M) as indicated. The converse statement follows by inserting the expression for m in the Lévy representation for ϕ. ◻

Note that m in (10.8) satisfies $\int_0^1 x\, m(x)\, \mathrm{d}x < \infty$. Therefore, by Corollary IV.4.16 it follows that the mixtures of Laplace distributions can be decomposed in a way that is well known for the Laplace distribution.

Corollary 10.9. *If the distribution of a random variable X is a mixture of Laplace distributions, then X can be written as*

$$X \stackrel{\mathrm{d}}{=} Y - Y', \quad \textit{with } Y \textit{ infinitely divisible and } \mathbb{R}_+\textit{-valued},$$

where Y' is independent of Y with $Y' \stackrel{\mathrm{d}}{=} Y$.

We proceed with considering *generalized* mixtures of Laplace distributions in the sense that also 'negative probabilities' in the mixing function are allowed.

Theorem 10.10. *Let H be a right-continuous bounded function satisfying $H(\lambda) = 0$ for $\lambda \le 0$ and such that for some $\lambda_0 > 0$*

$$(10.11) \qquad \begin{cases} H \text{ is nondecreasing on } [0, \lambda_0), \\ H \text{ is nonincreasing on } [\lambda_0, \infty) \text{ with } \lim_{\lambda\to\infty} H(\lambda) = 1. \end{cases}$$

With this H, let the function f on $\mathbb{R} \setminus \{0\}$ and its Ft ϕ be defined by

$$(10.12) \qquad f(x) = \int_{(0,\infty)} \tfrac{1}{2}\lambda\, e^{-\lambda|x|}\, dH(\lambda), \quad \phi(u) = \int_{(0,\infty)} \frac{\lambda^2}{\lambda^2 + u^2}\, dH(\lambda).$$

Then f can be viewed as a probability density iff $\varepsilon := \int_{(0,\infty)} \lambda\, dH(\lambda) \ge 0$. Moreover, if $\delta := \int_{(0,\infty)} \lambda^2\, dH(\lambda) \ge 0$, then $\varepsilon \ge 0$ and f is infinitely divisible; in fact, ϕ can then be written as

$$(10.13) \qquad \phi(u) = \phi_1(u)\, \phi_2(u),$$

where ϕ_1 is the characteristic function of the Laplace$\,(\lambda_0)$ distribution and ϕ_2 is the characteristic function of a mixture of Laplace distributions.

Proof. Note that $\int_{\mathbb{R}} f(x)\, \mathrm{d}x = \int_{(0,\infty)} \mathrm{d}H(\lambda) = 1$ and that $f(0+) = \frac{1}{2}\varepsilon$ and $\varepsilon \in [-\infty, \infty)$. So, if $\varepsilon < 0$, then f is not a density. On the other hand, if $\varepsilon \ge 0$, then by writing f on $\mathbb{R} \setminus \{0\}$ as

$$f(x) = \int_{(0,\infty)} \tfrac{1}{2}\lambda \left(\mathrm{e}^{-\lambda|x|} - \mathrm{e}^{-\lambda_0|x|}\right) \mathrm{d}H(\lambda) + \tfrac{1}{2}\varepsilon\, \mathrm{e}^{-\lambda_0|x|},$$

we see that $f(x) \ge 0$ for all x. As in Proposition 10.1 the function ϕ in (10.12) can be represented as

$$\phi(u) = \pi(u^2), \quad \text{with } \pi(s) = \int_{(0,\infty)} \frac{\lambda}{\lambda + s}\, \mathrm{d}G(\lambda),$$

where $G(\lambda) := H(\sqrt{\lambda})$ for $\lambda \ge 0$; note that G satisfies condition (3.27) with λ_0 replaced by λ_0^2. Let further $\delta \ge 0$. Then $\int_{(0,\infty)} \lambda\, \mathrm{d}G(\lambda) \ge 0$, so according to Theorem 3.10 the function π is an infinitely divisible pLSt. By Proposition 9.2 it follows that ($\varepsilon > 0$ and) ϕ is an infinitely divisible characteristic function. Also, factorization (3.29) for π yields (10.13) with

$\phi_i(u) := \pi_i(u^2)$ for $i = 1, 2$; ϕ_1 and ϕ_2 are characteristic functions of types as indicated in the theorem because of Theorems 9.5 and 9.7. □

By writing $e^{-\lambda|x|}$ as $\int_\lambda^\infty |x|\, e^{-t|x|}\, dt$ and changing the order of integration we see that the function f in (10.12) can also be written as

$$(10.14) \qquad f(x) = |x| \int_{(0,\infty)} e^{-\lambda|x|}\, C(\lambda)\, d\lambda \qquad [x \neq 0],$$

where $C(\lambda) := \frac{1}{2} \int_{(0,\lambda]} y\, dH(y)$ for $\lambda > 0$. If $\varepsilon \geq 0$, then C is *nonnegative*, so from (10.14) it is seen once more that $f(x) \geq 0$ for all x. Moreover, it follows that Theorem 10.10, like Theorem 10.4, gives examples of densities f of the attractive form (10.3) that are infinitely divisible.

11. Generalized sym-gamma convolutions

A probability distribution on $\mathbb{R}$ is said to be a *generalized sym-gamma convolution* if it is the weak limit of finite convolutions of sym-gamma distributions; the corresponding characteristic function ϕ can be obtained as in (9.11). We reformulate Corollary 9.10 in terms of transforms.

Proposition 11.1. *A function ϕ on $\mathbb{R}$ is the characteristic function of a generalized sym-gamma convolution iff ϕ has the form*

$$(11.1) \qquad \phi(u) = \pi(u^2) \qquad [u \in \mathbb{R}],$$

where π is the pLSt of a generalized gamma convolution.

This *representation* result will be used to translate results for generalized gamma convolutions into results in the present context. We start with doing so for Theorem 5.1 and Proposition 5.2; cf. Theorem 8.2 and Proposition 8.3.

Theorem 11.2 (Canonical representation). *A function ϕ on $\mathbb{R}$ is the characteristic function of a generalized sym-gamma convolution iff it has the form*

$$(11.2) \qquad \phi(u) = \exp\Big[-au^2 + \int_{(0,\infty)} \log \frac{\lambda^2}{\lambda^2 + u^2}\, dV(\lambda)\Big] \qquad [u \in \mathbb{R}],$$

where $a \geq 0$ and V is an LSt-able function with $V(0) = 0$ and, necessarily,

$$(11.3) \qquad \int_{(0,1]} \log \frac{1}{\lambda}\, dV(\lambda) < \infty, \quad \int_{(1,\infty)} \frac{1}{\lambda^2}\, dV(\lambda) < \infty.$$

In proving this theorem one observes that the *Thorin function* V is obtained from the Thorin function U of π in (11.1) by

$$(11.4) \qquad V(\lambda) = U(\lambda^2) \qquad [\lambda \in \mathbb{R}].$$

Of course, the canonical pair (a, V) of ϕ in (11.2) is uniquely determined by ϕ. If $a > 0$, then ϕ has a *normal* factor; we will often restrict ourselves to the case where $a = 0$.

Proposition 11.3. *If a sequence (ϕ_n) of characteristic functions of generalized sym-gamma convolutions converges (pointwise) to a characteristic function ϕ, then ϕ is the characteristic function of a generalized sym-gamma convolution.*

Clearly, the generalized sym-gamma convolutions are *infinitely divisible.* We will now *characterize* them among the infinitely divisible distributions using the Lévy canonical triple (a, σ^2, M); cf. Theorem IV.4.4.

Theorem 11.4. *A probability distribution on $\mathbb{R}$ is a generalized sym-gamma convolution without normal component iff it is infinitely divisible having a Lévy canonical triple $(0, 0, M)$ where M has an even density m such that $x \mapsto x\, m(x)$ is completely monotone on $(0, \infty)$. In this case m can be expressed in terms of the LSt $\widehat{V}$ of the Thorin function V of the distribution by*

$$(11.5) \qquad m(x) = \frac{1}{|x|}\, \widehat{V}(|x|) \qquad [x \neq 0].$$

Proof. Use Theorem 11.2. First, let ϕ be a characteristic function of the form (11.2) with $a = 0$ and with V as indicated. Use the fact (see Example IV.4.8) that the Laplace (λ) distribution has canonical triple $(0, 0, M_\lambda)$ where M_λ has density m_λ given by $m_\lambda(x) = \mathrm{e}^{-\lambda|x|}/|x|$ for $x \neq 0$, so

$$\log \frac{\lambda^2}{\lambda^2 + u^2} = \int_{\mathbb{R}\setminus\{0\}} \Big(\mathrm{e}^{iux} - 1 - \frac{iux}{1 + x^2}\Big)\, m_\lambda(x)\, \mathrm{d}x.$$

Inserting this in (11.2) and using Fubini's theorem, we see that ϕ can be written as

$$\phi(u) = \exp\Big[\int_{\mathbb{R}\setminus\{0\}} \Big(\mathrm{e}^{iux} - 1 - \frac{iux}{1 + x^2}\Big)\, m(x)\, \mathrm{d}x\Big]$$

with m given by (11.5). It follows that ϕ has canonical triple (a, σ^2, M) as indicated. The converse statement follows by noting that by Bernstein's theorem m can be represented as in (11.5), and then inserting this expression for m in the Lévy representation for ϕ. □

This theorem can be used to show that the generalized sym-gamma convolutions interpolate between the *symmetric stable* and the *self-decomposable* distributions on $\mathbb{R}$; just apply Theorems V.6.12 and V.7.9. Alternatively, one might use Proposition 11.1 together with Propositions 5.5 and 5.7, (9.5) and Corollary 9.4. Thus the following two propositions hold; the final statement in the first one is a consequence of Theorems V.6.14 and V.6.23.

Proposition 11.5. *A generalized sym-gamma convolution is self-decomposable. Moreover, it is absolutely continuous and unimodal.*

Proposition 11.6. *A symmetric stable distribution, with characteristic function ϕ of the form*

$$\phi(u) = \exp\left[-\lambda\, |u|^{\gamma}\right], \tag{11.6}$$

where $\lambda > 0$ and $\gamma \in (0, 2\,]$, is a generalized sym-gamma convolution.

Next we establish a connection with the *mixtures of sym-gamma distributions* from the preceding section. By Theorem 11.4 a generalized sym-gamma convolution with canonical pair (a, V) has a Lévy density m that satisfies (11.5). This relation can be rewritten as

$$m(x) = \int_0^\infty e^{-\lambda|x|}\, V(\lambda)\, d\lambda \qquad [\, x \neq 0\,], \tag{11.7}$$

and, conversely, if m is of the form (11.7) with V an LSt-able function, then m satisfies (11.5). Now, compare (11.7) with (10.8) where the Lévy density m of a mixture of Laplace distributions is given in terms of the *second canonical density* w satisfying $0 \leq w \leq 1$. Then one is led to the following proposition and the first corollary; the second corollary is easily obtained from the canonical representation in Theorem 11.2.

Proposition 11.7. *A generalized sym-gamma convolution with canonical pair (a, V) is a mixture of Laplace distributions iff $a = 0$ and V satisfies $\lim_{\lambda\to\infty} V(\lambda) \leq 1$. In this case for the second canonical density w of the mixture one may take $w = V$.*

Corollary 11.8. *A mixture of Laplace distributions with second canonical density w is a generalized sym-gamma convolution iff w can be chosen to be nondecreasing.*

Corollary 11.9. *For $r > 0$ a generalized sym-gamma convolution with canonical pair (a, V) is the r-fold convolution of a mixture of Laplace distributions iff $a = 0$ and V satisfies* $\lim_{\lambda\to\infty} V(\lambda) \le r$.

The generalized sym-gamma convolutions in this corollary, with r fixed, can also be viewed as *mixtures of sym-gamma (r) distributions.* This easily follows by translating Theorem 5.12 in the usual way: Apply Corollary 9.8, use Proposition 11.1 and observe that because of (11.4) the Thorin functions U of π and V of ϕ satisfy $\lim_{\lambda\to\infty} U(\lambda) = \lim_{\lambda\to\infty} V(\lambda)$. This leads to the following result; its corollary is a consequence of Corollary 11.9.

Theorem 11.10. *For $r > 0$ a generalized sym-gamma convolution with canonical pair (a, V) is a mixture of sym-gamma (r) distributions iff $a = 0$ and V satisfies* $\lim_{\lambda\to\infty} V(\lambda) \le r$.

Corollary 11.11. *For $r > 0$ a generalized sym-gamma convolution is a mixture of sym-gamma (r) distributions iff it is the r-fold convolution of a mixture of Laplace distributions.*

We are interested in determining classes of explicit densities f on $\mathbb{R}$ that are generalized sym-gamma convolutions, and hence are self-decomposable and infinitely divisible. Of course, translating Proposition 11.1 in terms of densities yields the following result.

Proposition 11.12. *A probability density f on $\mathbb{R}$ of the form*

$$f(x) = \int_0^\infty \frac{1}{2\sqrt{\pi t}} \exp\left[-x^2/(4t)\right] g(t)\, dt \qquad [x \neq 0] \tag{11.8}$$

is a generalized sym-gamma convolution if (and only if) g is a density of a generalized gamma convolution.

The main results of Section 5, i.e., Theorems 5.18, 5.24 and 5.30, now give important criteria in terms of *hyperbolic complete monotonicity*: If the density g itself, or the Lt π of g, or the distribution function G corresponding to g, is hyperbolically completely monotone, then f in (11.8) is the density of a generalized sym-gamma convolution. Such a density f can, however, seldom be obtained explicitly. Another procedure that sometimes leads to explicit densities of generalized sym-gamma convolutions, will be derived from the following analogue of Proposition 8.14.

Proposition 11.13. *Let Z and Y be independent random variables with Z nonnegative. Then the distribution of the product $\sqrt{Z}\,Y$ is a generalized sym-gamma convolution in each of the following cases:*

(i) *Z has a hyperbolically completely monotone density and the distribution of Y is a generalized sym-gamma convolution;*

(ii) *The distribution of Z is a generalized gamma convolution and Y satisfies $Y \stackrel{\mathrm{d}}{=} B(T)$ with T a random variable with a hyperbolically completely monotone density.*

PROOF. By Corollary 9.10 in case (i) we have $Y \stackrel{\mathrm{d}}{=} B(T)$ where the distribution of T is a generalized gamma convolution. Hence by (9.3)

$$\sqrt{Z}\,Y \stackrel{\mathrm{d}}{=} B(ZT), \quad \text{with } Z \text{ and } T \text{ independent,}$$

which also holds in case (ii), of course. Now apply Proposition 5.27; the distribution of ZT is a generalized gamma convolution, so $\sqrt{Z}\,Y$ has a generalized sym-gamma convolution as its distribution. □

If Z and Y have absolutely continuous distributions with densities g and f_Y, respectively, then $\sqrt{Z}\,Y$ has density f given by

$$(11.9)\qquad f(x) = \int_0^\infty \frac{1}{\sqrt{t}}\, f_Y(x/\sqrt{t})\, g(t)\,\mathrm{d}t \qquad [\,x \in \mathbb{R}\,].$$

So, by taking Y in part (i) of the proposition normal $(0,2)$ we get the special case of Proposition 11.12 with g hyperbolically completely monotone. Further, by taking Y in part (ii) standard sym-gamma (r) with $r \in \{1,2\}$, we obtain the following counterpart to Proposition 11.12.

Proposition 11.14. *Let g be a density of a generalized gamma convolution. Then each of the following two densities f_1 and f_2 corresponds to a generalized sym-gamma convolution:*

$$(11.10)\qquad f_1(x) = \int_0^\infty \frac{1}{2\sqrt{t}}\, \exp\big[-|x|/\sqrt{t}\,\big]\, g(t)\, dt \qquad [\,x \neq 0\,],$$

$$(11.11)\qquad f_2(x) = \int_0^\infty \frac{1}{4\sqrt{t}} \Big(1 + \frac{|x|}{\sqrt{t}}\Big) \exp\big[-|x|/\sqrt{t}\,\big]\, g(t)\, dt \qquad [\,x \neq 0\,].$$

Note that f_1 is of the form (9.9) and f_2 of the form (10.2), so for $r \in \{1, 2\}$ Proposition 11.14 gives a sufficient condition for a mixture of sym-gamma (r) distributions to be a generalized sym-gamma convolution.

In some special cases Propositions 11.12 and 11.14 can be used to determine explicit densities f that correspond to generalized sym-gamma convolutions. We now give one such example because it leads to the solution of a classical problem in infinite divisibility that was open for a long time; see Notes. Let $T \stackrel{d}{=} 1/Y$ with Y *gamma*$(r, \frac{1}{4})$ for some $r > 0$; then T has density g given by

$$g(t) = \frac{4}{\Gamma(r)} \left(\frac{1}{4t}\right)^{r+1} \exp\left[-1/(4t)\right] \qquad [\, t > 0\,].$$

Note that g is hyperbolically completely monotone (see, e.g., Theorem 5.22), so from Theorem 5.18 it follows that g is the density of a generalized gamma convolution. Now, use this g in (11.8), so consider the density f of $B(T)$ or, by (9.3), of $B/\sqrt{Y}$; it is easily computed and recognized as a density of the *student* distribution (with k degrees of freedom when $r = \frac{1}{2}k$ with $k \in \mathbb{N}$). Thus Proposition 11.12 leads to the following result.

Theorem 11.15. *For $r > 0$ the student (r) distribution with density f given by*

$$(11.12) \qquad f(x) = \frac{1}{B(r, \frac{1}{2})} \left(\frac{1}{1 + x^2}\right)^{r+\frac{1}{2}} \qquad [\, x \in \mathbb{R}\,],$$

is self-decomposable and hence infinitely divisible.

Note that by Proposition 9.2 for the infinite divisibility of f it is only necessary to know that T above is infinitely divisible; cf. Example IV.11.6. For further examples we refer to Section 12. In dealing with concrete examples it is sometimes convenient to make use of the following *closure property*; it is obtained by combining Propositions 5.25 and 11.1.

Proposition 11.16.

(i) *If π is the LSt of a generalized gamma convolution, then $u \mapsto \pi\big(|u|^\gamma\big)$ is the characteristic function of a generalized sym-gamma convolution for every $\gamma \in (0, 2\,]$.*

(ii) *If ϕ is the characteristic function of a generalized sym-gamma convolution, then so is $u \mapsto \phi\big(|u|^\gamma\big)$ for every $\gamma \in (0, 1\,]$.*

Part (ii) of this proposition shows, for instance, that the generalized sym-gamma convolutions that are *stable*, can be obtained in a simple way from the normal distribution.

12. Examples

In the preceding sections we have obtained rather special results on mixtures and convolutions of specific probability distributions: the *gamma* distributions on $\mathbb{R}_+$ and (via mixtures of *Poisson* and *normal* distributions) their counterparts on $\mathbb{Z}_+$ and on $\mathbb{R}$, the *negative-binomial* and the *sym-gamma* distributions. Therefore, it is no surprise that now many examples of explicit *infinitely divisible* distributions can be given. Of course, we present examples and counter-examples that illustrate the scope of the results given, but further we concentrate on examples of distributions the infinite divisibility of which can not easily be shown by the methods in the earlier chapters. Examples of this type are the *log-normal* and the *student* distribution, which were already given because of their historical relevance.

Example 12.1. Let $r > 2$ be fixed, and consider for $c > 0$ the following pLSt π_c of a *mixture of gamma* (r) *distributions*:

$$\pi_c(s) = \frac{c}{1+c} + \frac{1}{1+c}\Big(\frac{1}{1+s}\Big)^r.$$

Set $\phi(u) := (1 - iu)^{-r}$ for $u \in \mathbb{R}$. We will choose $c > 0$ such that $\phi(u) = -c$ for some $u \in \mathbb{R}$; the corresponding pLSt π_c then has a zero on the imaginary axis, and hence can*not* be infinitely divisible because of Theorem III.2.8. Now $\big|\phi(u)\big| = (1 + u^2)^{-r/2}$ and $\arg \phi(u) = r \arctan u$, so we need to solve the following pair of equations for $(c, u) \in (0, \infty) \times \mathbb{R}$:

$$(1 + u^2)^{-r/2} = c, \quad \arctan u = \pi/r.$$

Since $r > 2$, this can be done; we find $c = \{\cos(\pi/r)\}^r$ and $u = \tan(\pi/r)$. Conclusion: *For $r > 2$ not all mixtures of gamma (r) distributions are infinitely divisible.* Compare with Theorem 4.5. From Proposition III.2.2 it follows that for $r > 2$ not all mixtures of gamma (r) *densities* are infinitely divisible. □

Example 12.2. Let T be an $\mathbb{R}_+$-valued random variable with density f given by the following generalized *mixture of exponential densities*:

$$f(t) = 2\,\mathrm{e}^{-t} - 6\,\mathrm{e}^{-3t} + 5\,\mathrm{e}^{-5t} \qquad [\,t > 0\,];$$

since f can be written as $f(t) = 5\mathrm{e}^{-t}\,\big\{(\mathrm{e}^{-2t} - \frac{3}{5})^2 + \frac{1}{25}\big\}$, f is indeed non-negative. The pLSt π of T is given by

$$\pi(s) = \frac{2}{1+s} - \frac{6}{3+s} + \frac{5}{5+s} = \frac{15+s^2}{(1+s)\,(3+s)\,(5+s)}.$$

Since $\pi(z_0) = 0$ for $z_0 = i\sqrt{15}$, it follows from Theorem III.2.8 that T is *not* infinitely divisible. Conclusion: *Finite generalized mixtures of exponential distributions with two changes of sign in the sequence of mixing probabilities are not all infinitely divisible.*

Next, for $\theta > 0$ consider the random variable $N(\theta T)$ from Section 6; its distribution is a *mixture of Poisson distributions.* The pgf P_θ of $N(\theta T)$ is given by $P_\theta(z) = \pi\big(\theta\{1-z\}\big)$ which in our case yields

$$P_\theta(z) = \frac{\big(i\sqrt{15} + \theta\{1-z\}\big)\,\big(-i\sqrt{15} + \theta\{1-z\}\big)}{\big(1+\theta\{1-z\}\big)\,\big(3+\theta\{1-z\}\big)\,\big(5+\theta\{1-z\}\big)},$$

so for the R-function R_θ of P_θ, with $R_\theta(z) = \big(\log P_\theta(z)\big)'$, we have

$$R_\theta(z) = \sum_{j=1,3,5} \frac{\theta}{j+\theta\{1-z\}} - \sum_{j=1,2} \frac{\theta}{(-1)^j\,i\sqrt{15} + \theta\{1-z\}}.$$

It follows that R_θ has a power-series expansion with coefficients $r_k(\theta)$ for $k \in \mathbb{Z}_+$ given by

$$r_k(\theta) = \sum_{j=1,3,5} \Big(\frac{\theta}{\theta+j}\Big)^{k+1} - \sum_{j=1,2} \Big(\frac{\theta}{\theta+(-1)^j\,i\sqrt{15}}\Big)^{k+1} =$$

$$= \sum_{j=1,3,5} \Big(\frac{\theta}{\theta+j}\Big)^{k+1} - 2\,m_\theta^{k+1}\cos\,(k+1)\alpha_\theta,$$

where $m_\theta := \theta/\sqrt{\theta^2+15}$ and $\alpha_\theta := \arctan(\sqrt{15}/\theta)$. Now observe that for $\theta \le 1$ we have $\theta/(\theta+3) \ge m_\theta$, which suffices to make $r_k(\theta) \ge 0$ for all k, so R_θ absolutely monotone. Thus, for $\theta \in (0,1\,]$ the random variable $N(\theta T)$ is *infinitely divisible.* Conclusion: *A random variable of the form $N(T)$ may be infinitely divisible though T is not infinitely divisible.* Compare Theorems 6.3 and 6.4. ◫

Example 12.3. Let T be a $\mathbb{Z}_+$-valued random variable with pgf P given by

$$P(z) = \exp\left[-\{1 - Q(z)\}\right],$$

where Q is the gf of the sequence $(q_k)_{k=1,..,5}$ with $q_1 = q_2 = q_4 = q_5 = \alpha$ and $q_3 = -\beta$ where $\alpha := 0.26$ and $\beta := 0.04$. Indeed, $P(1) = 1$ and, since $Q + \frac{1}{2}Q^2$, Q^2 and Q^3 can be seen to be absolutely monotone, P is absolutely monotone as well. The R-function of P, which is given by $R = Q'$, is not absolutely monotone, so P is *not* infinitely divisible. Now, consider the random variable $B(T)$ from Section 9; its distribution is a *mixture of zero-mean normal distributions.* The characteristic function ϕ of $B(T)$ is given by

$$\phi(u) = P\big(\exp[-u^2]\big) = \exp\Big[\sum_{k=1}^{5} q_k \left(\exp[-ku^2] - 1\right)\Big].$$

Since here $u \mapsto \exp[-ku^2]$ can be viewed as the characteristic function of the normal $(0, 2k)$ distribution, ϕ can be written as

$$\phi(u) = \exp\Big[\int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{iux} - 1)\, m(x)\, \mathrm{d}x\Big],$$

where the function m is given by

$$m(x) = \sum_{k=1}^{5} q_k \frac{1}{2\sqrt{\pi k}} \exp\left[-x^2/(4k)\right] \qquad [x \neq 0].$$

Since $\alpha/\sqrt{5} > \beta/\sqrt{3}$, the function m is nonnegative; hence ϕ is *infinitely divisible* with Lévy density m. Conclusion: *A random variable of the form $B(T)$ may be infinitely divisible though T is not infinitely divisible.* Compare with Proposition 9.2. ◫

Example 12.4. Consider the *half-Cauchy* distribution, i.e., the distribution on $(0, \infty)$ with density f given by

$$f(x) = \frac{2}{\pi}\, \frac{1}{1 + x^2} \qquad [x > 0].$$

Is it infinitely divisible? Clearly, f is *not* completely monotone. Further, if f were hyperbolically completely monotone, then so would be f_n with $f_n(x) := (1 + x^2/n)^{-n}$, and then also $f_\infty := \lim_{n\to\infty} f_n$ for which $f_\infty(x) = \exp[-x^2]$; cf. Proposition 5.17. But since the *half-normal* distribution is not even infinitely divisible (it has too thin a tail; cf. Example III.11.2), from Theorem 5.18 we conclude that f is *not* hyperbolically completely monotone. Also, by Corollary 5.9 f does *not* correspond to a generalized gamma convolution; f is monotone but not completely monotone. Now, use the fact that $u \mapsto (1+u^2)^{-1}$ is the characteristic function of the standard Laplace distribution; then one sees that f can be written as

$$f(x) = \frac{2}{\pi}\Big(1 - \frac{1}{1+(1/x)^2}\Big) = \frac{2}{\pi}\Big(1 - \int_0^\infty \mathrm{e}^{-t} \cos(t/x)\,\mathrm{d}t\Big) =$$

$$= \frac{2}{\pi}\int_0^\infty x\,\mathrm{e}^{-\lambda x}\,(1 - \cos\lambda)\,\mathrm{d}\lambda = \int_0^\infty \lambda^2 x\,\mathrm{e}^{-\lambda x}\,g(\lambda)\,\mathrm{d}\lambda,$$

where g is the probability density given by $g(\lambda) = (2/\pi)\,(1-\cos\lambda)/\lambda^2$ for $\lambda > 0$. Application of Theorem 4.5 leads to the following conclusion: *The half-Cauchy distribution is a mixture of gamma (2) distributions, and is hence infinitely divisible.* It can even be shown to be *self-decomposable*; see Notes. ◫

Example 12.5. Consider the *half-Gumbel* distribution, i.e., the distribution on $(0,\infty)$ with density f given by

$$f(x) = \frac{\mathrm{e}}{\mathrm{e}-1}\,\exp\big[-(x + \mathrm{e}^{-x})\big] \qquad [\,x > 0\,].$$

In Example III.11.6 we observed that f is *not* completely monotone. It follows that f is *not* hyperbolically completely monotone and does *not* correspond to a generalized gamma convolution, because $f(0+) > 0$; cf. Corollary 5.9. Now, putting $c := \mathrm{e}/(\mathrm{e}-1)$ and using the dominated convergence theorem, we can write

$$f(x) = c\sum_{k=0}^\infty \frac{(-1)^k}{k!}\,\mathrm{e}^{-(k+1)x} = c\sum_{k=0}^\infty \frac{(-1)^k}{k!}\int_{k+1}^\infty x\,\mathrm{e}^{-\lambda x}\,\mathrm{d}\lambda =$$

$$= c\int_1^\infty x\,\mathrm{e}^{-\lambda x}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\,1_{[k+1,\infty)}(\lambda)\,\mathrm{d}\lambda =$$

$$= \int_1^\infty \lambda^2 x\,\mathrm{e}^{-\lambda x}\,g(\lambda)\,\mathrm{d}\lambda,$$

where g is the probability density on $(1,\infty)$ given by

$$g(\lambda) = \frac{\mathrm{e}}{\mathrm{e}-1}\,\frac{1}{\lambda^2}\sum_{k=0}^{\lfloor\lambda\rfloor-1} \frac{(-1)^k}{k!} \qquad [\,\lambda > 1\,].$$

By Theorem 4.5 we come to the following conclusion: *The half-Gumbel distribution is a mixture of gamma (2) distributions, and is hence infinitely divisible.* □□

Example 12.6. Let f be a probability density on $(0,\infty)$ of the form

$$f(x) = c\,x\,\psi(x) \qquad [\,x > 0\,]$$

with ψ positive, where $c > 0$ is a norming constant. From Corollary 4.6 we know that f is *infinitely divisible* if ψ is *completely monotone*. What if, more generally, ψ is *log-convex*? Take $\psi(x) = \exp\big[-h(x)\big]$ with h given by

$$h(x) = \begin{cases} x & , \text{ if } 0 < x \le 1, \\ \frac{1}{2}\,(x+1) & , \text{ if } x > 1; \end{cases}$$

then h is concave, so ψ is log-convex. Consider the corresponding density f; if it were infinitely divisible, then by Theorem III.4.17 it would satisfy the functional equation

$$x\,f(x) = \int_{[0,x)} f(x-u)\,\mathrm{d}K(u) \qquad [\,x > 0\,],$$

for some nondecreasing function K. It can be shown, however (see Notes), that f satisfies this equation with a function K that has a *downward* jump of size $\frac{1}{4}\mathrm{e}^{-2}$ at $u = 2$. This means that f is *not* infinitely divisible. □□

Example 12.7. Consider the pLSt π of a generalized *mixture of two gamma (2) distributions.* Then, for some λ_1, λ_2 with $0 < \lambda_1 < \lambda_2$,

$$\pi(s) = \alpha\Big(\frac{\lambda_1}{\lambda_1 + s}\Big)^2 + (1-\alpha)\Big(\frac{\lambda_2}{\lambda_2 + s}\Big)^2,$$

where α may be any positive number satisfying $\alpha\,\lambda_1^2 + (1-\alpha)\,\lambda_2^2 \ge 0$, i.e., $\alpha \le 1/\{1 - (\lambda_1/\lambda_2)^2\}$; cf. (3.22). When $\alpha \le 1$, π is infinitely divisible by Theorem 4.5. So take $\alpha > 1$. By choosing an appropriate scale, we may further take $\lambda_1 = 1$; then $\lambda := \lambda_2 > 1$. Putting $\alpha = a^2$ and $(\alpha - 1)\,\lambda^2 = b^2$, we see that π takes the form

$$\pi(s) = \frac{\big(a\lambda - b + (a-b)s\big)\,\big(a\lambda + b + (a+b)s\big)}{(1+s)^2\,(\lambda+s)^2},$$

where $a > 1$ and $b > 0$ may be any numbers satisfying $b^2 = (a^2 - 1)\,\lambda^2$ and $b \le a$. Now we can rewrite π as

$$\pi(s) = \Big(\frac{1}{1+s}\Big)^2 \Big(\frac{\lambda}{\lambda+s}\Big)^2 \Big/ \Big\{\frac{\mu}{\mu+s}\,\frac{\nu}{\nu+s}\Big\},$$

where $\mu := (a\lambda - b)/(a-b)$ and $\nu := (a\lambda + b)/(a+b)$; here $\mu := \infty$ if $b = a$, and $\mu/(\mu+s) := 1$ if $\mu = \infty$. Using (3.5) one sees that π can be put in the form (3.4) with k given by

$$k(x) = 2\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{-\lambda x} - \mathrm{e}^{-\mu x} - \mathrm{e}^{-\nu x} \qquad [\,x > 0\,],$$

where $\mathrm{e}^{-\mu x} := 0$ if $\mu = \infty$. As $\lambda > 1$, we have $\mu > 1$ and $\nu > 1$, so $k(x) \ge 0$ for all x. Conclusion: *Any distribution that is a generalized mixture of two gamma (2) distributions, is infinitely divisible.* ◫

Example 12.8. Let $X \stackrel{\mathrm{d}}{=} Y^{1/\alpha}$ with $\alpha \ne 0$ and Y standard gamma (r) distributed. Then X has density f given by

$$f(x) = \frac{|\alpha|}{\Gamma(r)}\, x^{\alpha r - 1} \exp\left[-x^{\alpha}\right] \qquad [\,x > 0\,].$$

The corresponding distribution is known as the *generalized-gamma* distribution. When $\alpha > 0$ and $r = 1$ one gets the *Weibull* distribution. For $\alpha = -1$ the *inverse-gamma* distribution is obtained; the density f then reduces to

$$f(x) = \frac{1}{\Gamma(r)}\, x^{-r-1} \exp\left[-1/x\right] \qquad [\,x > 0\,],$$

which for $r = \frac{1}{2}$ is recognized as a *stable* density with exponent $\gamma = \frac{1}{2}$ (cf. Example V.9.5). Now, by Proposition 5.19 (i), or Theorem 5.22, in case $|\alpha| \le 1$ the density f is *hyperbolically completely monotone*. Thus, Theorem 5.18 leads to the following conclusion: *The generalized-gamma distribution with $|\alpha| \le 1$, and hence the inverse-gamma distribution, is self-decomposable and infinitely divisible.* When $0 < \alpha \le 1$ and $r \le 1/\alpha$, f is *completely monotone* (and hence infinitely divisible); cf. Example III.11.3. The generalized-gamma distribution with $\alpha > 1$ is, however, *not* infinitely divisible; cf. Example III.11.2. It is unknown whether X is infinitely divisible for $\alpha < -1$. ◫

Example 12.9. Let $X \stackrel{d}{=} Y/Z$ with Y and Z independent, Y standard gamma (r) and Z standard gamma (s). Then X has density f given by

$$f(x) = \frac{1}{B(r,s)}\, x^{r-1} \Big(\frac{1}{1+x}\Big)^{r+s} \qquad [\, x > 0\,].$$

The corresponding distribution is known as the *beta* distribution *of the second kind.* For $r = 1$ one gets the *Pareto* distribution. By Proposition 5.19 (ii), or Theorem 5.22, the density f is *hyperbolically completely monotone.* Thus, Theorem 5.18 leads to the following conclusion: *The beta distribution of the second kind is self-decomposable and infinitely divisible.* When $r \le 1$, f is *completely monotone* (and hence infinitely divisible). □

Example 12.10. For $\varepsilon \in (0,1)$ let the function π_ε on $\mathbb{R}_+$ be given by

$$\pi_\varepsilon(s) = \frac{s^\varepsilon - 1}{s - 1} \qquad [\, s \ge 0\,],$$

where $\pi_\varepsilon(1) := \lim_{s\to 1} \pi_\varepsilon(s) = \varepsilon$. For $\varepsilon = \frac{1}{2}$ we have $\pi_{\frac{1}{2}}(s) = 1/(1+\sqrt{s})$, which we know to be an infinitely divisible pLSt; cf. Example III.11.11. To show that π_ε can be viewed as a pLSt for every $\varepsilon \in (0,1)$, we compute a density f_ε of the random variable $X\,Y/Z$, where X, Y and Z are independent, X is standard exponential, Y gamma $(1-\varepsilon)$ and Z gamma (ε); using f in Example 12.9 we find

$$f_\varepsilon(x) = c_\varepsilon \int_0^\infty \mathrm{e}^{-\lambda x}\, \frac{\lambda^\varepsilon}{\lambda + 1}\, \mathrm{d}\lambda \qquad [\, x > 0\,],$$

where $c_\varepsilon := 1/B(1-\varepsilon, \varepsilon)$. Note that integration over x proves the equality $\int_0^\infty \lambda^{\varepsilon-1}/(\lambda+1)\, \mathrm{d}\lambda = 1/c_\varepsilon$. Hence the Lt of f_ε is given by the following function of s (with $s \ne 1$):

$$c_\varepsilon \int_0^\infty \frac{\lambda^\varepsilon}{(\lambda+s)\,(\lambda+1)}\, \mathrm{d}\lambda = c_\varepsilon\, s^\varepsilon \int_0^\infty \frac{\lambda^{-\varepsilon}}{(\lambda+s)\,(\lambda+1)}\, \mathrm{d}\lambda =$$

$$= c_\varepsilon\, s^\varepsilon\, \frac{1 - s^{-\varepsilon}}{s-1} \int_0^\infty \frac{\lambda^{-\varepsilon}}{\lambda+1}\, \mathrm{d}\lambda = \frac{s^\varepsilon - 1}{s - 1} = \pi_\varepsilon(s).$$

Note that f_ε is a mixture of exponential densities, and hence is *completely monotone.* Also, by Proposition 5.19 (ii) f_ε is *hyperbolically completely monotone.* Conclusion: *For every $\varepsilon \in (0,1)$ the function π_ε is the pLSt of a self-decomposable and infinitely divisible distribution.*

Next consider the pLSt $s \mapsto \pi_\varepsilon(1+s)/\pi_\varepsilon(1)$, and let $\varepsilon \downarrow 0$. By the continuity theorem one then obtains the following pLSt π:

$$\pi(s) = \lim_{\varepsilon \downarrow 0} \frac{\pi_\varepsilon(1+s)}{\pi_\varepsilon(1)} = \lim_{\varepsilon \downarrow 0} \frac{(1+s)^\varepsilon - 1}{\varepsilon s} = \frac{\log(1+s)}{s},$$

with corresponding density f on $(0,\infty)$ given by

$$f(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\, \mathrm{e}^{-x} f_\varepsilon(x) = \int_1^\infty \mathrm{e}^{-\lambda x}\, \frac{1}{\lambda}\, \mathrm{d}\lambda = \int_x^\infty \frac{1}{t}\, \mathrm{e}^{-t}\, \mathrm{d}t.$$

Clearly, f is again a mixture of exponential densities, and hence *completely monotone*. Also, f is *hyperbolically completely monotone* as it is the limit of such functions. Conclusion: *The function π is the pLSt of a self-decomposable and infinitely divisible distribution.* ◫

Example 12.11. Consider the following pLSt π, which is a *shape mixture of standard gamma distributions*:

$$\pi(s) = \tfrac{1}{2} \int_0^2 \Big(\frac{1}{1+s}\Big)^r \mathrm{d}r = \frac{s\,(2+s)}{2\,(1+s)^2 \log(1+s)},$$

with corresponding density f given by

$$f(x) = \frac{1}{2x}\, \mathrm{e}^{-x} \int_0^2 \frac{x^r}{\Gamma(r)}\, \mathrm{d}r \qquad [\,x > 0\,].$$

According to Theorem 4.9 π is *infinitely divisible.* Note that π can be rewritten as

$$\pi(s) = \Big(\frac{1}{1+s}\Big)^2 \Big/ \Big\{\frac{\log(1+s)}{s}\, \frac{2}{2+s}\Big\},$$

which yields a curious decomposition of the gamma (2) distribution into three infinitely divisible distributions; cf. Example 12.10. ◫

Example 12.12. Let $X \overset{\mathrm{d}}{=} \mathrm{e}^Y - 1$ with Y standard gamma (r) distributed. Then X has density f given by

$$f(x) = \frac{1}{\Gamma(r)} \Big(\frac{1}{1+x}\Big)^2 \big\{\log(1+x)\big\}^{r-1} \qquad [\,x > 0\,].$$

First, let $r \le 1$. Then we write f as

$$f(x) = \frac{1}{\Gamma(r)} \Big(\frac{1}{x}\Big)^{1-r} \Big(\frac{1}{1+x}\Big)^{2r} \Big(\frac{2}{2+x}\Big)^{1-r} \big\{\pi_1(x)\big\}^{1-r},$$

with π_1 the pLSt π from Example 12.11. Since π_1 is infinitely divisible, π_1^{1-r} is completely monotone, and similarly for the other factors in f. Therefore, f is *completely monotone* as well. Next, let $r \geq 1$. Then we write f as

$$f(x) = \frac{1}{\Gamma(r)}\, x^{r-1} \Big(\frac{1}{1+x}\Big)^2 \big\{\pi_2(x)\big\}^{r-1},$$

with π_2 the pLSt π from Example 12.10. Now, π_2 is the pLSt of a generalized gamma convolution; hence so is π_2^{r-1}. By Theorem 5.24 it follows that π_2^{r-1} is hyperbolically completely monotone. Therefore, f is *hyperbolically completely monotone* as well; cf. Proposition 5.17. Conclusion: *For all values of r the random variable X is infinitely divisible; when $r \geq 1$, X is self-decomposable.* ◫

Example 12.13. Consider a product of two different exponential pLSt's:

$$\pi(s) = \frac{1}{1+s}\,\frac{1}{1+\theta s} = \frac{1}{\theta - 1}\Big\{\frac{\theta}{1+\theta s} - \frac{1}{1+s}\Big\},$$

where $\theta > 0$ with $\theta \neq 1$. Then for the corresponding density g we have

$$g(x) = \frac{1}{\theta - 1}\,(\mathrm{e}^{-x/\theta} - \mathrm{e}^{-x}) \qquad [\,x > 0\,].$$

Now suppose that g is hyperbolically completely monotone. Then the function ψ with $\psi(s) = \theta\, g(s)/s$ for $s > 0$ would be hyperbolically completely monotone as well. Since $\psi(0+) = 1$, it would follow from Theorem 5.24 that ψ is the pLSt of a generalized gamma convolution, and hence of an infinitely divisible distribution. The function ψ is, however, the pLSt of a uniform distribution, which is not infinitely divisible. Conclusion: *The convolution of two different exponential densities is not hyperbolically completely monotone.*

Next we use this π in Corollary 5.29, which says that for any $n \in \mathbb{N}$ the following function f_n is a density of a generalized gamma convolution:

$$f_n(x) = \frac{(-1)^n}{(n-1)!}\,(1/x)^{n+1}\,\pi^{(n)}(1/x) \qquad [\,x > 0\,].$$

An easy calculation shows that in the present case for f_n we get

$$f_n(x) = \frac{n}{\theta - 1}\Big\{\Big(\frac{\theta}{\theta + x}\Big)^{n+1} - \Big(\frac{1}{1+x}\Big)^{n+1}\Big\} \qquad [\,x > 0\,].$$

Conclusion: *This density f_n is self-decomposable and infinitely divisible.*

Note that f_n is *not* hyperbolically completely monotone for arbitrarily large n, since $f_n(\cdot/n)/n$ tends to g above as $n \to \infty$. As a special case we take $\theta = 2$ and $n = 1$; then for $f := f_1$ we get

$$f(x) = \frac{4}{(2+x)^2} - \frac{1}{(1+x)^2} = \int_0^\infty \lambda\,\mathrm{e}^{-\lambda x}\,\mathrm{e}^{-\lambda}(4\mathrm{e}^{-\lambda} - 1)\,\mathrm{d}\lambda,$$

which shows that f is a generalized *mixture of exponential densities.* ◻◻

Example 12.14. Though for gamma variables X the inverse $1/X$ is infinitely divisible (cf. Theorem 5.20), not all random variables X corresponding to generalized gamma convolutions have this property. To show this in case $\ell_X > 0$ is easy. We now prove the existence of a random variable X with $\ell_X = 0$ that corresponds to a generalized gamma convolution and is such that $1/X$ is (unbounded and) not infinitely divisible. For $n \in \mathbb{N}$ let Y_n and Z be independent with Y_n gamma (n, n) and Z standard exponential, and consider X_n such that

$$X_n \stackrel{\mathrm{d}}{=} Y_n + Z.$$

Then the distribution of X_n is a generalized gamma convolution; moreover, $\ell_{X_n} = 0$. Now, let $n \to \infty$; since $Y_n \xrightarrow{\mathrm{d}} 1$, we have $X_n \xrightarrow{\mathrm{d}} 1 + Z$, and therefore $1/X_n \xrightarrow{\mathrm{d}} 1/(1+Z)$, which is bounded by one and hence is *not* infinitely divisible. It follows that $1/X_n$ cannot be infinitely divisible for all n, i.e., there must be an n_0 such that $1/X_{n_0}$ is not infinitely divisible. Now take $X = X_{n_0}$. ◻◻

Example 12.15. Let X_1 and X_2 be independent $\mathbb{Z}_+$-valued random variables with pgf's P_1 and P_2, respectively, given by

$$P_1(z) = \tfrac{1}{2}\,(1+z), \quad P_2(z) = \tfrac{3}{2}\,\frac{1}{2-z} - \tfrac{1}{2}.$$

Then P_1 is *not* infinitely divisible, and for the R-function R_2 of P_2 we find

$$R_2(z) = \tfrac{1}{2}\,\frac{2}{2-z} + \frac{1}{1+z},$$

so by Theorem II.4.3 the pgf P_2 is *not* infinitely divisible either. The pgf P_3 of the product X_1X_2, however, can be written as

$$P_3(z) = \tfrac{1}{2} + \tfrac{1}{2}\,P_2(z) = \tfrac{1}{4} + \tfrac{3}{4}\,\frac{1}{2-z},$$

which is a *mixture of geometric pgf's*, and hence is *infinitely divisible*; see Section 7. This can also be verified directly; for the R-function R_3 of P_3 we find an absolutely monotone function:

$$R_3(z) = \frac{1}{2-z} + \frac{1}{5-z}.$$

Conclusion: *The product of two random variables, neither of which is infinitely divisible, may be infinitely divisible.*
In contrast, the product $N_1 N_2$ of two independent random variables N_1 and N_2 with the same geometric distribution is *not* infinitely divisible. The proof of this is practically identical to that of Example II.11.3. On the other hand, for the pgf P_4 of the discrete product $N_1 \odot N_2$ we find

$$P_4(z) = \sum_{k=0}^{\infty} (1-p)\, p^k \frac{1-p}{1-p+kp\,(1-z)},$$

which is a *mixture of geometric pgf's*; so $N_1 \odot N_2$ is *infinitely divisible.* ◻◻

Example 12.16. Consider the following generalized *mixture of two geometric pgf's*:

$$P(z) = 2\frac{1-\frac{17}{24}}{1-\frac{17}{24}z} - \frac{1}{2-z} = \frac{4+3z}{(24-17z)\,(2-z)}.$$

Use the quantities ε and δ from Theorem 7.9. In the present case we have $\varepsilon = P(0) = \frac{1}{12}$, so $\varepsilon > 0$; hence P is a pgf. But $\delta = -\frac{3}{17}$, so $\delta < 0$, and P is *not* a mixture of Poisson distributions. Moreover, P has radius of convergence $\rho = \frac{24}{17}$, and $P(z_0) = 0$ for $z_0 = -\frac{4}{3}$. Since $|z_0| < \rho$, P can*not* be infinitely divisible because of Theorem II.2.8. Also, the coefficients r_k in the power-series expansion of the R-function of P are given by

$$r_k = \left(\tfrac{17}{24}\right)^{k+1} + \left(\tfrac{1}{2}\right)^{k+1} - \left(-\tfrac{3}{4}\right)^{k+1} \qquad [\,k \in \mathbb{Z}_+\,],$$

which is negative for large odd k. ◻◻

Example 12.17. Here we illustrate Corollary 5.9 and the fact that this result has no analogue for generalized negative-binomial convolutions. Let the distribution of the random variable T be a generalized gamma convolution with canonical pair $(0, U)$, where $U(\lambda) = \frac{1}{2}\sqrt{\lambda}$ for $\lambda > 0$. Since $\lim_{\lambda\to\infty} U(\lambda) = \infty$, by Proposition 5.8 the continuous density f of T is *not* completely monotone, and hence by Corollary 5.9 *not* monotone. Now consider the random variable $N(T)$ from Section 6; according to the beginning of Section 8 its distribution (p_k) is a generalized negative-binomial convolution with canonical pair $(0, V)$, where $V(p) = -U(1/p - 1)$ for $p \in (0,1)$. Since $-V(0+) = \infty$, by Proposition 8.8 (p_k) is *not* completely monotone. But one easily shows that

$$\int_{(0,1)} p \, \mathrm{d}V(p) = -\int_0^1 V(p)\, \mathrm{d}p = \tfrac{1}{2} \int_0^1 \sqrt{\frac{1-p}{p}}\, \mathrm{d}p = \tfrac{1}{4}\,\pi,$$

so, as noted just after Proposition 8.8, (p_k) is *monotone.* A simpler example illustrating this, is obtained by taking T standard gamma (2); then the distribution (p_k) of $N(T)$ is negative-binomial $(2, \frac{1}{2})$, which is *monotone* but *not* completely monotone. Even simpler, since the Poisson (t) distribution is monotone for $t \in (0,1]$, $N(T)$ has a *monotone* distribution for every T with values in $[0,1]$. Compare with Theorem 6.12. ◻◻

Example 12.18. Let $r > 0$, $\lambda > 0$ and $\gamma > r$. Consider the function g on $(0, \infty)$ with

$$g(x) = c\, x^{\lambda-1} \Big(\frac{1}{1+x}\Big)^{\lambda-r+1} \big\{\log\,(1 + 1/x)\big\}^{\gamma-1} \qquad [\,x > 0\,];$$

one easily verifies that g is a probability density for some $c > 0$. Write g as

$$g(x) = c\, x^{\lambda-\gamma} \Big(\frac{1}{1+x}\Big)^{\lambda-r+1} \Big(\frac{\log\,(1+1/x)}{1/x}\Big)^{\gamma-1},$$

and use the fact (cf. Example 12.12) that the function π_2^t with $\pi_2(s) := \big\{\log\,(1+s)\big\}/s$ is hyperbolically completely monotone for all $t > 0$. In view of some further properties of hyperbolically completely monotone functions we conclude that g is hyperbolically completely monotone if $\lambda \geq r - 1$ and $\gamma \geq 1$. For these values of the parameters one can use g in Proposition 8.15, which says that (p_k) with

$$p_k = \binom{k+r-1}{k} \int_0^\infty \frac{x^k}{(1+x)^{k+r}}\, g(x)\, \mathrm{d}x \qquad [\,k \in \mathbb{Z}_+\,]$$

is a generalized negative-binomial convolution. In the present case we find, using the substitution $x/(1+x) = \mathrm{e}^{-t}$:

$$p_k = c \binom{k+r-1}{k} \Big(\frac{1}{\lambda+k}\Big)^{\gamma} \qquad [\,k \in \mathbb{Z}_+\,].$$

Conclusion: *If* $r > 1$, $\lambda \geq r - 1$, $\gamma > r$ *or* $r = 1$, $\lambda > 0$, $\gamma > 1$ *or* $r \in (0, 1)$, $\lambda > 0$, $\gamma \geq 1$, *then the distribution* (p_k) *is self-decomposable and infinitely divisible.* When $r \in (0, 1]$, $\lambda > 0$, $\gamma > r$, (p_k) is *completely monotone* and hence *infinitely divisible* as well; cf. the beta distribution in Section B.3. Note that for $r = \lambda = 1$ the *discrete Pareto* distribution is obtained. ◻◻

Example 12.19. Let the function ϕ on $\mathbb{R}$ be given by

$$\phi(u) = \frac{1}{u^2}\left(1 - \exp\left[-u^2\right]\right) \qquad [\,u \in \mathbb{R}\,].$$

Is it a characteristic function? If so, is it infinitely divisible? Note that ϕ can be written as $\phi(u) = \pi(u^2)$ with π given by

$$\pi(s) = \frac{1}{s}\left(1 - \mathrm{e}^{-s}\right) \qquad [\,s > 0\,].$$

Since π is recognized as the pLSt of the uniform distribution on $(0, 1)$, it follows that ϕ is indeed a characteristic function; it corresponds to a *mixture of zero-mean normal distributions.* Proposition 9.2 implies, however, that ϕ is *not* infinitely divisible. ◻◻

Example 12.20. Let U and V be independent, U standard gamma$\,(r)$ with $r > 1$ and V stable$\,(2)$ on $\mathbb{R}_+$ with exponent $\gamma = \frac{1}{2}$, so V has density f_V and pLSt π_V given by

$$f_V(x) = \frac{1}{\sqrt{\pi}}\, x^{-3/2}\, \mathrm{e}^{-1/x}, \quad \pi_V(s) = \exp\left[-2\sqrt{s}\right];$$

cf. Example V.9.5. Consider the random variable $W := U^2 V$. Since U and V have hyperbolically completely monotone densities, W has a hyperbolically completely monotone density h, say, as well; cf. Propositions 5.16 and 5.19. Moreover, the pLSt π of W is given by

$$\pi(s) = \int_0^\infty \pi_V(su^2)\, \mathrm{d}F_U(u) = \pi_U(2\sqrt{s}) = \left(\frac{1}{1 + 2\sqrt{s}}\right)^r.$$

Now, apply Proposition 11.12 with g on $(0, \infty)$ given by

$$g(t) = (r-1)\sqrt{\pi}\,\sqrt{t}\, f_{1/W}(t) = (r-1)\sqrt{\pi}\,(1/t)^{3/2}\, h(1/t);$$

indeed, g is a probability density on $(0, \infty)$ because $\mathbb{E}(1/U) = 1/(r-1)$ and $\mathbb{E}(1/\sqrt{V}) = 1/\sqrt{\pi}$, and g is hyperbolically completely monotone, and hence the density of a generalized gamma convolution, because of Propositions 5.16, 5.17 and Theorem 5.18. For the density f in (11.8) we find

$$f(x) = \tfrac{1}{2}(r-1)\,\pi(\tfrac{1}{4}x^2) = \tfrac{1}{2}(r-1)\,\frac{1}{\big(1+|x|\big)^r} \qquad [x \in \mathbb{R}].$$

We conclude that the *double-Pareto (r)* distribution is a generalized sym-gamma convolution, and hence is *self-decomposable* and *infinitely divisible.* Its infinite divisibility was already proved in Example IV.11.16; f is *completely monotone* on $(0,\infty)$. □

Example 12.21. Let $X \stackrel{d}{=} -\log Y$ with Y *beta* (α,β) distributed. Then X has density f on $(0,\infty)$ and pLSt π given by

$$f(x) = \frac{1}{B(\alpha,\beta)}\,\mathrm{e}^{-\alpha x}\,(1-\mathrm{e}^{-x})^{\beta-1}, \quad \pi(s) = \frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+s)}{\Gamma(\alpha)\,\Gamma(\alpha+\beta+s)}.$$

Using a well-known expression for the gamma function (see Section A.5), we can rewrite π in the form (3.4) with k given by

$$k(x) = \mathrm{e}^{-\alpha x}\,(1-\mathrm{e}^{-\beta x})\,(1-\mathrm{e}^{-x})^{-1} \qquad [x>0];$$

since k is nonnegative, it follows that X is *infinitely divisible* with canonical density k. Further, it can be shown that k is nonincreasing iff $2\alpha+\beta \ge 1$, so by Theorem V.2.11 this condition is necessary and sufficient for X to be *self-decomposable.* Finally, it is easily shown that the function $x \mapsto k(x)/x$ is the Lt of the function v on $(0,\infty)$ given by

$$v(\lambda) = \sum_{n=0}^{\infty} 1_{[n+\alpha,n+\alpha+\beta)}(\lambda) \qquad [\lambda>0],$$

so from Theorem 3.8 it follows that f is a *mixture of exponential densities* iff $\beta \le 1$, and from (5.7) and Theorem 5.3 that f corresponds to a *generalized gamma convolution* iff $\beta \in \mathbb{N}$. Hence f is *completely monotone* iff $\beta \le 1$, and k is *completely monotone* iff $\beta \in \mathbb{N}$. □

13. Notes

Mixing of probability distributions is a very natural procedure; it occurs in many situations, most of which can be viewed as randomizing a parameter. Though there is no obvious connection between mixing and infinite divisibility, many authors have considered infinite divisibility of mixtures. One of the first instances can be found in Feller (1971), who considers power mixtures in the context of subordination of processes.

Mixtures of exponential distributions were first studied by Goldie (1967) in connection with queueing theory and by Steutel (1967). The canonical representation in Theorem 3.5, the characterization in Theorem 3.8 and the generalizations in Theorems 3.10 and 3.13 are given in Steutel (1970). Here also mixtures of gamma distributions are considered; the basic Lemma 4.4 conjectured there, was proved by Kristiansen (1994).

Generalized gamma convolutions were introduced by Thorin (1977a), and used by him (1977b) to prove the infinite divisibility of the log-normal distribution. They are extensively studied in a book by Bondesson (1992). This book, in which the useful concept of hyperbolic complete monotonicity is introduced, seems to mark the last 'great leap forward' in the study of infinitely divisible distributions on the real line. Practically all results in Section 5 are from this source. The functions occurring in Lemma 5.14 are known as Pick functions. Early results in the vein of Theorem 5.20 are given by Grosswald (1976), Goovaerts et al. (1977, 1978) and Thorin (1978b); they are generalized in Bondesson (1979). The mixtures of exponential distributions as well as the generalized gamma convolutions are contained in the class of so-called generalized convolutions of mixtures of exponential distributions. This class is studied in Bondesson (1981, 1992); it consists of the infinitely divisible distributions on $\mathbb{R}_+$ that have a canonical density k such that $x \mapsto k(x)/x$ is completely monotone; cf. Example 12.21.

Mixtures of Poisson distributions have been studied for a long time, especially in the context of insurance mathematics and reliability theory; see the book by Grandell (1997), which contains a lot of general information, for references. A paper by Puri and Goldie (1979) on the subject is partly devoted to infinite divisibility; it contains Theorem 6.4; related results are given in Forst (1981). Statements 6.6 through 6.8, including their generalizations with respect to semigroups, can be found in van Harn et al. (1982), or van Harn and Steutel (1993); a different proof of Theorem 6.6 is given in Forst (1979). Berg and Forst (1983) use Poisson mixtures to relate multiple self-decomposability on $\mathbb{Z}_+$ to that on $\mathbb{R}_+$. The log-convexity result in Theorem 6.13 is due to Hirsch (1975), whose proof is less detailed than ours; some of the tools used in the proof are taken from Widder (1972).

Mixtures of geometric distributions occur in Steutel (1967). Mixtures of negative-binomial distributions are considered by Bondesson (1992) in the context of his study of generalized negative-binomial convolutions. Most results in Section 8 can be found there. Steutel and van Eenige (1997) show that the set of mixtures of negative-binomial distributions together with their limits equals the set of Poisson mixtures; this in contrast to the set of mixtures of gamma distributions and limits thereof, which equals the set of *all* distributions on $\mathbb{R}_+$.

Mixtures of normal distributions are studied by, among others, Kelker (1971), Keilson and Steutel (1974), Grosswald (1976) and Wolfe (1978b). Keilson and Steutel also consider mixtures of more general stable distributions. Wolfe (1978b) shows that the mixing distributions in infinitely divisible normal mixtures have tails very similar to those of infinitely divisible distributions. That the mixing distributions need not be infinitely divisible, is shown in an ingenious example (our Example 12.3) by Kelker (1971). Our statement (9.5) was proved earlier by Halgreen (1979), and by Ismail and Kelker (1979); an extension to Brownian motion with drift is given by Sato (2001).

Mixtures of Laplace distributions are considered in Steutel (1970). The generalized sym-gamma convolutions in Section 11 coincide with the symmetric distributions among the so-called extended generalized gamma convolutions, which were proposed by Thorin (1978a) and studied by Bondesson (1992). The infinite divisibility of the student distribution, which was examined by so many authors (cf. the Notes of Chapter IV), is now a simple consequence of results on generalized sym-gamma convolutions.

Example 12.2 can be found in Puri and Goldie (1979). The infinite divisibility of the half-Cauchy distribution in Example 12.4 is proved by Bondesson (1987); our proof was suggested by C. Berg (personal communication to Bondesson); the self-decomposability is demonstrated in Diédhiou (1998). The counter-example in Example 12.6 is due to Kristiansen (1995). Example 12.10 is given in Bondesson (1992); a special case of it occurs as a limit distribution in heavy-traffic queueing theory (J.W. Cohen, personal communication). Examples 12.12, 12.13, 12.18, 12.20 and 12.21 are adaptations of examples in Bondesson (1992).

Chapter VII

INFINITE DIVISIBILITY IN STOCHASTIC PROCESSES

1. Introduction

This chapter is devoted to stochastic processes that give rise to infinitely divisible distributions or that are otherwise connected to infinite divisibility. The processes with stationary and independent increments (*sii-processes*), which are at the heart of infinite divisibility, were discussed in Chapter I, and turned up in several subsequent chapters; here they will only be considered in passing. Since some of the subjects discussed in this chapter are of a rather specialistic nature, we shall not give full proofs of all results presented; in fact, we shall sometimes give no proof at all.

In Section 2 we consider *first-passage times* in discrete- and continuous-time *Markov chains*. These are often compound-exponential or (shifted) compound-geometric, and hence infinitely divisible, and this fact can be used to prove the infinite divisibility of a given distribution, if it is known to correspond to such a first-passage time. In Section 3 a similar phenomenon is shown to occur for limiting *waiting-time* distributions in *queueing processes*. As it turns out, the results of Sections 2 and 3 do not often lead to proofs of the infinite divisibility of distributions that were not known to be infinitely divisible already. Still it is interesting that infinite divisibility appears to be so widespread.

Branching processes are the subject of Section 4. It will be shown that limit distributions of such processes in *continuous time* are often (generalized) self-decomposable and hence infinitely divisible; in discrete-time processes only a kind of 'partial self-decomposability' occurs. More specifically, in the supercritical case with positive off-spring the limit distribution as $t \to \infty$ of the number of individuals at time t divided by its mean is classically self-decomposable. In the subcritical case with immigration according to an sii-process, obeying a logarithmic moment condition, the number of individuals at time t has a limit distribution that is self-decomposable with respect to the composition semigroup $\mathcal{F}$ of pgf's governing the branching process; cf. Section V.8.

In Section 5 *renewal processes* are considered. We first show that some aspects of the renewal function can be connected to infinite divisibility properties of the underlying lifetime distribution. Next, we consider the limiting lifetime spanning x as $x \to \infty$ and find that it can be decomposed as the ordinary lifetime plus an independent 'extra' time iff the ordinary lifetime is infinitely divisible. This leads to an extreme case of the *inspection paradox*, also known as the *waiting-time paradox*, and to a simple derivation of limit distributions connected with the passage by an sii-process on $\mathbb{R}_+$ of a level x as $x \to \infty$. In the special case of compound-exponential lifetimes there is a similar decomposition of the limiting remaining lifetime at x.

A much used stochastic process in physics, finance and queueing theory is the so-called *shot noise*. In Section 6 we will show that its one-dimensional distributions are infinitely divisible; special choices for the parameters lead to well-known subclasses of infinitely divisible distributions. Section 7 is devoted to examples, and Section 8 contains notes, bibliographic and otherwise.

2. First-passage times in Markov chains

We start with considering the *Bernoulli walk* $(X_n)_{n \in \mathbb{Z}_+}$ on $\mathbb{Z}$ with parameter p, so $X_n = X_0 + Y_1 + \cdots + Y_n$, where $X_0, Y_1, Y_2, \ldots$ are independent, X_0 is $\mathbb{Z}$-valued, and $\mathbb{P}(Y_i = 1) = p$ and $\mathbb{P}(Y_i = -1) = 1 - p =: q$ for all i. Set $\mathbb{P}_j := \mathbb{P}(\cdot \mid X_0 = j)$ for $j \in \mathbb{Z}$, and let T_k be the *first-passage time* to state k:

$$T_k := \inf \{n \in \mathbb{N} : X_n = k\}.$$

We are interested in the $\mathbb{P}_j$-distribution of T_k for $k > j$; let $P_{j,k}$ denote the corresponding pgf. Suppose that $p \geq \frac{1}{2}$; then (and only then) we have $\mathbb{P}_j(T_k < \infty) = 1$. We first take $k = j + 1$; then it is easily verified and intuitively obvious that $P_{j,j+1}(z) = pz + qz\, P_{j-1,j}(z)\, P_{j,j+1}(z)$, so

$$P_{j,j+1}(z) = \frac{pz}{1 - qz\, P_{j-1,j}(z)}. \tag{2.1}$$

We conclude that $P_{j,j+1}$ is shifted *compound-geometric* and hence is the pgf of an *infinitely divisible* distribution on $\mathbb{N}$ (note that it has no mass at zero). Similarly, for $k > j+1$ we get $P_{j,k} = \prod_{\ell=j}^{k-1} P_{\ell,\ell+1}$, so $P_{j,k}$ corresponds to an *infinitely divisible* distribution on $\mathbb{N}$, too. Also, $P_{j,j+1}$ is easily seen not to depend on j, so (2.1) yields a quadratic equation for $P_{0,1}$. Solving this equation shows that the pgf P and (hence) the distribution $(p_k)_{k\in\mathbb{Z}_+}$ of the 'reduced' first-passage time $\frac{1}{2}(T_1 - 1)$ from 0 to 1 are given by

$$P(z) = \frac{1 - \sqrt{1 - 4pqz}}{2qz}, \quad p_k = \frac{1}{k+1}\binom{2k}{k} p^{k+1} q^k. \tag{2.2}$$

For the R-function and the canonical sequence $(r_k)_{k\in\mathbb{Z}_+}$ of P we find

$$R(z) = \frac{1}{2z}\Big(\frac{1}{\sqrt{1-4pqz}} - 1\Big), \quad r_k = \tfrac{1}{2}\binom{2k+2}{k+1} (p\,q)^{k+1}. \tag{2.3}$$

Now, from the considerations in Example II.11.11, where we discussed the special case $p = \frac{1}{2}$, it easily follows that both (p_k) and (r_k) are *completely monotone*, also when $p > \frac{1}{2}$. So (p_k) is a *generalized negative-binomial convolution*, and hence is *self-decomposable*; cf. Theorem VI.8.4 and Proposition VI.8.6.

The infinite divisibility of the first-passage time from j to k above is a very special instance of infinite divisibility of *first-passage times* in (discrete-time) *Markov chains*; in fact, in a similar way one can prove the following general result.

Theorem 2.1. *Let $(X_n)_{n\in\mathbb{Z}_+}$ be a Markov chain on $\mathbb{Z}$ with stationary transition probabilities p_{jk}, $j,k \in \mathbb{Z}$, satisfying*

$$p_{j,j+1} > 0 \ \text{ for } j \in \mathbb{Z}; \quad p_{jk} = 0 \ \text{ for } j,k \in \mathbb{Z} \text{ with } k > j+1. \tag{2.4}$$

Then for $k > j$ the $\mathbb{P}_j$-distribution of the first-passage time T_k to state k, if not defective, is infinitely divisible on $\mathbb{N}$, and even shifted compound-geometric if $k = j+1$.

Proof. We first take $k = j+1$. Then $\mathbb{P}_j(T_{j+1} = 1) = p_{j,j+1}$, and for $n \geq 2$ we have

$$\mathbb{P}_j(T_{j+1} = n) = \sum_{\ell \le j} p_{j\ell}\, \mathbb{P}_\ell(T_{j+1} = n-1) =$$

$$= p_{jj}\, \mathbb{P}_j(T_{j+1} = n-1) +$$

$$+ \sum_{\ell < j} p_{j\ell} \sum_{m=1}^{n-1} \mathbb{P}_\ell(T_j = m)\, \mathbb{P}_j(T_{j+1} = n-1-m),$$

so for the pgf $P_{j,j+1}$ of the $\mathbb{P}_j$-distribution of T_{j+1} we get

$$P_{j,j+1}(z) = p_{j,j+1}\, z + z \Big\{ p_{jj} + \sum_{\ell<j} p_{j\ell}\, P_{\ell,j}(z) \Big\}\, P_{j,j+1}(z). \tag{2.5}$$

It follows that $P_{j,j+1}$ can be written as

$$P_{j,j+1}(z) = \frac{p_{j,j+1}\, z}{1 - (1 - p_{j,j+1})\, Q_j(z)}, \tag{2.6}$$

where Q_j is the $\mathbb{P}_j$-pgf of $(T_j \mid X_1 \le j)$. We conclude that $P_{j,j+1}$ is shifted compound-geometric and hence is the pgf of an infinitely divisible distribution on $\mathbb{N}$. Similarly, for $k > j+1$ we get $P_{j,k} = \prod_{\ell=j}^{k-1} P_{\ell,\ell+1}$, so $P_{j,k}$ corresponds to an infinitely divisible distribution on $\mathbb{N}$, too. □

There is a similar result for Markov chains *in continuous time*. Consider a Markov chain $X(\cdot) = \big(X(t)\big)_{t\ge 0}$ on $\mathbb{Z}$ with stationary transition probabilities described by a vector $(\lambda_j)_{j\in\mathbb{Z}}$ and a transition matrix $(p_{jk})_{j,k\in\mathbb{Z}}$ with the following interpretation: If the process is in state j at a given time, then it stays there for an exponential time with mean $1/\lambda_j$; when the process leaves state j, then it changes to state k with probability p_{jk}. Without essential restriction we take $p_{jj} = 0$ for all j. Then we have the following result for the *first-passage time* T_k to state k, which now, of course, is defined by $T_k := \inf\{t > 0 : X(t) = k\}$.

Theorem 2.2. *Let $X(\cdot)$ be a Markov chain as described above and such that the p_{jk} satisfy (2.4). Then for $k > j$ the $\mathbb{P}_j$-distribution of the first-passage time T_k to state k, if not defective, is infinitely divisible, and even compound-exponential if $k = j+1$.*

Proof. Again, it is sufficient to consider the case where $k = j+1$. Let $F_{j,j+1}$ be the $\mathbb{P}_j$-distribution function of T_{j+1}, and let g_j be the $\mathbb{P}_j$-density of the sojourn time τ_j in state j, so $g_j(t) = \lambda_j e^{-\lambda_j t}$ for $t > 0$. Then for

$x > 0$ we have

$$F_{j,j+1}(x) = \int_0^x \mathbb{P}_j(T_{j+1} \le x \,|\, \tau_j = t)\, g_j(t)\, \mathrm{d}t =$$
$$= \int_0^x \Big\{ p_{j,j+1} + \sum_{\ell<j} p_{j\ell}\, \mathbb{P}_\ell(T_{j+1} \le x - t) \Big\}\, g_j(t)\, \mathrm{d}t =$$
$$= p_{j,j+1} \int_0^x g_j(t)\, \mathrm{d}t + \sum_{\ell<j} p_{j\ell} \int_0^x F_{\ell,j+1}(x-t)\, g_j(t)\, \mathrm{d}t;$$

here the $\mathbb{P}_\ell$-distribution function $F_{\ell,j+1}$ of T_{j+1} can be written as

$$F_{\ell,j+1}(y) = \int_{[0,y]} F_{j,j+1}(y-z)\, \mathrm{d}F_{\ell,j}(z) \qquad [\, y > 0\,].$$

For the pLSt $\pi_{j,j+1} := \widehat{F}_{j,j+1}$ it follows that

$$\pi_{j,j+1}(s) = p_{j,j+1}\, \frac{\lambda_j}{\lambda_j + s} + \sum_{\ell<j} p_{j\ell}\, \pi_{\ell,j}(s)\, \pi_{j,j+1}(s)\, \frac{\lambda_j}{\lambda_j + s}. \tag{2.7}$$

Solving for $\pi_{j,j+1}$ we obtain

$$\pi_{j,j+1}(s) = \frac{\lambda_j}{\lambda_j + s}\, \frac{p_{j,j+1}}{1 - (1 - p_{j,j+1})\, \big\{\lambda_j/(\lambda_j + s)\big\}\, \pi_j(s)}, \tag{2.8}$$

where π_j is the pLSt given by $\pi_j(s) := \sum_{\ell<j} p_{j\ell}\, \pi_{\ell,j}(s)/(1 - p_{j,j+1})$. Now, observing that $\pi_{j,j+1}$ is the product of an exponential pLSt and a compound-geometric one, we conclude that $\pi_{j,j+1}$ is infinitely divisible. Also, it is easily verified that $\pi_{j,j+1}$ is compound-exponential with an underlying pLSt π_0 that is shifted compound-Poisson:

$$\pi_0(s) = \exp\big[-\big(1/\{\lambda_j p_{j,j+1}\}\big)\, s - (1/p_{j,j+1} - 1)\, \{1 - \pi_j(s)\} \big].$$

Alternatively, one can apply Theorem III.5.1. ◫

Return to condition (2.4) on the transition probabilities of a Markov chain; it says that the process is *skipfree to the right*: To go from state j to state k with $k > j$ it has to pass through the intermediate states $j+1, \ldots, k-1$. A process that is skipfree in both directions, is just called *skipfree*. A continuous analogue of skipfree-ness for processes not on $\mathbb{Z}$ is *path-continuity*. This plays an important role in the infinite divisibility of first-passage times in one-dimensional *diffusion processes*. We only give a heuristic argument for this; for references with more information see Notes.

If $T_{a,b}$ is the first-passage time from a to b in such a process (more precisely, the random variable $T_{a,b}$ represents the $\mathbb{P}_a$-distribution of T_b), then by the strong Markov property for any $c \in (a,b)$ it can be written as

$$T_{a,b} \stackrel{\mathrm{d}}{=} T_{a,c} + T_{c,b}, \quad \text{with } T_{a,c} \text{ and } T_{c,b} \text{ independent.}$$

Since diffusion processes have continuous paths, more and more points can be inserted between a and b, in such a way that $T_{a,b}$ emerges as a limit in an infinitesimal triangular array, and hence is *infinitely divisible*; cf. Section I.5. In this way the following result can be proved.

Theorem 2.3. *Let $X(\cdot)$ be a one-dimensional diffusion process. Then for $b \neq a$ the first-passage time $T_{a,b}$ from state a to state b, if not defective, is infinitely divisible.*

Consider, for instance, standard *Brownian motion* $X(\cdot)$ started at zero, and for $a > 0$ let T_a be the first-passage time to state a. We will show that T_1 is even *stable*. To this end we take $n \in \mathbb{N}$ and note that by an argument similar to the one above

$$T_1 \stackrel{\mathrm{d}}{=} T_{n,1} + \cdots + T_{n,n}, \quad \text{with } T_{n,1}, \ldots, T_{n,n} \text{ independent,}$$

where now, by space-homogeneity, $T_{n,j} \stackrel{\mathrm{d}}{=} T_{1/n}$ for all j. But, since the well-known self-similarity of $X(\cdot)$ implies that $n\,X(\cdot) \stackrel{\mathrm{d}}{=} X(n^2\,\cdot)$, we also have $T_{1/n} \stackrel{\mathrm{d}}{=} (1/n^2)\,T_1$, so

$$T_1 \stackrel{\mathrm{d}}{=} \frac{1}{n^2}\,\big(T_1^{(1)} + \cdots + T_1^{(n)}\big),$$

where $T_1^{(1)}, \ldots, T_1^{(n)}$ are independent with $T_1^{(j)} \stackrel{\mathrm{d}}{=} T_1$ for all j. From Corollary V.3.2 we conclude that T_1 is *stable with exponent* $\gamma = \frac{1}{2}$. Hence from Theorem V.3.5 it follows that T_1 has pLSt π_{T_1} given by

$$\pi_{T_1}(s) = \exp\big[-\lambda\sqrt{s}\,\big]$$

with a certain $\lambda > 0$; it can be shown (this is not quite trivial) that $\lambda = \sqrt{2}$. The first-passage time T_1 can be viewed as the first-exit time from $(-\infty, 1)$. In Section 7 we will consider the *first-exit time* T from $(-1, 1)$ with pLSt given by

$$\pi_T(s) = \frac{2}{\mathrm{e}^{\sqrt{2s}} + \mathrm{e}^{-\sqrt{2s}}};$$

in a similar way we will show that T is *infinitely divisible* and even *self-decomposable.*

3. Waiting times in queueing processes

We shall mainly be concerned with the G/G/1 queue (general input, general service, one counter), and we start with a brief description. Customers, numbered $1, 2, \ldots$, arrive at a counter to be served; the first one arrives at time $t = 0$. Their *interarrival times* $A_1, A_2, \ldots$ (A_n is the time between the arrivals of the n-th and the $(n+1)$-st customer) are distributed as A, and their *service times* $B_1, B_2, \ldots$ are distributed as B; all A_n and B_n are independent. Arriving customers who find the counter occupied by another customer, wait their turn before they get to be served. Let W_n denote the *waiting time* of the n-th customer. Then $W_1 \equiv 0$, and for $n \in \mathbb{N}$ the waiting time W_{n+1} is easily seen to be obtained from W_n and the difference $D_n := B_n - A_n$ as follows:

$$W_{n+1} = \max\{0, W_n + D_n\}.$$

Iterating this equation leads to

$$W_{n+1} = \max\{0, D_n, D_n + D_{n-1}, \ldots, D_n + \cdots + D_1\},$$

so putting $S_n := \sum_{j=1}^n D_j$ for $n \in \mathbb{Z}_+$ with $S_0 :\equiv 0$, we conclude that W_{n+1} can be written (in distribution) as

$$\text{(3.1)} \qquad W_{n+1} \stackrel{\mathrm{d}}{=} \max\{S_0, S_1, \ldots, S_n\} \qquad [\, n \in \mathbb{Z}_+ \,].$$

Now, suppose that $\mathbb{E}B < \mathbb{E}A$. Then by the strong law of large numbers $S_n \to -\infty$ as $n \to \infty$ with probability one. It follows that W_n converges in distribution to an $\mathbb{R}_+$-valued random variable W, say, where

$$\text{(3.2)} \qquad W \stackrel{\mathrm{d}}{=} \max\{S_0, S_1, S_2, \ldots\}.$$

We will show that W is *infinitely divisible*. To do so we let N be the number of *records* among the sequence $(S_1, S_2, \ldots)$, so

$$N = \#\big(\{n \in \mathbb{N} : S_n > S_k \text{ for } k = 0, 1, \ldots, n-1\}\big).$$

Then N is finite with probability one; in fact, N has a *geometric* distribution with parameter p given by $p = \mathbb{P}(S_n > 0 \text{ for some } n \in \mathbb{N})$. If $N = 0$, then $W = 0$ (and conversely). If $N \geq 1$, then W can be viewed as the sum of N sums: The first positive partial sum $D_1 + \cdots + D_{L_1}$, say, the next positive partial sum $D_{L_1+1}+\cdots+D_{L_2}$, say, and so on, until the last positive sum $D_{L_{N-1}+1}+\cdots+D_{L_N}$. Thus we see that W is of the form

$$W \stackrel{\mathrm{d}}{=} X_1 + X_2 + \cdots + X_N,$$

where the X_n are defective random variables: $X_n := D_{L_{n-1}+1}+\cdots+D_{L_n}$ if $N \geq n$, and $:= \infty$ if $N \leq n-1$. Now, it can be shown that, given $N=n$, $X_1, \ldots, X_n$ are (conditionally) independent and identically distributed with (non-defective) distribution function G, say, not depending on n; G is given by $G = F_{X_1}(\cdot \mid N \geq 1)$. It follows that

$$F_W(w) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\, G^{\star n}(w) \qquad [\, w \in \mathbb{R}\,],$$

so W has a *compound-geometric*, and hence *infinitely divisible* distribution on $\mathbb{R}_+$ with pLSt given by

$$\widehat{F}_W(s) = \frac{1-p}{1-p\,\widehat{G}(s)}. \tag{3.3}$$

We state the result in a formal way as follows.

Theorem 3.1. *Consider a G/G/1 queue with interarrival time A and service time B such that $\mathbb{E}B < \mathbb{E}A$. Then the limiting distribution as $n \to \infty$ of the waiting time W_n of the n-th customer is compound-geometric, and hence infinitely divisible.*

In general it is not possible to obtain an explicit expression for $\widehat{F}_W$, since mostly p and $\widehat{G}$ in (3.3) can not be found. In some special cases, however, $\widehat{F}_W$ can be calculated; see Section 7. Here we only note, without proof, that if the interarrival time A is exponential (λ) (*Poisson arrivals*) (and the service time B satisfies $\mu_B := \mathbb{E}B < 1/\lambda$), then (3.3) takes the form of the well-known *Pollaczek-Khintchine formula*, for which

$$p = \lambda\,\mu_B, \quad \widehat{G}(s) = \frac{1}{\mu_B\, s}\,\big\{1-\widehat{F}_B(s)\big\}. \tag{3.4}$$

As an alternative way to show the infinite divisibility of the limit W above, one can use *Spitzer's identity* (see Notes), which reads

$$\sum_{n=0}^{\infty} \widehat{F}_{\max\{0,S_1,\ldots,S_n\}}(s)\, z^n = \exp\Big[\sum_{n=1}^{\infty} \frac{1}{n}\, \widehat{F}_{\max\{0,S_n\}}(s)\, z^n\Big]; \tag{3.5}$$

here $S_1, S_2, \ldots$ are again the partial sums at a sequence (D_j) of independent, identically distributed random variables. Now apply (3.5) in the present context and set $S_n^+ := \max\{0, S_n\}$, then by (3.1) it follows that

$$(1-z)\sum_{n=0}^{\infty} \widehat{F}_{W_{n+1}}(s)\, z^n = \exp\Big[-\sum_{n=1}^{\infty} \frac{1}{n}\, \{1-\widehat{F}_{S_n^+}(s)\}\, z^n\Big].$$

By letting here $z \uparrow 1$ we see that $\widehat{F}_W = \lim_{n\to\infty} \widehat{F}_{W_n}$ can be written as

$$(3.6) \qquad \widehat{F}_W(s) = \exp\Big[-\sum_{n=1}^{\infty} \frac{1}{n} \int_{\mathbb{R}_+} (1-\mathrm{e}^{-sx})\, \mathrm{d}F_{S_n^+}(x)\Big].$$

Comparing this with the canonical representation in Theorem III.4.3 and using Propositions III.2.2 and III.4.6 (i) we conclude that W is *infinitely divisible* with canonical function K given by

$$(3.7) \qquad K(x) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{[0,x]} y\, \mathrm{d}F_{S_n^+}(y) \qquad [\,x \geq 0\,].$$

Limiting *queue sizes* are *not* always infinitely divisible. As an example, let N_n be the number of customers left behind by the n-th customer at the end of his service. Then N_n is equal to the number of customers that have arrived during his sojourn period of length $W_n + B_n$. For instance, in the situation of Poisson arrivals with intensity λ (and with $\mathbb{E}B < 1/\lambda$) it easily follows that $N := \text{d-}\lim_{n\to\infty} N_n$ exists and has pgf P given by

$$(3.8) \qquad P(z) = \widehat{F}_W\big(\lambda(1-z)\big)\, \widehat{F}_B\big(\lambda(1-z)\big)$$

with $\widehat{F}_W$ as in (3.3) and (3.4). By Theorem VI.6.4, however, such a pgf P is in general not infinitely divisible if B is not; for instance, take B such that $\widehat{F}_B$ has a zero on the imaginary axis $\big($cf. Theorem III.2.8 (i)$\big)$.

Other distributions of interest connected to queueing theory are the *busy period* and *idle period* distributions; the former are quite often infinitely divisible. We do not pursue this.

4. Branching processes

Several limit distributions in continuous-time branching processes turn out to be (generalized) *self-decomposable* and hence *infinitely divisible*. In discrete-time branching processes only a kind of 'partial self-decomposability' occurs. Nevertheless we start with considering the discrete-time case as it serves as a good introduction to its continuous-time counterpart.

A *discrete-time* branching process $(Z_n)_{n\in\mathbb{Z}_+}$ with *off-spring* distribution $(f_k)_{k\in\mathbb{Z}_+}$ can be recursively defined by

$$(4.1) \qquad Z_{n+1} = U_1^{(n)} + \cdots + U_{Z_n}^{(n)} \qquad [\, n \in \mathbb{Z}_+ \,],$$

where Z_0 and all the $U_j^{(n)}$ are independent, Z_0 is $\mathbb{Z}_+$-valued, and the $U_j^{(n)}$ are distributed as U with distribution (f_k). The random variable Z_n is interpreted as the number of individuals in the n-th generation and $U_j^{(n)}$ as the number of off-spring of the j-th individual in the n-th generation. If the individuals are supposed to live for one unit of time, then Z_n can be viewed as the number of individuals present at time n. From (4.1) it follows that (Z_n) is a Markov chain with state space $\mathbb{Z}_+$ and transition probabilities p_{ij} for which

$$(4.2) \qquad \sum_{j=0}^{\infty} p_{ij}\, z^j = \big\{F(z)\big\}^i, \quad \text{and hence } P_{Z_n}(z) = P_{Z_0}\big(F^{\circ n}(z)\big),$$

where P_{Z_n} is the pgf of Z_n and $F^{\circ n}$ is the n-fold composition of the off-spring pgf F with itself with $F^{\circ n}(z) := z$ when $n = 0$. An important quantity is the expected number of off-spring, i.e.,

$$m := \mathbb{E}\, U = \sum_{k=1}^{\infty} k\, f_k = F'(1),$$

which is supposed to be finite. The process is called *subcritical* if $m < 1$, *critical* if $m = 1$, and *supercritical* if $m > 1$. Suppose that $Z_0 \equiv 1$. Then it is not hard to see that the *extinction probability*, i.e.,

$$\rho := \mathbb{P}(Z_n = 0 \;\text{ for some } n),$$

is the smallest nonnegative root of the equation $F(z) = z$; hence $\rho = 1$ in the subcritical and critical cases, and $\rho < 1$ in the supercritical case. Moreover, it can easily be verified that in all cases $Z_n \to Z$ as $n \to \infty$ (in distribution, or even a.s.) with $\mathbb{P}(Z = 0) = \rho$, $\mathbb{P}(Z = \infty) = 1 - \rho$. There are, however, several ways to normalize Z_n in such a way that finite non-degenerate limits appear. We will show this in two cases.

First, consider the *supercritical* case with $Z_0 \equiv 1$. Since we then have $\mathbb{E}\, Z_n = m^n$, we normalize by dividing by m^n; define

$$W_n := Z_n / m^n \qquad [\, n \in \mathbb{Z}_+ \,].$$

One easily verifies that (W_n) is a (nonnegative) martingale; so by a well-known martingale convergence theorem we have $W_n \to W$ a.s., and hence in distribution, as $n \to \infty$ for some $\mathbb{R}_+$-valued random variable W. Since by (4.2) the pLSt of W_n can be written as

$$\pi_{W_n}(s) = P_{Z_n}(\mathrm{e}^{-s/m^n}) = F^{\circ n}(\mathrm{e}^{-s/m^n}) = F\big(\pi_{W_{n-1}}(s/m)\big),$$

the limit W has a pLSt π that satisfies the following functional equation:

$$(4.3) \qquad \pi(s) = F\big(\pi(s/m)\big), \quad \text{so } \pi(s) = F^{\circ k}\big(\pi(s/m^k)\big) \text{ for } k \in \mathbb{N}.$$

Now, suppose that $f_0 = 0$; so there is no zero off-spring, and $\rho = 0$. Then for all $k \in \mathbb{N}$ the function $z \mapsto F^{\circ k}(z)/z$ is a pgf, and hence (cf. the beginning of Section III.3) the function $\pi^{(k)}$ with $\pi^{(k)}(s) := F^{\circ k}\big(\pi(s/m^k)\big)/\pi(s/m^k)$ is a pLSt. Since by (4.3) π can be written as

$$(4.4) \qquad \pi(s) = \pi(s/m^k)\,\pi^{(k)}(s) \qquad [\,k \in \mathbb{N}\,],$$

one might say that the limit W is 'partially self-decomposable'. And, more important, one might hope that a continuous-time analogue leads to (non-partially) self-decomposable, and hence infinitely divisible limits. In view of this we note that the limit W may be degenerate at zero. It can be shown, however, that it is *not* iff the off-spring variable U satisfies $\mathbb{E}\, U \log^+ U < \infty$. In this case it follows from (4.3) by letting $s \to \infty$ that $\mathbb{P}(W = 0) = \rho = 0$, so W is $(0, \infty)$-valued.

In the (discrete-time) *subcritical* case a similar phenomenon occurs. Here we normalize by allowing *immigration*; the dwindling population is supplemented by an influx of individuals arriving in batches of sizes $B_1, B_2, \ldots$, which are supposed to be independent and distributed as B with distribution $(q_k)_{k \in \mathbb{Z}_+}$. This leads to a process $(X_n)_{n \in \mathbb{Z}_+}$ recursively defined by

$$(4.5) \qquad X_{n+1} = U_1^{(n)} + \cdots + U_{X_n}^{(n)} + B_n \qquad [\,n \in \mathbb{Z}_+\,],$$

where X_0, the $U_j^{(n)}$ and the B_n are independent. Let P_n be the pgf of X_n; then (4.5) can be translated in terms of the off-spring pgf F and the batch-size pgf Q as follows:

$$(4.6) \qquad P_{n+1}(z) = P_n\big(F(z)\big)\, Q(z) \qquad [\,n \in \mathbb{Z}_+\,].$$

Iterating this and assuming that $X_0 \equiv 0$ (no essential restriction), we see that

$$(4.7) \qquad P_{n+1}(z) = \prod_{k=0}^{n} Q\big(F^{\circ k}(z)\big) \qquad [\,n \in \mathbb{Z}_+\,].$$

From this it can be deduced that $X_n \xrightarrow{\mathrm{d}} X$ as $n \to \infty$ for some $\mathbb{Z}_+$-valued random variable X iff the batch size B has a finite logarithmic moment: $\mathbb{E} \log^+ B < \infty$. Consider the pgf P of the limit X; it satisfies

$$(4.8) \qquad P(z) = \prod_{k=0}^{\infty} Q\big(F^{\circ k}(z)\big).$$

Clearly, if Q is infinitely divisible, then so is P; cf. Sections II.2 and II.3. In general, however, P in (4.8) is *not* infinitely divisible; cf. Theorem II.2.8. On the other hand, from (4.8) it follows that $P(z) = P\big(F(z)\big)\, Q(z)$, and iteration of this shows that there exist pgf's $P^{(1)}, P^{(2)}, \ldots$ such that

$$(4.9) \qquad P(z) = P\big(F^{\circ k}(z)\big)\, P^{(k)}(z) \qquad [\, k \in \mathbb{Z}_+ \,].$$

Now, compare this with relation (V.8.14) with $t > 0$, which characterizes $\mathcal{F}$-self-decomposability of P; here $\mathcal{F} = (F_t)_{t \geq 0}$ is a composition semigroup of pgf's. Then one might say that P in (4.9) is self-decomposable with respect to $(F^{\circ k})_{k \in \mathbb{Z}_+}$ and, if this sequence is embeddable in a composition semigroup $\mathcal{F}$ as above, that P is 'partially $\mathcal{F}$-self-decomposable'. Again, this suggests that a continuous-time analogue of (4.5) might lead to (non-partially) *$\mathcal{F}$-self-decomposable*, and hence *infinitely divisible* limits.

Therefore, we now make time continuous, and consider a *continuous-time* branching process $Z(\cdot) = \big(Z(t)\big)_{t \geq 0}$ where branching is governed by a continuous composition semigroup $\mathcal{F} = (F_t)_{t \geq 0}$ of pgf's. This means that $Z(\cdot)$ is a Markov chain with state space $\mathbb{Z}_+$ and with stationary transition probabilities $p_{ij}(\cdot)$ for which

$$(4.10) \qquad \sum_{j=0}^{\infty} p_{ij}(t)\, z^j = \big\{F_t(z)\big\}^i, \quad \text{and hence} \;\; P_{Z(t)}(z) = P_{Z(0)}\big(F_t(z)\big).$$

The random variable $Z(t)$ can be interpreted as the number of individuals alive at time t; the individuals live an exponential period of time with mean $1/a$, for some $a > 0$, and at the moment of their deaths they produce off-spring independent of each other and according to a distribution $(h_k)_{k \in \mathbb{Z}_+}$, with $h_1 = 0$ and pgf H. The infinitesimal quantities a and H determine and are determined by the infinitesimal *generator* U of $\mathcal{F}$ according to

$$(4.11) \qquad U(z) = a\,\big\{H(z) - z\big\};$$

for the definition of U and further relations we refer to Section V.8.

In the *supercritical* case where $m := F_1'(1) > 1$ (and m finite), we suppose that $Z(0) \equiv 1$, and look at the limit

$$W := \text{d-}\lim_{t\to\infty} Z(t)/m^t,$$

which can be seen to exist (in $\mathbb{R}_+$) by a martingale argument again. Now, it is well known that for every $h > 0$ the process $(Z_n^{(h)})$ with $Z_n^{(h)} := Z(nh)$ is a discrete-time branching process with off-spring pgf F_h and $Z_0^{(h)} \equiv 1$. From (4.3) and (4.4) applied to $(Z_n^{(h)})$ it easily follows that the pLSt π of W satisfies

$$(4.12) \qquad \pi(s) = F_h\big(\pi(s/m^h)\big) \qquad [\,h > 0\,],$$

and that if $F_1(0) = 0$, and hence $F_h(0) = 0$, there then exists a pLSt $\pi^{(h)}$ such that

$$(4.13) \qquad \pi(s) = \pi(s/m^h)\,\pi^{(h)}(s) \qquad [\,h > 0\,];$$

note that $1/m^h$ here takes *all* values in $(0,1)$ when h varies in $(0,\infty)$. In a similar way one also finds a necessary and sufficient condition for W to be not degenerate at zero. We summarize the results in the following theorem.

Theorem 4.1. *Let $Z(\cdot)$ be a continuous-time supercritical $\mathcal{F}$-branching process on $\mathbb{Z}_+$ with $Z(0) \equiv 1$; let $m := \mathbb{E}\,Z(1)$. Then*

$$Z(t)/m^t \xrightarrow{\text{d}} W \qquad [\,t \to \infty\,],$$

with W $\mathbb{R}_+$-valued, and if $\mathbb{P}\big(Z(1) = 0\big) = 0$, then W is self-decomposable and hence infinitely divisible. Moreover, W is not degenerate at zero iff $\mathbb{E}\,Z(1)\log^+ Z(1) < \infty$, in which case W is $(0,\infty)$-valued.

Finally, we consider the (continuous-time) *subcritical* case where $m := F_1'(1) < 1$; we shall take $m = 1/\mathrm{e}$. We allow *immigration* according to an sii-process $Y(\cdot)$, i.e., according to a compound-Poisson process with, say, intensity $\lambda > 0$ and batch-sizes $B_1, B_2, \ldots$, which are supposed to be independent and distributed as B with distribution $(q_k)_{k\in\mathbb{Z}_+}$ and pgf Q. Denote the resulting *branching process with immigration* by $X(\cdot) = \big(X(t)\big)_{t\geq 0}$, and let P_t be the pgf of $X(t)$. In order to determine P_t we condition on the number $N(t)$ of arriving batches in $(0,t\,]$, and use the well-known fact that, given $N(t) = n$, the times at which the batches arrive, are distributed as the order statistics of n independent random variables $T_1, \ldots, T_n$ with a uniform distribution on $(0, t)$. Assuming that $X(0) \equiv 0$, we thus obtain

$$P_t(z) = \sum_{n=0}^{\infty} \mathbb{P}\big(N(t) = n\big)\, \mathbb{E}\, z^{Z_1(t-T_1)+\cdots+Z_n(t-T_n)},$$

where $Z_1(\cdot), \ldots, Z_n(\cdot)$ are independent $\mathcal{F}$-branching processes (without immigration) satisfying $Z_j(0) \overset{\mathrm{d}}{=} B_j$ for $j = 1, \ldots, n$. Since by (4.10), with $P_{Z(0)} = Q$, the pgf of $Z_j(t - T_j)$ is given by $\int_0^t Q \circ F_s \,\mathrm{d}s/t$ for all j, it follows that

$$\begin{aligned} P_t(z) &= \exp\Big[-\lambda \int_0^t \big\{1 - Q\big(F_s(z)\big)\big\}\, \mathrm{d}s\Big] = \\ &= \exp\Big[\int_0^t \log P_0\big(F_s(z)\big)\, \mathrm{d}s\Big], \end{aligned} \tag{4.14}$$

where P_0 is the pgf of $Y(1)$. Now, pgf's of this form we already encountered in Section V.8 on $\mathcal{F}$-self-decomposable distributions on $\mathbb{Z}_+$; from Theorem V.8.3 and Corollary V.8.4 it follows that $X(t) \xrightarrow{\mathrm{d}} X$ as $t \to \infty$ for some $\mathbb{Z}_+$-valued random variable X iff $\mathbb{E} \log^+ Y(1) < \infty$ or, equivalently (cf. Proposition A.4.2), iff the batch size has a finite logarithmic moment: $\mathbb{E} \log^+ B < \infty$, in which case the pgf P of X can be written as

$$\begin{aligned} P(z) &= \exp\Big[-\lambda \int_0^\infty \big\{1 - Q\big(F_s(z)\big)\big\}\, \mathrm{d}s\Big] = \\ &= \exp\Big[\int_0^\infty \log P_0\big(F_s(z)\big)\, \mathrm{d}s\Big]. \end{aligned} \tag{4.15}$$

So we can now state the following result on infinite divisibility in connection with $X(\cdot)$; here $\odot_{\mathcal{F}}$ is the 'shrinking' operator introduced in Section V.8.

Theorem 4.2. *Let $X(\cdot)$ be a continuous-time subcritical $\mathcal{F}$-branching process on $\mathbb{Z}_+$ with $X(0) \equiv 0$ and with immigration according to an sii-process on $\mathbb{Z}_+$. Then $X(t)$ is infinitely divisible for all t, and if*

$$X(t) \xrightarrow{\mathrm{d}} X \qquad [\,t \to \infty\,]$$

for some $\mathbb{Z}_+$-valued random variable X (which is the case iff the batch size has a finite logarithmic moment), then X is $\mathcal{F}$-self-decomposable (and hence infinitely divisible):

$$X \overset{\mathrm{d}}{=} e^{-t} \odot_{\mathcal{F}} X + X(t) \qquad [\,t > 0\,], \tag{4.16}$$

where in the right-hand side X and $X(t)$ are independent. Conversely, any $\mathcal{F}$-self-decomposable distribution on $\mathbb{Z}_+$ can be obtained as the limit distribution of such an $\mathcal{F}$-branching process with immigration.

Consider the special case where the semigroup $\mathcal{F} = (F_t)_{t\geq 0}$ is given by $F_t(z) = 1 - \mathrm{e}^{-t}\,(1-z)$. Then the branching process $X(\cdot)$ above is a *pure-death process* with immigration, and $X(t)$ can also be interpreted as the number of customers present at time t in an M/M/∞-queue with batch arrivals (both the interarrival time and the service time are exponential; cf. Section 3). Moreover, $\odot_{\mathcal{F}}$ is the closest analogue to ordinary multiplication, and $\mathcal{F}$-self-decomposability reduces to self-decomposability as considered in Section V.4. Therefore, $X := \mathrm{d}\text{-}\lim_{t\to\infty} X(t)$ has a *unimodal* distribution; cf. Theorem V.4.20. For further special cases we refer to Section 7.

There is an analogue of Theorem 4.2 for branching processes with *continuous* state space $\mathbb{R}_+$. In such a process branching is governed by a continuous composition semigroup $\mathcal{C} = (C_t)_{t\geq 0}$ of cumulant generating functions: $C_t = -\log \eta_t$, where η_t is an *infinitely divisible* pLSt. If there is no immigration, this means that in stead of (4.10) we have

$$\pi_{Z(t)}(s) = \pi_{Z(0)}\big(C_t(s)\big). \tag{4.17}$$

Allowing immigration in the subcritical case where $-\eta_1'(0) < 1$, one obtains limit distributions on $\mathbb{R}_+$ that are $\mathcal{C}$*-self-decomposable*, and hence *infinitely divisible*; cf. the end of Section V.8. The special case where $\mathcal{C}$ is given by $C_t(s) = \mathrm{e}^{-t}s$ for $t \geq 0$, corresponds to ordinary multiplication of $\mathbb{R}_+$-valued random variables and therefore yields limit distributions that are classically self-decomposable, and hence are *unimodal*; cf. Theorem V.2.17. Of course, for the same reason the limit W in Theorem 4.1 has a unimodal distribution.

5. Renewal processes

The results to be given here are somewhat different from those in the preceding sections; we do not prove infinite divisibility of interesting quantities such as limit distributions, but we require infinite divisibility of the underlying lifetime distribution and present some interesting consequences of this assumption. We shall mainly deal with lifetime distributions that are non-lattice; about the lattice case we will be brief and consider only aperiodic distributions on $\mathbb{Z}_+$ with positive mass at zero.

We start with a brief description of the renewal context. Let $X_1, X_2, \ldots$ be independent nonnegative random variables distributed as X with distribution function F and mean $\mu \in (0, \infty)$. Define $S_n := X_1 + \cdots + X_n$ for $n \in \mathbb{N}$, and set $S_0 :\equiv 0$; the process $(S_n)_{n \in \mathbb{Z}_+}$ can be viewed as a *random walk* or as a discrete-time *sii-process* on $\mathbb{R}_+$. The X_i are interpreted as the lifetimes of items that are replaced (renewed) immediately after they break down; S_n is then the n-th renewal epoch. We are interested in the number of renewals in $[0, x]$ and define

$$N(x) := \#\big(\{n \in \mathbb{N} : S_n \le x\}\big) \qquad [\,x \ge 0\,];$$

from the strong law of large numbers it follows that $N(x) < \infty$ a.s. for all x. The counting process $N(\cdot) = \big(N(x)\big)_{x \ge 0}$ is called the *renewal process* generated by X (or by F). Sometimes one also counts S_0 as a renewal epoch and then considers

$$N_0(x) := 1 + N(x) = \#\big(\{n \in \mathbb{Z}_+ : S_n \le x\}\big) =$$

$$= \inf\{n \in \mathbb{N} : S_n > x\} =: T_x,$$

where T_x can be interpreted as the time at which the random walk (S_n) jumps over level x. One might ask whether $N(x)$ or, equivalently, T_x is infinitely divisible. Of course, for exponential lifetimes this is the case, since then $N(\cdot)$ is an sii-process (and, in fact, a Poisson process). For lifetimes that are gamma (2), however, one can show that $N(x)$ is not infinitely divisible; see Section 7.

An important quantity is the *renewal function* generated by F; this is the (finite) function U on $\mathbb{R}$, with $U(x) = 0$ for $x < 0$, on $\mathbb{R}_+$ defined by and with LSt $\widehat{U}$ given by

$$(5.1) \qquad U(x) := \mathbb{E}\, N(x) = \sum_{n=1}^{\infty} F^{\star n}(x) \ \text{ for } x \ge 0, \quad \widehat{U}(s) = \frac{\widehat{F}(s)}{1 - \widehat{F}(s)}.$$

We start with a curiosity. The famous *renewal theorem* of Blackwell states that if the lifetime X is non-lattice then

$$\lim_{x \to \infty} \big\{U(x+h) - U(x)\big\} = \frac{h}{\mu} \ \text{ for } h > 0, \ \text{ so } \lim_{x \to \infty} \frac{U(x)}{x} = \frac{1}{\mu}.$$

Clearly, we may replace U here by the function U_0 defined by

$$(5.2) \qquad U_0(x) := \mathbb{E}\, N_0(x) = 1 + U(x) = \sum_{n=0}^{\infty} F^{\star n}(x) \qquad [\, x \geq 0\,],$$

where $F^{\star 0} := 1_{[0,\infty)}$, and $U_0(x) := 0$ for $x < 0$, of course. Now, an easy computation shows that the LSt $\widehat{U}_0$ of U_0 can be written in the form

$$\widehat{U}_0(s) = \frac{1}{1 - \widehat{F}(s)} = \frac{\widehat{G}(s)}{1 - \widehat{G}(s)}, \quad \text{with } \widehat{G}(s) = \frac{1}{2 - \widehat{F}(s)}.$$

In view of (5.1) we can now formulate the following result.

Proposition 5.1. *If U is the renewal function generated by F, then the function U_0 in (5.2) can be viewed as the renewal function, in the sense of (5.1), generated by a compound-geometric distribution function G, say, with $\widehat{G}(s) = 1/\{2 - \widehat{F}(s)\}$.*

This means that Blackwell's theorem needs to be proved only for *compound-geometric* lifetimes, and hence only for *infinitely divisible* ones.

The somewhat more general *compound-exponential* lifetimes yield a second curiosity on renewal functions. Let U be the renewal function generated by F, and let $c > 0$. Then using (5.1) twice, one easily verifies that $c\,U$ is a renewal function iff the function π_c defined by

$$(5.3) \qquad \pi_c(s) := \frac{c\,\widehat{F}(s)}{1 - (1-c)\,\widehat{F}(s)},$$

is a pLSt. Clearly, π_c is a pLSt if $c \leq 1$. Now, using a 'self-decomposability' characterization of compound-exponentiality similar to Theorem III.6.3, one can show that π_c is a pLSt for *all* $c > 0$ iff F is compound-exponential. Thus we arrive at the following result.

Proposition 5.2. *Let U be the renewal function generated by F. Then $c\,U$ is a renewal function for all $c \in (0,1\,]$. Moreover, $c\,U$ is a renewal function for all $c > 0$ iff F is compound-exponential, in which case $c\,U$ is generated by F_c with $\widehat{F}_c = \pi_c$ as in (5.3).*

Next we turn to some interesting quantities in connection with the *inspection paradox*. At a fixed time $x > 0$ we inspect the item in service, and define V_x as the *spent lifetime* of that item, W_x as the *remaining lifetime*, and Z_x as the *total lifetime*. More precisely, we have (see Figure 1)

$$V_x := x - S_{N(x)}, \; W_x := S_{N(x)+1} - x, \; Z_x := V_x + W_x = X_{N(x)+1}.$$

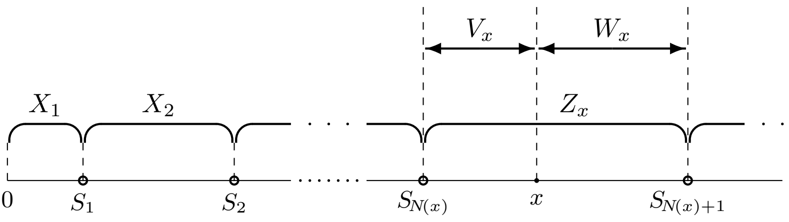

Figure 1. Quantities in connection with the inspection paradox.

We let $x \to \infty$, and, until otherwise stated, we take the ordinary lifetime X *non-lattice*, so there is no $h > 0$ such that $X \in h\,\mathbb{Z}_+$ a.s. It is well known that then, by Blackwell's theorem, the following limit result holds.

Lemma 5.3. *The quantities* (V_x, W_x) *and* Z_x *converge in distribution:*

$$(V_x, W_x) \xrightarrow{\text{d}} (V, W), \quad Z_x \xrightarrow{\text{d}} Z \qquad [\,x \to \infty\,],$$

where the limits (V, W) *and* Z *have distributions determined by*

$$(5.4) \qquad \mathbb{P}(V > v;\, W > w) = \frac{1}{\mu} \int_{v+w}^{\infty} \big\{1 - F(t)\big\}\, dt \qquad [\,v \geq 0,\, w \geq 0\,],$$

$$(5.5) \qquad \mathbb{P}(Z > z) = \frac{1}{\mu} \int_{(z,\infty)} t\, dF(t) \qquad [\,z \geq 0\,].$$

Moreover, $V \stackrel{\text{d}}{=} W$*, and* (V, W) *and* Z *determine each other:*

$$(5.6) \qquad Z \stackrel{\text{d}}{=} V + W, \quad (V, W) \stackrel{\text{d}}{=} \big((1-U)Z, UZ\big),$$

where U *is uniform on* $(0, 1)$ *and independent of* Z.

Let us consider the limiting total lifetime Z. Using (5.5) one can show that $Z \geq X$ in distribution, i.e., $\mathbb{P}(Z > z) \geq \mathbb{P}(X > z) = 1 - F(z)$ for all z. This implies that on an appropriate sample space we have

$$(5.7) \qquad Z \stackrel{\text{d}}{=} X + Y,$$

where Y is nonnegative and, in general, not independent of X. Now the question arises for what lifetimes X we have (5.7) with X and Y *independent*. In terms of distribution functions we look for those F for which there exists $Y \geq 0$ such that $F_Z = F \star F_Y$ or by (5.5)

$$\int_{[0,z]} t\, \mathrm{d}F(t) = \int_{[0,z]} F(z - y)\, \mathrm{d}K(y) \qquad [\,z \geq 0\,],$$

with $K = \mu F_Y$. Since this is precisely the functional equation occurring in Theorem III.4.10, and by (III.7.2) the canonical function K of an infinitely divisible X satisfies $\lim_{x\to\infty} K(x) = \mathbb{E}X$, we arrive at the following result; it can also be read as a *characterization* of infinitely divisible (non-lattice) distributions on $\mathbb{R}_+$ with finite mean.

Theorem 5.4. *Let Z be the limiting total lifetime in a renewal process generated by X with mean μ. Then Z can be decomposed as*

$$Z \stackrel{d}{=} X + Y, \quad \text{with } X \text{ and } Y \geq 0 \text{ independent}, \tag{5.8}$$

iff X is infinitely divisible. The distribution function of Y is then given by $F_Y = K/\mu$ with K the canonical function of X.

Let the ordinary lifetime X be infinitely divisible with canonical function K and mean μ. Then X, and the discrete-time sii-process (S_n) generated by X, can be embedded in a continuous-time sii-process $X(\cdot)$; so $X \stackrel{d}{=} X(1)$ and $S_n \stackrel{d}{=} X(n)$ for all n. In order to determine how the limiting total lifetime depends on the ordinary lifetime, we replace X in Theorem 5.4 by $X(a)$ with $a > 0$ and conclude that the corresponding limiting total lifetime $Z(a)$, say, satisfies

$$Z(a) \stackrel{d}{=} X(a) + Y(a), \quad \text{with } X(a) \text{ and } Y(a) \geq 0 \text{ independent}.$$

Now, the question is what $Y(a)$ will be. The answer is surprising at first sight. Since by Proposition III.4.5 (i) $X(a)$ has canonical function $a\,K$ and $\mathbb{E}\,X(a) = a\,\mu$, the distribution function of $Y(a)$ is given by

$$F_{Y(a)} = \frac{1}{\mathbb{E}\,X(a)}\,K_{X(a)} = \frac{1}{a\,\mu}\,a\,K = \frac{1}{\mu}\,K,$$

which does *not* depend on a. We summarize.

Theorem 5.5. *Let X be nonnegative and infinitely divisible with canonical function K and mean μ, and let $X(\cdot)$ be the sii-process generated by X. Then for $a > 0$ the limiting total lifetime $Z(a)$ in a renewal process generated by $X(a)$ can be decomposed as*

$$Z(a) \stackrel{d}{=} X(a) + Y, \quad \text{with } X(a) \text{ and } Y \geq 0 \text{ independent}, \tag{5.9}$$

and the distribution function of Y is given by $F_Y = K/\mu$.

This result has two interesting implications. First, letting $a \downarrow 0$ we have $X(a) \xrightarrow{\text{d}} 0$, and one might expect that then also $Z(a) \xrightarrow{\text{d}} 0$. From (5.9) we see, however, that this is *not* the case: $Z(a) \xrightarrow{\text{d}} Y$ as $a \downarrow 0$. So we have obtained an *extreme case of the inspection paradox*: For small a the total lifetime $Z(a)$ is very much larger than the ordinary lifetime $X(a)$. Next, recall that, in an obvious notation, $Z(a)$ can be obtained as d-$\lim_{x\to\infty} Z_x(a)$; cf. Lemma 5.3. So we have

$$\text{(5.10)} \qquad \operatorname*{d\text{-}lim}_{a\downarrow 0}\, \operatorname*{d\text{-}lim}_{x\to\infty} Z_x(a) = Y.$$

Now, formally, interchange the two limit operations in (5.10); let first $a \downarrow 0$. Since $Z_x(a)$ can be viewed as the height of the jump by which the random walk $\big(X(na)\big)_{n\in\mathbb{Z}_+}$ passes x, and this random walk seems to approximate $X(\cdot)$ well for a small, one may expect that

$$\text{(5.11)} \qquad \operatorname*{d\text{-}lim}_{a\downarrow 0} Z_x(a) = Z^{(x)} \qquad [x > 0],$$

where $Z^{(x)} := X(T_x) - X(T_x-)$, with $T_x := \inf\{t > 0 : X(t) > x\}$, is the height of the *jump* by which the sii-process $X(\cdot)$ passes x. Similarly, one may expect that the spent lifetime $V_x(a)$ and the remaining lifetime $W_x(a)$ at x in a renewal process generated by $X(a)$ satisfy

$$\text{(5.12)} \qquad \operatorname*{d\text{-}lim}_{a\downarrow 0} \big(V_x(a), W_x(a)\big) = (V^{(x)}, W^{(x)}) \qquad [x > 0],$$

where $V^{(x)} := x - X(T_x-)$ and $W^{(x)} := X(T_x) - x$ are the lengths of, respectively, the *undershoot* and the *overshoot* of $X(\cdot)$ at x. It is not clear, though, that these two d-limits exist. Further, from (5.10) and the final assertion in (5.6) it follows that

$$\text{(5.13)} \qquad \operatorname*{d\text{-}lim}_{a\downarrow 0}\, \operatorname*{d\text{-}lim}_{x\to\infty} \big(V_x(a), W_x(a)\big) = \big((1-U)Y, UY\big),$$

where U is uniform on $(0,1)$ and independent of Y. Now, letting $x \to \infty$ in the right-hand sides of (5.11) and (5.12) one may hope to obtain the same limits as in (5.10) and (5.13), respectively. This is indeed the case, as one can prove by using Blackwell's theorem and some general results on sii-processes; see Notes. So we have the following limit theorem.

Theorem 5.6. *Let $X(\cdot)$ be an sii-process on (a non-lattice part of) $\mathbb{R}_+$ with $\mu := \mathbb{E}\, X(1) \in (0,\infty)$. Then the jump $Z^{(x)}$, the undershoot $V^{(x)}$ and the overshoot $W^{(x)}$ of $X(\cdot)$ at x satisfy*

$$\text{(5.14)} \qquad (V^{(x)}, W^{(x)}) \xrightarrow{\text{d}} \big((1-U)Y, UY\big), \quad Z^{(x)} \xrightarrow{\text{d}} Y \qquad [x \to \infty],$$

where U is uniform on $(0,1)$, independent of Y, and Y has distribution function $F_Y = K/\mu$ with K the canonical function of $X(1)$.

We return to the decomposition $Z \stackrel{d}{=} X + Y$ of the limiting total lifetime Z as given by (5.8): Z has the ordinary lifetime X as an independent component. We now wonder whether there are (infinitely divisible) lifetimes X for which the corresponding Z has *two* ordinary lifetimes as independent components: $Z \stackrel{d}{=} X + X' + Y_0$ with $X' \stackrel{d}{=} X$ and X, X' and $Y_0 \geq 0$ independent. By (5.5) this decomposition of Z amounts to the following requirement for F:

$$\int_{[0,z]} t\,\mathrm{d}F(t) = \int_{[0,z]} F^{\star 2}(z-y)\,\mathrm{d}K_0(y) \qquad [\,z \geq 0\,],$$

with $K_0 = \mu F_{Y_0}$. Since this is precisely the functional equation occurring in Theorem III.5.3, only *compound-exponential* lifetimes X may yield the desired decomposition of Z. On the other hand, recall from Section III.5 that if X is compound-exponential, then F satisfies the functional equation above with K_0 the canonical function of the underlying (infinitely divisible) distribution of X, i.e., $\widehat{K}_0 = \rho_0$ with

$$\rho_0(s) := \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{\widehat{F}(s)} = \frac{-\widehat{F}'(s)}{\{\widehat{F}(s)\}^2};$$

letting here $s \downarrow 0$ shows that $\lim_{x\to\infty} K_0(x) = \mathbb{E}\,X$. Hence we can reverse matters, and state the following extension of Theorem 5.4; it can also be read as a *characterization* of compound-exponential (non-lattice) distributions on $\mathbb{R}_+$ with finite mean.

Theorem 5.7. *Let Z be the limiting total lifetime in a renewal process generated by X with mean μ; let $X' \stackrel{d}{=} X$. Then Z can be decomposed as*

(5.15) $\qquad Z \stackrel{d}{=} X + X' + Y_0,$ *with X, X' and $Y_0 \geq 0$ independent,*

iff X is compound-exponential. The distribution function of Y_0 is then given by $F_{Y_0} = K_0/\mu$ with K_0 the canonical function of the underlying distribution of X.

If X is compound-exponential, then so is $X(a)$ for every $a \in (0,1]$; here $X(\cdot)$ is the sii-process generated by X. This follows by use of the criterion in

Theorem III.5.1 and the easily verified fact that the ρ_0-function of $X(a)$ is related to that of X by

$$\rho_{a,0}(s) = a\,\{\widehat{F}(s)\}^{1-a}\,\rho_0(s).$$

Hence one can apply Theorem 5.7 with X replaced by $X(a)$. Since, in an obvious notation, the resulting component $Y_0(a)$ satisfies

$$\widehat{F}_{Y_0(a)}(s) = \frac{1}{a\mu}\,\widehat{K}_{a,0}(s) = \frac{1}{a\mu}\,\rho_{a,0}(s) = \{\widehat{F}(s)\}^{1-a}\,\widehat{F}_{Y_0}(s),$$

this leads to the following counterpart to Theorem 5.5; note that $Z(\frac{1}{2})$ has *three* ordinary lifetimes as independent components.

Theorem 5.8. *Let X be nonnegative and compound-exponential with finite mean, and let $X(\cdot)$ be the sii-process generated by X. Then for $a \in (0,1]$ the limiting total lifetime $Z(a)$ in a renewal process generated by $X(a)$ can be decomposed as*

$$Z(a) \stackrel{d}{=} X(a) + X'(a) + X''(1-a) + Y_0, \tag{5.16}$$

where all random variables in the right-hand side are independent, $X'(a) \stackrel{d}{=} X(a)$, $X''(1-a) \stackrel{d}{=} X(1-a)$, and Y_0 is determined by X as in Theorem 5.7.

We make a brief remark concerning the limiting spent lifetime V and the limiting remaining lifetime W, for which $Z \stackrel{d}{=} V + W$ with $V \stackrel{d}{=} W$; cf. Lemma 5.3. If Z can be decomposed as in (5.15), then it can also be written as

$$Z \stackrel{d}{=} \{X + (1-U)Y_0\} + \{X' + UY_0\}$$

with U uniform on $(0,1)$, so with $X + (1-U)Y_0 \stackrel{d}{=} X' + UY_0$. Therefore, one might wonder whether a decomposition similar to (5.8) then holds for V and W:

$$V \stackrel{d}{=} W \stackrel{d}{=} X + UY_0, \quad \text{with } X, U \text{ and } Y_0 \geq 0 \text{ independent.} \tag{5.17}$$

Now, by use of (5.4) and Theorem A.3.13 it can be shown indeed that this decomposition holds iff X is compound-exponential. We do not give the details, and note that it is not known whether X is compound-exponential if one only requires that W can be decomposed as $W \stackrel{d}{=} X + B$ with X and $B \geq 0$ independent.

So far, we restricted attention to the non-lattice case. Now, we briefly consider *lattice* lifetimes X, and suppose (this is no essential restriction) that X is $\mathbb{Z}_+$-valued with *aperiodic* distribution $(p_k)_{k\in\mathbb{Z}_+}$; cf. Section II.8. We inspect the item in service at a fixed time k with $k\in\mathbb{N}$, and consider the *spent lifetime* V_k, the *remaining lifetime* W_k and the *total lifetime* Z_k at k, for $k\to\infty$. Then the following well-known limit result holds; here and in the sequel we use without further comment the notations

$$\overline{W} := W-1, \quad \overline{Z} := Z-1,$$

which random variables are more convenient to look at than the $\mathbb{N}$-valued quantities W and Z.

Lemma 5.9. *The quantities (V_k, W_k) and Z_k converge in distribution:*

$$(V_k, W_k) \xrightarrow{\mathrm{d}} (V, W), \quad Z_k \xrightarrow{\mathrm{d}} Z \qquad [\,k\to\infty\,],$$

where the limits (V, W) and Z have distributions determined by

$$\text{(5.18)} \qquad \mathbb{P}(V=i;\, \overline{W}=j) = \frac{1}{\mu}\, p_{i+j+1} \qquad [\,i,j\in\mathbb{Z}_+\,],$$

$$\text{(5.19)} \qquad \mathbb{P}(\overline{Z}=n) = \frac{1}{\mu}\,(n+1)\, p_{n+1} \qquad [\,n\in\mathbb{Z}_+\,].$$

Moreover, $V \overset{\mathrm{d}}{=} \overline{W}$, and (V, W) and Z determine each other:

$$\text{(5.20)} \qquad \overline{Z} \overset{\mathrm{d}}{=} V+\overline{W}, \quad (V, \overline{W}) \overset{\mathrm{d}}{=} \big((1-U)\odot \overline{Z},\, U\odot \overline{Z}\big),$$

where U is uniform on $(0,1)$ and independent of $\overline{Z}$, and $\odot$ is the standard discrete multiplication from Section A.4.

Clearly, the distribution of the limiting total lifetime Z as determined by (5.19) can be put in the form (5.5). Therefore, Theorems 5.4 and 5.7 remain true if the (tacitly assumed) non-lattice condition there is replaced by our aperiodicity condition on (p_k). When $p_0>0$, however, these theorems can be formulated and proved by only using quantities and results from the discrete case considered in Chapter II. Recall that the condition $p_0>0$ is automatically satisfied if (p_k) is compound-exponential or, equivalently, compound-geometric; cf. Theorem II.3.6.

Theorem 5.10. *Let Z be the limiting total lifetime in a renewal process generated by X with mean μ and aperiodic distribution $(p_k)_{k\in\mathbb{Z}_+}$ with $p_0>0$; let $X' \overset{\mathrm{d}}{=} X$. Then:*

(i) $\overline{Z} \stackrel{d}{=} X + Y$, *with* X *and* Y *independent, iff* X *is infinitely divisible. The distribution of* Y *is then given by* $(r_k/\mu)_{k\in\mathbb{Z}_+}$ *where* (r_k) *is the canonical sequence of* X.

(ii) $\overline{Z} \stackrel{d}{=} X + X' + Y_0$, *with* X, X' *and* Y_0 *independent, iff* X *is compound-exponential. The distribution of* Y_0 *is then given by* $(r_{0,k}/\mu)_{k\in\mathbb{Z}_+}$ *where* $(r_{0,k})$ *is the canonical sequence of the underlying (infinitely divisible) distribution of* X.

PROOF. From (5.19) it follows that $\overline{Z}$ can be decomposed as in (i) iff (p_k) satisfies the recurrence relations

$$(n+1)\, p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \qquad [\, n \in \mathbb{Z}_+ \,]$$

with $r_k = \mu\, \mathbb{P}(Y = k)$ for $k \in \mathbb{Z}_+$. From Theorem II.4.4 and the fact, stated in (II.7.2), that the canonical sequence (r_k) of an infinitely divisible X satisfies $\sum_{k=0}^{\infty} r_k = \mathbb{E}X$, we now conclude that part (i) holds. Part (ii) is proved similarly; $\overline{Z}$ can be decomposed as in (ii) iff (p_k) satisfies

$$(n+1)\, p_{n+1} = \sum_{k=0}^{n} p_k^{*2}\, r_{0,n-k} \qquad [\, n \in \mathbb{Z}_+ \,]$$

with $r_{0,k} = \mu\, \mathbb{P}(Y_0 = k)$ for $k \in \mathbb{Z}_+$. Now use Theorem II.5.8, and observe that if X is compound-exponential, then the sequence $(r_{0,k})$ satisfies $\sum_{k=0}^{\infty} r_{0,k} = \mathbb{E}X$; just let $z \uparrow 1$ in (II.5.13). □

Of course, also Theorems 5.5 and 5.8 and their proofs have lattice analogues; we don't spell them out. The distribution of (V, W) as determined by (5.18) can*not* be put in the form (5.4); in the latter case both V and W have absolutely continuous distributions. So there is no immediate extension of the decomposition result around (5.17) to our situation. Nevertheless, there exists a discrete counterpart, suggested by Theorem 5.10 (ii) and (5.20); by use of (5.18) and Theorems A.4.11 and A.4.12 it can be shown that

$$(5.21) \qquad V \stackrel{d}{=} \overline{W} \stackrel{d}{=} X + U \odot Y_0, \quad \text{with } X,\ U \text{ and } Y_0 \text{ independent},$$

iff X is compound-exponential. Again, we do not give the details, and note that it is not known whether X is compound-exponential if one only requires that $\overline{W}$ can be decomposed as $\overline{W} \stackrel{d}{=} X + B$ with X and B independent.

To conclude this section we say a few words on infinite divisibility of the limiting total lifetime and the limiting remaining lifetime. Consider first the non-lattice case, and suppose that the ordinary lifetime X has an absolutely continuous distribution function F with density f (and finite mean μ). Then according to Lemma 5.3 the random variables Z and W have densities given by

$$(5.22) \qquad f_Z(x) = \frac{1}{\mu}\, x\, f(x), \quad f_W(x) = \frac{1}{\mu}\,\{1 - F(x)\} \qquad [\,x > 0\,].$$

Even if X is infinitely divisible, then Z and W need *not* be infinitely divisible. Sufficient conditions can be found in Proposition III.10.10, Corollary VI.4.6 and Proposition VI.5.4, and counter-examples are presented in Examples VI.12.1 and VI.12.6. Turning to the lattice case, we suppose that X is $\mathbb{Z}_+$-valued with aperiodic distribution $(p_k)_{k\in\mathbb{Z}_+}$. Then according to Lemma 5.9 the random variables $\overline{Z}$ and $\overline{W}$ have distributions given by

$$(5.23) \qquad \mathbb{P}(\overline{Z} = k) = \frac{1}{\mu}\,(k+1)\,p_{k+1}, \;\; \mathbb{P}(\overline{W} = k) = \frac{1}{\mu}\sum_{j=k+1}^{\infty} p_j \quad [\,k \in \mathbb{Z}_+\,].$$

Again, $\overline{Z}$ and $\overline{W}$ need *not* be infinitely divisible, but there are sufficient conditions; see Proposition II.10.7, Theorem VI.7.4 and Proposition VI.8.5.

6. Shot noise

A much used stochastic process in physics, finance and queueing theory is the so-called *shot noise* or *shot effect process* $X(\cdot)$, which occurs in the literature in various degrees of generality. One of the more general forms is the one that is also called a *filtered Poisson process*:

$$(6.1) \qquad X(t) = \sum_{k=1}^{N(t)} h(t, S_k, C_k);$$

here the S_k are the points in a Poisson process $N(\cdot)$, and the C_k are mutually independent, independent of the Poisson process, and distributed as C. The quantity $h(t, s, c)$ is interpreted as the effect at time t of a 'shot' of intensity c fired at time s. In a much used special case of (6.1) one takes $h(t, s, c) = c\, h_0(t - s)$, which leads to

$$(6.2) \qquad X(t) = \sum_{k=1}^{N(t)} C_k\, h_0(t - S_k).$$

The characteristic function of $X(t)$ in (6.1) can be seen to equal

$$\phi_{X(t)}(u) = \exp\Big[\lambda \int_0^t \big\{\mathbb{E}\,\mathrm{e}^{iu\,h(t,s,C)} - 1\big\}\,\mathrm{d}s\Big], \tag{6.3}$$

where λ is the intensity of the Poisson process. In fact, one can argue as in the computation of the pgf P_t of $X(t)$ in Section 4 as follows. Let $U_1, \ldots, U_n$ form a sample from a uniform distribution on $(0,t)$; then $\phi_{X(t)}$ can be written as

$$\begin{aligned}\phi_{X(t)}(u) &= \sum_{n=0}^{\infty} \mathbb{P}(N(t)=n)\,\mathbb{E}\exp\Big[iu\sum_{k=1}^{n} h(t,U_k,C_k)\Big] = \\ &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}\,\mathrm{e}^{-\lambda t}\Big(\frac{1}{t}\int_0^t \mathbb{E}\,\mathrm{e}^{iu\,h(t,s,C)}\,\mathrm{d}s\Big)^n,\end{aligned}$$

which is equal to the right-hand side of (6.3). Since λ may have any positive value, it follows that $X(t)$ is *infinitely divisible.* The argument can be generalized to time-dependent Poisson processes, and also more-dimensional generalizations have been considered; we shall not pursue this. The characteristic function of $X(t)$ in (6.2) reads

$$\phi_{X(t)}(u) = \exp\Big[\lambda \int_0^t \big\{\mathbb{E}\,\mathrm{e}^{iu\,C\,h_0(x)} - 1\big\}\,\mathrm{d}x\Big]. \tag{6.4}$$

Special choices for h_0 or C in (6.4) lead to various well-known subclasses of infinitely divisible distributions. As an example we choose $h_0(x) = \mathrm{e}^{-\rho x}$ for some $\rho > 0$, and after a little algebra we obtain

$$\phi_{X(t)}(u) = \exp\Big[\frac{\lambda}{\rho}\int_{u\mathrm{e}^{-\rho t}}^{u} \frac{\phi_C(v)-1}{v}\,\mathrm{d}v\Big].$$

This means that the characteristic function of $X := \text{d-lim}_{t\to\infty} X(t)$, if this limit exists, has precisely the form (V.6.8), from which it follows that X is *self-decomposable.* For references to other examples see Notes.

7. Examples

Here we examine a few special cases of the stochastic processes considered in the preceding sections, and obtain in this way some new infinitely divisible distributions or get a new perspective on some old ones. Also a few illustrative counter-examples are presented.

We start with considering a continuous-time analogue of the introductory example in Section 2, i.e., the Bernoulli walk with parameter $p \geq \frac{1}{2}$.

Example 7.1. Let $X(\cdot)$ be a Markov chain on $\mathbb{Z}$ as in Theorem 2.2, and suppose that the sojourn parameters λ_j are all equal to one and that the transition matrix (p_{jk}) is given by $p_{j,j+1} = p$ and $p_{j,j-1} = q := 1-p$, for some $p \geq \frac{1}{2}$; note that indeed the chain is skipfree to the right. Then the first-passage time $T_{0,1}$ from 0 to 1 is not defective, and by (2.8) its pLSt π satisfies

$$\pi(s) = \frac{1}{1+s}\,\frac{p}{1 - q\,\{1/(1+s)\}\,\pi(s)} = \frac{p}{1+s-q\,\pi(s)}.$$

Solving this equation for π, we obtain two solutions, only one of which is a pLSt:

$$\pi(s) = \frac{(1+s) - \sqrt{(1+s)^2 - 4pq}}{2q} = \frac{2p}{(1+s) + \sqrt{(1+s)^2 - 4pq}}. \tag{7.1}$$

From Theorem 2.2 we conclude that π is *compound-exponential*, and hence *infinitely divisible*. We can say somewhat more. Let I_r be the modified Bessel function of the first kind of order r; cf. Section A.5. In Example III.11.9 we used I_1 to prove that in case $p = \frac{1}{2}$ the function π is the Lt of a completely monotone density. Now, this density can be used to obtain a density f also when $p > \frac{1}{2}$; just note that π can be considered as a function of $1+s$, and then change the scale by a factor $2\sqrt{pq}$. Thus we find

$$f(x) = \sqrt{p/q}\,\frac{1}{x}\,\mathrm{e}^{-x}\,I_1(2\sqrt{pq}\,x) \qquad [\,x>0\,], \tag{7.2}$$

and f is *completely monotone*. In a similar way one uses Example III.11.9 to show that the corresponding canonical density k is *completely monotone*; the Lt ρ of k, which is the ρ-function of π, and k are given by

$$\rho(s) = \frac{1}{\sqrt{(1+s)^2 - 4pq}}, \qquad k(x) = \mathrm{e}^{-x}\,I_0(2\sqrt{pq}\,x) \ \text{ for } x>0.$$

An alternative way of showing that π has a completely monotone canonical density k, is observing that, in case $p > \frac{1}{2}$, the ρ-function of π can be written as

$$\rho(s) = c_p\,\Big(\frac{\lambda_1}{\lambda_1+s}\Big)^{\frac{1}{2}}\Big(\frac{\lambda_2}{\lambda_2+s}\Big)^{\frac{1}{2}},$$

where $c_p := 1/\sqrt{\lambda_1\lambda_2}$ and $\lambda_{1,2} := 1 \pm 2\sqrt{pq}$, and using the canonical representation of mixtures of exponential pLSt's as given by Theorem VI.3.5; since $s \mapsto \lambda_i/(\lambda_i+s)$ has the form (VI.3.14) with $v = v_i := 1_{(\lambda_i,\infty)}$, also

ρ/c_p has this form with $v = \frac{1}{2}(v_1 + v_2)$, which has its values in $[0,1]$. The complete monotonicity of k implies that π corresponds to a generalized gamma convolution, and hence is *self-decomposable*; cf. Theorem VI.5.3 and Proposition VI.5.5. In Example V.9.2 this self-decomposability, in case $p = \frac{1}{2}$, was proved more directly. The first-passage time $T_{0,k}$ from 0 to $k \in \mathbb{N}$ has pLSt π^k, and hence is *infinitely divisible*, and even *self-decomposable*, as well. It can be seen to have density f_k given by

$$f_k(x) = k\,\big(\sqrt{p/q}\,\big)^k \,\frac{1}{x}\, \mathrm{e}^{-x}\, I_k(2\sqrt{pq}\,x) \qquad [\,x > 0\,]. \tag{7.3}$$

Finally we look, for a fixed t, at the distribution of $X(t)$ itself. As is well known, $X(t)$ can be obtained as

$$X(t) \stackrel{\mathrm{d}}{=} N_1(p\,t) - N_2(q\,t),$$

where $N_1(\cdot)$ and $N_2(\cdot)$ are independent Poisson processes of rate one. It follows that $X(t)$ is *infinitely divisible*, and that its distribution is given by

$$\mathbb{P}\big(X(t) = k\big) = \mathrm{e}^{-t}\,\big(\sqrt{p/q}\,\big)^k\, I_{|k|}(2\sqrt{pq}\,t) \qquad [\,k \in \mathbb{Z}\,]. \tag{7.4}$$

Note that for $t > 0$ and $k \in \mathbb{N}$ we have $\mathbb{P}\big(X(t) = k\big) = (t/k)\, f_k(t)$ with f_k as in (7.3). ◻◻

The first-passage time $T_{0,1}$ for $X(\cdot)$ in this example has the same distribution as the absorption time $\widetilde{T}_{1,0}$ for the process $\widetilde{X}(\cdot)$ that is obtained from $X(\cdot)$ by making state 0 absorbing, interchanging p and q, and starting at 1. In a similar way one sees that Theorem 2.2 can also be applied to the *absorption time* for a *birth-death process* $X(\cdot)$ on $\mathbb{Z}_+$ with 0 as an absorbing state and with birth and death rates $\mu_1, \mu_2, \ldots$ and $\nu_1, \nu_2, \ldots$, say, satisfying $\mu_j + \nu_j > 0$ for all j; $X(\cdot)$ is a Markov chain on $\mathbb{Z}_+$ which on $\mathbb{N}$ behaves as before: For $j \in \mathbb{N}$ the sojourn parameter λ_j and the transition probability p_{jk} are given by

$$\lambda_j = \mu_j + \nu_j, \quad p_{jk} = \begin{cases} \mu_j/(\mu_j + \nu_j) & \text{, if } k = j+1, \\ \nu_j/(\mu_j + \nu_j) & \text{, if } k = j-1. \end{cases}$$

Using these relations in (2.8), adapted to this case, one sees that if $X(0) \equiv 1$ then the pLSt π of the absorption time $T := \inf\big\{t > 0 : X(t) = 0\big\}$, if not defective, can be written as

$$\pi(s) = \frac{\nu_1}{\mu_1 + \nu_1 + s - \mu_1 \widetilde{\pi}(s)}, \tag{7.5}$$

where $\widetilde{\pi}$ is the pLSt of the first-passage time from 2 to 1 if $\mu_1 > 0$. In special cases relation (7.5) leads to explicit expressions for π. We have seen this in Example 7.1; we now give another example.

Example 7.2. Let the *birth-death process* $X(\cdot)$, with $X(0) \equiv 1$, have the property that, for some $n \in \mathbb{N}$, $\mu_1 > 0, \ldots, \mu_{n-1} > 0$, $\mu_n = 0$ and $\nu_1 > 0, \ldots, \nu_n > 0$, so $X(\cdot)$ is a process on $\{0, 1, \ldots, n\}$. Let π_n denote the pLSt of the *absorption time* T; then by (7.5) we have

$$\text{(7.6)} \qquad \pi_1(s) = \frac{\nu_1}{\nu_1 + s}, \quad \pi_n(s) = \frac{\nu_1}{\mu_1 + \nu_1 + s - \mu_1\, \widetilde{\pi}_{n-1}(s)} \quad \text{for } n \geq 2,$$

where $\widetilde{\pi}_{n-1}$ is the pLSt of the absorption time in a birth-death process on the set $\{0, 1, \ldots, n-1\}$ started at one and with birth and death rates $\mu_2, \ldots, \mu_n$ and $\nu_2, \ldots, \nu_n$. Hence $\widetilde{\pi}_1(s) = \nu_2/(\nu_2 + s)$, so for π_2 we find

$$\text{(7.7)} \qquad \pi_2(s) = \frac{\nu_1(\nu_2 + s)}{\nu_1\nu_2 + (\mu_1 + \nu_1 + \nu_2)s + s^2}.$$

In turn, this formula for π_2 can be used to compute $\widetilde{\pi}_2$, which yields π_3. Proceeding in this way it is possible to compute π_n for every n; note that π_n, and hence $\widetilde{\pi}_n$, is of the form Q_{n-1}/P_n with Q_{n-1} and P_n polynomials of degree $n-1$ and n, respectively. From Theorem 2.2 we know that π_n is *compound-exponential*, but the construction above and the considerations in Section VI.3 suggest that π_n also corresponds to a *mixture of n exponential densities*, i.e., that π_n can be put in the form

$$\text{(7.8)} \qquad \pi_n(s) = \sum_{j=1}^{n} p_j \frac{a_j}{a_j + s},$$

with $a_j > 0$ and $p_j > 0$ for all j, and $\sum_{j=1}^n p_j = 1$. Indeed, this can be proved by induction as follows. The function π_2 can be written as

$$\pi_2(s) = \frac{a_1}{a_1 + s}\,\frac{a_2}{a_2 + s} \Big/ \frac{\nu_2}{\nu_2 + s},$$

for some a_1, a_2 with $0 < a_1 < \nu_2 < a_2$, so π_2 is infinitely divisible with canonical density $k_2(x) = \mathrm{e}^{-a_1 x} + \mathrm{e}^{-a_2 x} - \mathrm{e}^{-\nu_2 x}$ for $x > 0$; by Proposition VI.3.4 it follows that π_2 is a mixture of two exponential pLSt's (with parameters a_1 and a_2). By using (7.6) the induction step can be made similarly; we omit the details. Finally, we note that, conversely, every mixture of n exponential densities can be viewed as the density of the absorption time in a birth-death process on $\{0, 1, \ldots, n\}$ as above. We do not give a proof of this result; it is interesting to see, though, that in combination with Theorem 2.2 this leads to an alternative proof of Theorem VI.3.3. □□

Sometimes it is more convenient to look at the distribution function F of the absorption time T than at its pLSt. Since state 0 is absorbing, F can be written as

$$(7.9) \qquad F(t) = \mathbb{P}(T \le t) = \mathbb{P}\big(X(t) = 0\big) \qquad [\, t \ge 0\,].$$

Now, it is well known that the transition probability $p_k(t) := \mathbb{P}\big(X(t) = k\big)$ satisfies the following difference-differential equations for $t \ge 0$:

$$(7.10) \qquad \begin{cases} p_0'(t) = \nu_1\, p_1(t), \\ p_1'(t) = -(\mu_1 + \nu_1)\, p_1(t) + \nu_2\, p_2(t), \\ p_k'(t) = -(\mu_k + \nu_k)\, p_k(t) + \mu_{k-1}\, p_{k-1}(t) + \nu_{k+1}\, p_{k+1}(t), \end{cases}$$

for $k \ge 2$, with initial condition $p_k(0) = \delta_{k,n}$ if $X(0) \equiv n$ with $n \in \mathbb{N}$. By standard generating function techniques these equations can be solved in special cases; we give an example in which new infinitely divisible distributions appear.

Example 7.3. Let the *birth-death process* $X(\cdot)$ be *linear*, i.e., $\mu_j = j\,\mu$ and $\nu_j = j\,\nu$ for all j, and some $\mu > 0$ and $\nu > 0$. Then one finds for $t \ge 0$

$$(7.11) \qquad p_0(t) = \begin{cases} \Big(\dfrac{1 - \mathrm{e}^{-(\nu-\mu)\,t}}{1 - (\mu/\nu)\,\mathrm{e}^{-(\nu-\mu)\,t}}\Big)^n & \text{, if } \mu \ne \nu, \\ \Big(\dfrac{\mu t}{1 + \mu t}\Big)^n & \text{, if } \mu = \nu. \end{cases}$$

Note that $\lim_{t\to\infty} p_0(t) = 1$ iff $\mu \le \nu$. Now apply Theorem 2.2 and use (7.9); then the choice $\mu < \nu$ shows that for any $a > 0$, $b \in (0,1)$ and $n \in \mathbb{N}$ the (nondefective) distribution function F on $\mathbb{R}_+$ given by

$$(7.12) \qquad F(x) = \Big(\frac{1 - \mathrm{e}^{-ax}}{1 - b\,\mathrm{e}^{-ax}}\Big)^n \qquad [\, x \ge 0\,],$$

is *infinitely divisible*. By choosing $\mu = \nu$, or by putting $b = \mathrm{e}^{-a}$ in (7.12) and then letting $a \downarrow 0$, one sees that also the following distribution function F on $\mathbb{R}_+$ is *infinitely divisible* for any $n \in \mathbb{N}$:

$$(7.13) \qquad F(x) = \Big(\frac{x}{1 + x}\Big)^n \qquad [\, x \ge 0\,].$$

From Theorem 2.2 it also follows that in case $n = 1$ both (7.12) and (7.13) are *compound-exponential*. In fact, both have a *completely monotone* density because the function $1-F$ is easily seen to be completely monotone. The distribution in (7.13), for general n, is a special case of the *beta distribution of the second kind*, which in Example VI.12.9 was shown to be a *generalized gamma convolution* and hence *self-decomposable*. ◫

The preceding example shows how an interpretation in a stochastic process can be used to prove that a given distribution is infinitely divisible. The same is illustrated by the next example.

Example 7.4. Let $X(\cdot)$ be standard *Brownian motion* started at zero, and consider the *first-exit time* T from $(-1,1)$ for $X(\cdot)$. Is T infinitely divisible? Use the technique from the end of Section 2. Let S denote the first-exit time from $(-\frac{1}{2},\frac{1}{2})$. Then by the self-similarity, the strong Markov property and the space-homogeneity of $X(\cdot)$ it is seen that

$$S \stackrel{\mathrm{d}}{=} \tfrac{1}{4}T, \quad T \stackrel{\mathrm{d}}{=} S + S' + A\,T', \tag{7.14}$$

where all random variables in the right-hand side are independent, $S' \stackrel{\mathrm{d}}{=} S$, $T' \stackrel{\mathrm{d}}{=} T$ and A is uniform on $\{0,1\}$. It follows that the pLSt π of T satisfies

$$\pi(s) = \bigl\{\pi(\tfrac{1}{4}s)\bigr\}^2\,\tfrac{1}{2}\bigl\{1+\pi(s)\bigr\} \qquad [\,s \ge 0\,]. \tag{7.15}$$

To solve this functional equation we rewrite it by putting $\phi(x) := 1/\pi(x^2)$; extending the domain $\mathbb{R}_+$ of ϕ to $\mathbb{R}$ we get

$$\phi(x) = 2\,\bigl\{\phi(\tfrac{1}{2}x)\bigr\}^2 - 1 \qquad [\,x \in \mathbb{R}\,]. \tag{7.16}$$

Next we note that $\pi(0) = 1$ and that π has derivatives of all orders at zero; this follows from the fact that $\mathbb{P}(T > t) = O(\mathrm{e}^{-ct})$ as $t \to \infty$, for some $c > 0$; see Notes. Since $\phi''(0) = -2\,\pi'(0) = 2\,\mathbb{E}\,T$, we therefore look for all solutions ϕ of (7.16) having derivatives of all orders on $\mathbb{R}$, with $\phi(0) = 1$ and $\phi''(0) > 0$. Now, it is known (see Notes) that for every $\lambda \neq 0$ there exists exactly one such solution with $\phi''(0) = \lambda^2$ and that it is given by

$$\phi(x) = \cosh\lambda x = \tfrac{1}{2}\,(\mathrm{e}^{\lambda x} + \mathrm{e}^{-\lambda x}) \qquad [\,x \in \mathbb{R}\,]. \tag{7.17}$$

In our situation $\lambda = \sqrt{2}$, since it can be shown that $\mathbb{E}\,T = \mathbb{V}\mathrm{ar}\,X(T) = 1$. We conclude that (7.15) is solved by

$$\pi(s) = \frac{1}{\phi(\sqrt{s})} = \frac{1}{\cosh\sqrt{2s}} = \frac{2}{\mathrm{e}^{\sqrt{2s}} + \mathrm{e}^{-\sqrt{2s}}}. \tag{7.18}$$

Proving infinite divisibility of T, however, just from this expression for π, is not easy; also, the square-root is essential, as was shown in Example III.11.12. We therefore return to equation (7.15) for π. Putting

$$\pi_0(s) := \frac{1}{2 - \{\pi(s/4)\}^2},$$

which is the pLSt of a *compound-geometric*, and hence *infinitely divisible* distribution on $\mathbb{R}_+$, we see that π satisfies

$$(7.19) \qquad \pi(s) = \big\{\pi(\tfrac{1}{4}s)\big\}^2 \pi_0(s), \;\text{ so }\; \pi(s) = \big\{\pi(s/4^n)\big\}^{2^n} \prod_{j=0}^{n-1} \big\{\pi_0(s/4^j)\big\}^{2^j},$$

for every $n \in \mathbb{N}$. Since by the explicit expression (7.18) for π the first factor in the right-hand side tends to one as $n \to \infty$, π can be obtained as the limit of a sequence of infinitely divisible pLSt's, and hence is *infinitely divisible*, too.

It is much simpler, however, to prove that π is self-decomposable (and therefore infinitely divisible). To do so we return to T, and for $\alpha \in (0,1)$ we let $T_{\sqrt{\alpha}}$ be the first-exit time from $(-\sqrt{\alpha}, \sqrt{\alpha}\,)$. Then by the strong Markov property we have

$$(7.20) \qquad T \stackrel{\mathrm{d}}{=} T_{\sqrt{\alpha}} + T^{(\alpha)},$$

where $T^{(\alpha)}$ is nonnegative and independent of $T_{\sqrt{\alpha}}$. Now use again self-similarity: $a\,X(\cdot) \stackrel{\mathrm{d}}{=} X(a^2\,\cdot)$ for every $a > 0$; then it follows that $T_{\sqrt{\alpha}} \stackrel{\mathrm{d}}{=} \alpha\,T$, and T is *self-decomposable* by definition. □

Next we turn to the *G/G/1 queue*, as considered in Section 3, with interarrival time A and service time B such that the *traffic intensity* $\rho := \mathbb{E}B/\mathbb{E}A$ satisfies $\rho < 1$. We mention some special cases where something more can be said about the *infinitely divisible* limit W (in distribution) of the *waiting time* W_n of the n-th customer as $n \to \infty$; cf. Theorem 3.1.

Example 7.5. Let B be exponentially distributed with parameter μ. Then by using the lack-of-memory property of B one can show that for any A (such that $\rho < 1$) the distribution of W is a mixture of the degenerate distribution at zero and an exponential distribution; in fact, one has

$$\mathbb{P}(W > x) = p\,\mathrm{e}^{-(1-p)\,\mu\,x} \qquad [\,x \geq 0\,],$$

where p is the unique solution of the equation $\widehat{F}_A\big((1-p)\,\mu\big) = p$. □

Example 7.6. Let A be exponentially (λ) distributed, so the distribution of W is given by the Pollaczek-Khintchine formula in (3.3) and (3.4). Then it is easily verified that a standard gamma (r) distribution for B, with $r \in \mathbb{N}$, leads to a pLSt $\widehat{F}_W$ of W which is a rational function of degree r. We give examples for $r = 2$ and $r = 3$. First we take $r = 2$ and $\lambda = \frac{1}{6}$. Then we find

$$\widehat{F}_W(s) = \frac{4\,(1+s)^2}{4+11s+6s^2} = \tfrac{2}{3} + \tfrac{1}{3}\,\frac{2\,(2+s)}{(1+2s)(4+3s)},$$

which by partial fraction expansion leads to the following result for W:

$$\mathbb{P}(W > x) = \tfrac{2}{5}\,\mathrm{e}^{-\frac{1}{2}x} - \tfrac{1}{15}\,\mathrm{e}^{-\frac{4}{3}x} \qquad [\,x \ge 0\,].$$

Next we take $r = 3$ and $\lambda = \frac{1}{39}$. Then $\widehat{F}_W$ is given by

$$\widehat{F}_W(s) = \frac{36\,(1+s)^3}{36+114s+116s^2+39s^3} =$$
$$= \tfrac{12}{13} + \tfrac{1}{13}\,\frac{12\,(3+3s+s^2)}{13\,(2+3s)(\mu+s)(\overline{\mu}+s)},$$

where $\mu := \frac{3}{13}(5+i)$. Partial fraction expansion again shows that

$$\mathbb{P}(W > x) = \tfrac{3}{17}\,\mathrm{e}^{-\frac{2}{3}x} - \tfrac{22}{221}\,\mathrm{e}^{-\frac{15}{13}x}\,\{\cos \tfrac{3}{13}x + \tfrac{10}{11}\sin \tfrac{3}{13}x\},$$

where $x \ge 0$. Though both infinitely divisible distributions can be regarded as (very) generalized mixtures of exponential distributions, the second with complex-valued parameters, neither of them is infinitely divisible by the results on mixtures in Section VI.3. Similar examples can be obtained by taking a mixture of exponential distributions for B. ◫

Example 7.7. Let A be exponentially distributed. Since we want to send the traffic intensity ρ to one, we denote the limiting waiting time by $W(\rho)$. From the Pollaczek-Khintchine formula it easily follows that

$$\frac{W(\rho)}{\mathbb{E}\,W(\rho)} \xrightarrow{\mathrm{d}} X \qquad [\,\rho \uparrow 1\,],$$

where X is standard exponential. If the customary queue discipline 'first-come-first-served' is changed to 'service in random order', then one can show that the limiting waiting time $\widetilde{W}(\rho)$ still exists and has finite mean. Moreover, in this case

$$\frac{\widetilde{W}(\rho)}{\mathbb{E}\,\widetilde{W}(\rho)} \xrightarrow{\mathrm{d}} X_1 X_2 \qquad [\,\rho \uparrow 1\,],$$

where X_1 and X_2 are independent and standard exponential. From Theorem VI.3.3 it follows that $X_1 X_2$ is *infinitely divisible* with a *completely monotone* density; see also Notes. □

The results on *branching processes* in Theorems 4.1 and 4.2 are illustrated by the next three examples. In each of them we start from a special birth-death process; cf. the description preceding Example 7.2.

Example 7.8. Consider the birth-death process $Z(\cdot)$ (sometimes called the *Yule process*) with $Z(0) \equiv 1$ and with birth and death rates $\mu_j = j\mu$ and $\nu_j = 0$ for $j \in \mathbb{N}$, where $\mu > 0$; so the sojourn parameters λ_j and the transition probabilities p_{jk} of the Markov chain $Z(\cdot)$ are given by $\lambda_j = j\mu$ and $p_{j,j+1} = 1$ for $j \in \mathbb{N}$. Then $Z(\cdot)$ can be viewed as a branching process where each individual lives an exponential (μ) period of time and is replaced by two individuals at the moment of its death, so the infinitesimal quantities a and H are given by $a = \mu$ and $H(z) = z^2$. Because of (4.11) for the generator U of the process we then have

$$U(z) = \mu\,\{z^2 - z\} = \mu\, z\,(z-1).$$

Inserting this in (V.8.9) and solving the resulting differential equation, one finds the pgf F_t of $Z(t)$:

$$F_t(z) = \frac{\mathrm{e}^{-\mu t}\, z}{1 - (1 - \mathrm{e}^{-\mu t})\, z} \qquad [\,t \geq 0\,].$$

Hence $Z(t)$ is geometrically distributed with $\mathbb{E}\, Z(t) = m^t$ where $m = \mathrm{e}^{\mu}$; the process is supercritical. According to Theorem 4.1 we then have: $Z(t)/m^t \xrightarrow{\mathrm{d}} W$ as $t \to \infty$ with W positive and *self-decomposable.* This is easily verified directly by considering the pLSt π_t of $Z(t)/m^t$ and applying the continuity theorem:

$$\lim_{t\to\infty} \pi_t(s) = \lim_{t\to\infty} F_t\big(\exp\,[-s\,\mathrm{e}^{-\mu t}]\big) = \lim_{a \downarrow 0} \frac{a\,\mathrm{e}^{-as}}{1 - (1-a)\,\mathrm{e}^{-as}} = \frac{1}{1+s},$$

so W has a standard exponential distribution. □

Example 7.9. Consider the birth-death process $Z(\cdot)$ with $Z(0) \equiv 1$ and with birth and death rates $\mu_j = j\mu$ and $\nu_j = j\nu$ for $j \in \mathbb{N}$, where $\mu > \nu > 0$. Then, as in Example 7.8, $Z(\cdot)$ can be viewed as a branching process with infinitesimal quantities $a = \mu + \nu$ and $H(z) = (\nu + \mu z^2)/(\mu + \nu)$, so the generator U is given by

$$U(z) = (\mu z - \nu)\,(z - 1).$$

From (V.8.9) one obtains the pgf F_t of $Z(t)$:

$$F_t(z) = \frac{\nu\,(z-1) - \mathrm{e}^{-(\mu-\nu)\,t}\,(\mu z - \nu)}{\mu\,(z-1) - \mathrm{e}^{-(\mu-\nu)\,t}\,(\mu z - \nu)} \qquad [\,t \geq 0\,],$$

so $\mathbb{E}\,Z(t) = m^t$ with $m = \mathrm{e}^{\mu-\nu} > 1$; the process is supercritical. Now for the pLSt π_W of the limit W of $Z(t)/m^t$ as $t \to \infty$ one easily finds:

$$\pi_W(s) = \frac{\mu - \nu + \nu s}{\mu - \nu + \mu s} = 1 - \alpha + \alpha\,\frac{\alpha}{\alpha + s}, \quad \text{with } \alpha := (\mu - \nu)/\mu.$$

It follows that the distribution of W is a mixture of exponential distributions, which is *infinitely divisible* but *not* self-decomposable; note that the conditions of Theorem 4.1 are not satisfied: $\mathbb{P}\big(Z(1) = 0\big) > 0$. ◻◻

Example 7.10. Consider a branching process $X(\cdot)$ with immigration and $X(0) \equiv 0$, where branching is governed by the semigroup $\mathcal{F} = (F_t)_{t \geq 0}$ from Example 7.9, but now with $\nu = \mu + 1$; then $m = 1/\mathrm{e}$, so the process is subcritical. Putting $p := \mu/\nu \in (0,1)$, for U and F_t we get

$$U(z) = (1 - z)\,\frac{1 - p\,z}{1 - p}, \quad F_t(z) = \frac{1 - z - \mathrm{e}^t\,(1 - p\,z)}{p\,(1 - z) - \mathrm{e}^t\,(1 - p\,z)}.$$

Suppose that the immigration occurs according to a compound-Poisson process $Y(\cdot)$ with intensity $\lambda > 0$ and batch-size distribution (q_k) with pgf Q and finite logarithmic moment. Then according to Theorem 4.2 and Corollary V.8.4 $X(t) \xrightarrow{\mathrm{d}} X$ as $t \to \infty$, where X is $\mathcal{F}$*-self-decomposable*, and hence *infinitely divisible*, with pgf P satisfying

$$-\frac{1}{\lambda}\log P(z) = \int_z^1 \frac{1 - Q(x)}{U(x)}\,\mathrm{d}x = \int_z^1 \frac{1 - Q(x)}{1 - x}\,\frac{1 - p}{1 - p\,x}\,\mathrm{d}x.$$

Now, taking Q specific, one can obtain explicit expressions for P. For instance, if $q_0 = q_1 = \frac{1}{2}$, then P is the pgf of a negative-binomial (r,p) distribution with $r = \frac{1}{2}\lambda\,(1 - p)/p$. Taking (q_k) geometric (p) leads to a limit X with $X \stackrel{\mathrm{d}}{=} Y(1)$. Note that, because of Theorem II.3.2, for $p \downarrow 0$ the formula for P above tends to the canonical representation of self-decomposable pgf's as given in (V.4.13); by using this representation, however, one easily verifies that for arbitrary $p \in (0,1)$ the pgf P above is *self-decomposable* iff $p \leq q_1/(1 - q_0)$. ◻◻

Finally we consider *renewal processes* $N(\cdot)$, as in Section 5, generated by some special lifetimes X. The number $N(x)$ of renewals in $[\,0,x\,]$ is often *not* infinitely divisible; the case where X is exponential and hence $N(x)$ is Poisson, seems to be an exception. In many other cases the distribution of $N(x)$ has a tail that is too thin for infinite divisibility. We will make this more precise in two special cases; here $S_1, S_2, \ldots$ denote, as before, the renewal epochs for $N(\cdot)$.

Example 7.11. Let X have a *gamma* (2) distribution. Then we have $S_k \stackrel{d}{=} Y_1 + \cdots + Y_{2k}$ for all k, where $Y_1, Y_2, \ldots$ are independent random variables with the same exponential distribution, so

$$\mathbb{P}\big(N(x) \geq k\big) = \mathbb{P}(Y_1 + \cdots + Y_{2k} \leq x) = \mathbb{P}\big(M(x) \geq 2k\big),$$

with $M(\cdot)$ a Poisson process. From Lemma II.9.1 (iii) it now follows that

$$-\log \mathbb{P}\big(N(x) \geq k\big) \sim 2\,k \log k \qquad [\,k \to \infty\,],$$

so by Corollary II.9.5 $N(x)$ can*not* be infinitely divisible. □

Example 7.12. Let X be *stable* (λ) with exponent $\gamma = \frac{1}{2}$. Then we have $S_k/k^2 \stackrel{d}{=} X$ for all k, and from Example V.9.5 it follows that $X \stackrel{d}{=} 1/U^2$ with U normal $(0, 2/\lambda^2)$. Now we can write

$$\mathbb{P}\big(N(x) \geq k\big) = \mathbb{P}(S_k \leq x) = \mathbb{P}(X \leq x/k^2) = \mathbb{P}(U \geq k/\sqrt{x}),$$

and hence by (IV.9.9)

$$-\log \mathbb{P}\big(N(x) \geq k\big) \sim \frac{\lambda^2}{4x}\,k^2 \qquad [\,k \to \infty\,].$$

So here $N(x)$ has a tail that is even thinner than in the previous example; again, $N(x)$ is *not* infinitely divisible. □

A possible candidate for generating an infinitely divisible $N(x)$ is a lifetime X with a gamma $(\frac{1}{2})$ distribution; it is then easily seen that $N(x)$ has a fatter tail than the Poisson distribution. It does not seem easy, however, to obtain an explicit formula for the distribution of $N(x)$ in this case. We do not pursue this, and conclude this section with an illustration of the results in Section 5 around the inspection paradox.

Example 7.13. Let X be *exponential* (λ); by Example III.4.8 it is infinitely divisible with canonical density k given by $k(x) = \mathrm{e}^{-\lambda x}$ for $x > 0$. Let $X(\cdot)$ be the sii-process generated by X; then for $a > 0$ the random variable $X(a)$ is *gamma* (a, λ) distributed. According to Theorem 5.5 the limiting total lifetime $Z(a)$ in a renewal process generated by $X(a)$ can be decomposed as

$$(7.21) \qquad Z(a) \stackrel{\mathrm{d}}{=} X(a) + Y, \quad \text{with } X(a) \text{ and } Y \text{ independent,}$$

where Y has density $\lambda\, k$, so $Y \stackrel{\mathrm{d}}{=} X$. It follows that $Z(a)$ is gamma $(1+a, \lambda)$ distributed. Taking $a = 1/(n-1)$ with $n \geq 2$ we conclude: *For any $n \in \mathbb{N}$ there exists a lifetime X such that the corresponding limiting total lifetime Z has n ordinary lifetimes as independent components.* Let further $a \leq 1$. Then by Theorem 5.8 we also have decomposition (5.16) for $Z(a)$ with $Y_0 \equiv 0$ (cf. Example III.5.4), so under the usual conditions:

$$Z(a) \stackrel{\mathrm{d}}{=} X(a) + X'(a) + X''(1-a);$$

but this is, of course, a special case of (7.21). Since $X(a)$ is compound-exponential, we also have decomposition (5.17) for the limiting remaining lifetime $W(a)$ in a renewal process generated by $X(a)$, so

$$(7.22) \qquad W(a) \stackrel{\mathrm{d}}{=} X(a) + B(a), \quad \text{with } X(a) \text{ and } B(a) \text{ independent,}$$

where $B(a) \stackrel{\mathrm{d}}{=} U\,X(1-a)$ with U an independent uniform random variable on $(0,1)$. Note that $W(a) \stackrel{\mathrm{d}}{\longrightarrow} B(0)$ as $a \downarrow 0$ with $B(0) \stackrel{\mathrm{d}}{=} UX$. For $\lambda = 1$ it follows that $B(a)$ and $B(0)$ have pLSt's given by

$$\pi_a(s) = \frac{(1+s)^a - 1}{as}, \qquad \pi_0(s) = \frac{\log(1+s)}{s},$$

respectively. Combining Proposition VI.4.2 and Theorem VI.3.3 immediately shows that π_a and π_0 are Lt's of *completely monotone*, and hence *infinitely divisible* densities; compare Example VI.12.10, where this was proved in a different way. □

8. Notes

The results on first-passage times in Markov chains can be found in Miller (1967); similar results are given in Keilson (1979). Theorem 2.3 on first-passage times in diffusion processes is taken from Kent (1978), with the same very brief proof. Here and in some other papers (Kent (1982), Barndorff-Nielsen et al. (1978)) some distributions are shown to be infinitely divisible by the fact that they are first-passage time distributions in diffusion processes; one of those is the inverse-Gaussian distribution, which is, in fact, the stable distribution on $\mathbb{R}_+$ with exponent $\gamma = \frac{1}{2}$, 'exponentially tilted'. The computation of these first-passage time densities involves solving differential equations connected to the differential operator that generates the diffusion process.

Formula (3.3) occurs in Kingman (1965), where the random variables X_n are interpreted as the idle periods in the so-called dual queue; an earlier proof of (3.3) was given by Runnenburg (1960). The proof of the infinite divisibility of W by way of Spitzer's identity has been given by J. Keilson and J. Th. Runnenburg; see the discussion of the paper by Kingman. Conversely, Heinrich (1985) proves Spitzer's identity starting from (3.3).

The self-decomposability of the limit W in Theorem 4.1 is mentioned, without proof, in Yamazato (1975); a similar theorem is proved in Biggins and Shanbhag (1981). The necessary and sufficicient condition for W to be non-degenerate can be found in Athreya and Ney (1972). The result on branching processes with immigration is taken from van Harn et al. (1982); related results are given in Steutel et al. (1983).

The remark on Blackwell's renewal theorem in Proposition 5.1 was made by Runnenburg in the discussion of Kingman's paper above. Proposition 5.2 occurs in Daley (1965) in a less precise form and is also proved in van Harn (1978). Most results in the rest of Section 5 are taken from van Harn and Steutel (1995), sometimes slightly sharpened; Theorem 5.6 is proved in Bertoin et al. (1999). Theorem 5.4 is of some interest to statisticians; the distribution of Z in (5.5) is called the length-biased distribution, and the question for what X one has relation (5.8), arises quite naturally; see Arratia and Goldstein (1998), and Oluyede (1999).

Shot-noise processes of various degrees of generality are discussed in Bondesson (1992); another good reference, also for examples, is Parzen (1962).

Example 7.1 occurs in Feller (1971); the function given in (7.3) is also an infinitely divisible density for arbitrary positive, noninteger k. The distribution in (7.4) of the 'randomized random walk' is also taken from Feller (1971). Example 7.2 concerns a personal communication to us by

W. Vervaat; see also Sumita and Masuda (1985). Example 7.3 is taken from Miller (1967). Information on functional equations of the type (7.16) in Example 7.4 can be found in Kuczma (1968), and the tail behaviour of T in Feller (1971); the pLSt of T is given, e.g., in Breiman (1968). Example 7.5 occurs in many textbooks on queueing theory; see, e.g., Cohen (1982). For Example 7.6, see Kleinrock (1975). The second part of Example 7.7 is taken from Kingman (1965); this was the start of the work by Goldie, Steutel, Bondesson and Kristiansen on mixtures of exponential and gamma distributions. The branching semigroups $\mathcal{F}$ in Examples 7.8, 7.9 and 7.10 can also be found in Athreya and Ney (1972). Example 7.13 is taken from van Harn and Steutel (1995).

Appendix A

Prerequisites from probability and analysis

1. Introduction

We assume the reader to be familiar with the basic concepts and results from probability theory and analysis, including measure and integration theory. Still, it will be useful to recall a number of facts in order to establish terminology and notation, and to have on record some results that will be used in the main text. Sections 2, 3 and 4 mainly concern distributions on $\mathbb{R}$, on $\mathbb{R}_+$ and on $\mathbb{Z}_+$, respectively. Other, more special tools are collected in Section 5, and as usual we close with a section containing notes. For notational conventions and elementary properties concerning a number of well-known distributions, we refer to Appendix B.

2. Distributions on the real line

Random variable, distribution, distribution function. Let X be a *random variable*, i.e., X is an $\mathbb{R}$-valued measurable function on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then the (probability) *distribution* of X is the probability measure $\mathbb{P}_X$ on the Borel σ-field $\mathcal{B}(\mathbb{R})$ in $\mathbb{R}$ defined by

$$\mathbb{P}_X(B) = \mathbb{P}(X \in B) \qquad \big[\, B \in \mathcal{B}(\mathbb{R}) \,\big].$$

Equality in distribution of two random variables X and Y, possibly on different probability spaces, is denoted by $X \overset{\mathrm{d}}{=} Y$. The distribution $\mathbb{P}_X$ determines the (probability) *distribution function* of X, this is the function $F_X : \mathbb{R} \to [\,0, 1\,]$ with

$$F_X(x) = \mathbb{P}(X \le x) \qquad [\, x \in \mathbb{R} \,].$$

Conversely, $\mathbb{P}_X$ is determined by $F = F_X$; it is equal to the *Stieltjes measure* induced by F, i.e., the unique measure m_F on $\mathcal{B}(\mathbb{R})$ for which $m_F\big((a,b]\big) = F(b) - F(a)$ for $a < b$. The (generalized) *inverse* of a distribution function F is the function $F^{-1}\colon (0,1) \to \mathbb{R}$ defined by

$$F^{-1}(u) := \inf\big\{x \in \mathbb{R} : F(x) \geq u\big\} \qquad [0 < u < 1];$$

if U has a uniform distribution on $(0,1)$, then the random variable $F^{-1}(U)$ has distribution function F. A distribution or distribution function is said to be *symmetric* (around zero) if a random variable X having this distribution satisfies $X \stackrel{\mathrm{d}}{=} -X$.

Stieltjes measure, Lebesgue decomposition, support. Let m_G be the Stieltjes measure on $\mathcal{B}(\mathbb{R})$ induced by a right-continuous nondecreasing function G on $\mathbb{R}$; the identity on $\mathbb{R}$ induces *Lebesgue measure* m. An *atom* of m_G is a point $a \in \mathbb{R}$ such that $m_G\big(\{a\}\big) > 0$ or, equivalently, such that G is discontinuous at a; the set of atoms of G is countable. The measure m_G and the function G are called *discrete* if there is a countable set A such that $m_G(A^c) = 0$; in this case

$$m_G(B) = \sum_{a \in B} g(a) \qquad \big[\, B \in \mathcal{B}(\mathbb{R}) \,\big],$$

where g, with $g(a) := m_G\big(\{a\}\big)$, is called the *density* of m_G and of G. If m_G has no atoms or, equivalently, G is a continuous function, then m_G is said to be *continuous*. Any measure m_G can be decomposed as $m_G = m_{G,\mathrm{d}} + m_{G,\mathrm{c}}$ with $m_{G,\mathrm{d}}$ a discrete and $m_{G,\mathrm{c}}$ a continuous Stieltjes measure. The measure m_G and the function G are called *singular* (with respect to m) if there is a measurable set B with $m(B) = 0$ such that $m_G(B^c) = 0$. They are called *absolutely continuous* (with respect to m) if

$$m_G(B) = \int_B g(t)\,\mathrm{d}t \qquad \big[\, B \in \mathcal{B}(\mathbb{R}) \,\big]$$

for some nonnegative function g, which is called a *density* of m_G and of G. Densities are not unique, but often there is a most obvious one, which then is sometimes called *the* density. According to 'Radon-Nikodym' the (σ-finite) measure m_G is absolutely continuous iff $m_G(B) = 0$ for every measurable set B with $m(B) = 0$. Any measure m_G can be decomposed as $m_G = m_{G,\mathrm{s}} + m_{G,\mathrm{ac}}$ with $m_{G,\mathrm{s}}$ a singular and $m_{G,\mathrm{ac}}$ an absolutely continuous Stieltjes measure. Combining the two decompositions we obtain the so-called *Lebesgue decomposition*:

$$m_G = m_{G,\mathrm{d}} + m_{G,\mathrm{cs}} + m_{G,\mathrm{ac}}, \tag{2.1}$$

where $m_{G,\mathrm{d}}$ is the *discrete part* of m_G, $m_{G,\mathrm{cs}}$ the *continuous-singular part*, and $m_{G,\mathrm{ac}}$ the *absolutely continuous part*. Applying this result to probability distributions, we see that any distribution function F can be written as a mixture of three distribution functions 'of pure types', the *discrete component* F_d, the *continuous-singular component* F_cs and the *absolutely continuous component* F_ac of F; for some $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ with $\alpha + \beta + \gamma = 1$ we have

$$F = \alpha\, F_\mathrm{d} + \beta\, F_\mathrm{cs} + \gamma\, F_\mathrm{ac}. \tag{2.2}$$

By the *support* $S(G)$ of a function G as above we understand the set of *points of increase* of G:

$$S(G) := \bigl\{a \in \mathbb{R} : G(a+\varepsilon) - G(a-\varepsilon) > 0 \ \text{ for all } \varepsilon > 0\bigr\}.$$

Note that the support of a discrete G need not be countable. The *left extremity* ℓ_G of G is defined as $\ell_G := \inf\, S(G)$, and $r_G := \sup\, S(G)$ is called the *right extremity* of G; note that $\ell_G \in [-\infty, \infty)$ and $r_G \in (-\infty, \infty\,]$. For the support of the distribution function F of a random variable X we can write

$$S(F) = \bigl\{a \in \mathbb{R} : \mathbb{P}\bigl(|X-a| < \varepsilon\bigr) > 0 \ \text{ for all } \varepsilon > 0\bigr\},$$

and hence: $S(F)$ is the smallest *closed* subset S of $\mathbb{R}$ for which $X \in S$ a.s. (*almost surely*, i.e., with probability one). The left extremity ℓ_F of F is also called the *left extremity* of X, and is then denoted by ℓ_X; similarly for the *right extremity* r_X of X. Sometimes we allow the function G to be somewhat more general, for instance (right-continuous and) nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$; the results on G and m_G above and some results below can be adapted in an obvious way.

Random vector, random sequence, independence. Let (X, Y) be a *random vector*, i.e., (X, Y) is an $\mathbb{R}^2$-valued measurable function on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ or, equivalently, X and Y are random variables on the same probability space. Then, similarly to the one-dimensional case, the *distribution* $\mathbb{P}_{X,Y}$ and the *distribution function* $F_{X,Y}$ of (X, Y) are given by

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}\big((X,Y) \in B\big) \qquad \big[\, B \in \mathcal{B}(\mathbb{R}^2) \,\big],$$

$$F_{X,Y}(x,y) = \mathbb{P}(X \le x;\ Y \le y) \qquad \big[\, (x,y) \in \mathbb{R}^2 \,\big].$$

In the special case where $\mathbb{P}_{X,Y}$ is equal to the product measure $\mathbb{P}_X \otimes \mathbb{P}_Y$ of the one-dimensional marginal distributions $\mathbb{P}_X$ and $\mathbb{P}_Y$, the random variables X and Y are called (mutually) *independent*; equivalent to this is each of the following statements:

$$\mathbb{P}(X \in C;\ Y \in D) = \mathbb{P}(X \in C)\, \mathbb{P}(Y \in D) \qquad \big[\, C, D \in \mathcal{B}(\mathbb{R}) \,\big],$$

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y) \qquad [\, x, y \in \mathbb{R} \,].$$

Random vectors $(X_1, \ldots, X_n)$ with $n \ge 3$ are treated similarly; note that independence is stronger than pairwise independence. A *random sequence* $(X_n)_{n \in \mathbb{N}}$ is a sequence of random variables on the same probability space; $X_1, X_2, \ldots$ are called *independent* if $X_1, \ldots, X_n$ are independent for all $n \in \mathbb{N}$.

Convolution. The distribution $\mathbb{P}_{X+Y}$ and the distribution function F_{X+Y} of the sum of two *independent* random variables X and Y are given by the *convolutions* $\mathbb{P}_X \star \mathbb{P}_Y$ and $F_X \star F_Y$, respectively. Here we define the measure $\mu \star \nu$ for σ-finite measures μ and ν on $\mathcal{B}(\mathbb{R})$ and the function $G \star H$ for right-continuous nondecreasing functions G and H on $\mathbb{R}$ by

$$(\mu \star \nu)(B) := (\mu \otimes \nu)\big(\{(x,y) : x + y \in B\}\big) = \int_{\mathbb{R}} \mu(B - y)\, \nu(\mathrm{d}y),$$

$$(G \star H)(x) := \int_{\mathbb{R}} G(x - y)\, \mathrm{d}H(y),$$

where $B - y := \{x - y : x \in B\}$. If G is continuous, then so is $G \star H$. If G is absolutely continuous with density g, then $G \star H$ is absolutely continuous with density $g \star H$ (extending the definition of $\star$); if in this case also H is absolutely continuous with density h, then $g \star H = g * h$ with

$$(g * h)(x) := \int_{\mathbb{R}} g(x - y)\, h(y)\, \mathrm{d}y \qquad [\, x \in \mathbb{R} \,].$$

Similarly, for sequences $q = (q_k)_{k \in \mathbb{Z}}$ and $r = (r_k)_{k \in \mathbb{Z}}$ of nonnegative numbers we define $q * r$ as the sequence with elements

$$(q * r)_k := \sum_{j \in \mathbb{Z}} q_{k-j}\, r_j \qquad [\, k \in \mathbb{Z} \,].$$

The operation $*$ is also called *convolution*; in practice it will always be clear which one of the convolutions is meant. The convolutions $\star$ and $*$ are commutative and associative; for $n \in \mathbb{Z}_+$ the n-fold convolution of G with itself is denoted by $G^{\star n}$ with $G^{\star 1} := G$ and $G^{\star 0} := 1_{\mathbb{R}_+}$. Similarly for the function g^{*n} and for the sequence q^{*n} with $n \in \mathbb{N}$. We next state a result on the support of convolutions; since it does not seem to be well known, we give a proof in a special case. Here the *direct sum* $A \oplus B$ of two subsets A and B of $\mathbb{R}$ is defined as the set of all $c \in \mathbb{R}$ of the form $c = a + b$ with $a \in A$ and $b \in B$, and $A^{\oplus n} := A \oplus \cdots \oplus A$ (n times).

Proposition 2.1. *The support of the convolution of two right-continuous nondecreasing functions G and H on $\mathbb{R}$ is equal to the closure of the direct sum of their supports:*

$$S(G \star H) = \overline{S(G) \oplus S(H)}.$$

More generally, for right-continuous nondecreasing functions $G_1, \ldots, G_n$:

$$S(G_1 \star \cdots \star G_n) = \overline{S(G_1) \oplus \cdots \oplus S(G_n)}, \quad \text{so} \quad S(G^{\star n}) = \overline{S(G)^{\oplus n}}.$$

PROOF. We restrict ourselves to distribution functions. Let X and Y be independent random variables with distribution functions G and H, respectively. By the triangle inequality, for $a, b \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\mathbb{P}\big(|(X+Y) - (a+b)| < \varepsilon\big) \geq \mathbb{P}\big(|X-a| < \tfrac{1}{2}\varepsilon\big)\, \mathbb{P}\big(|Y-b| < \tfrac{1}{2}\varepsilon\big).$$

It follows that $a + b \in S(G \star H)$ if $a \in S(G)$ and $b \in S(H)$, and hence, since supports are closed, $S(G \star H)$ contains the closure of the direct sum of $S(G)$ and $S(H)$.

To prove the converse, we take c in the complement of the closure of the direct sum. Then there is an $\varepsilon_0 > 0$ such that this complement, and hence the complement of $S(G) \oplus S(H)$, contains the interval $(c - \varepsilon_0, c + \varepsilon_0)$. Now, for any $b \in S(H)$, the interval $\big((c-b) - \varepsilon_0, (c-b) + \varepsilon_0\big)$ is contained in the complement of $S(G)$; otherwise $(c - \varepsilon_0, c + \varepsilon_0)$ would contain a point of $S(G) \oplus S(H)$. Thus we have $\mathbb{P}\big(|X - (c-b)| < \varepsilon_0\big) = 0$ for $b \in S(H)$. Hence

$$\mathbb{P}\big(|(X+Y) - c| < \varepsilon_0\big) = \int_{S(H)} \mathbb{P}\big(|X - (c-b)| < \varepsilon_0\big)\, \mathrm{d}H(b) = 0,$$

and it follows that $c \notin S(G \star H)$. We conclude that also $S(G \star H)$ is contained in the closure of the direct sum of $S(G)$ and $S(H)$.

The final, more general statement is now easily proved. □

We finally note that integration with respect to (the Stieltjes measure induced by) a convolution $G \star H$ can be carried out as follows:

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}(G \star H)(x) = \int_{\mathbb{R}^2} f(u+v)\,\mathrm{d}G(u)\,\mathrm{d}H(v). \tag{2.3}$$

This is seen by writing $(G \star H)(x) = \int_{\mathbb{R}^2} 1_{[u+v,\infty)}(x)\,\mathrm{d}G(u)\,\mathrm{d}H(v)$, and is clear for distribution functions: both the integrals in (2.3) equal $\mathbb{E} f(X+Y)$ if X and Y are independent with $F_X = G$ and $F_Y = H$.

Expectation, moment inequalities. For the *expectation* $\mathbb{E}\,g(X)$ of a measurable function g of a random variable X on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ we have

$$\mathbb{E}\,g(X) := \int_{\Omega} g(X)\,\mathrm{d}\mathbb{P} = \int_{\mathbb{R}} g(x)\,\mathbb{P}_X(\mathrm{d}x),$$

provided that these integrals exist (possibly ∞ or $-\infty$). For $n \in \mathbb{Z}_+$ the special case with $g(x) = x^n$ yields the *n-th moment* or *moment of order n* of X. If the n-th moment of X (exists and) is finite, then so is the k-th moment for all $k \le n$; this follows from the inequality $\mathbb{E}\,|X|^r \le 1 + \mathbb{E}\,|X|^s$ if $0 < r \le s$. Here for arbitrary $r > 0$ the quantity $\mathbb{E}\,|X|^r$ is called the *absolute moment of order r* of X. The first moment or *mean* of X can be written in terms of the distribution function F of X as

$$\mathbb{E}X = \int_0^{\infty} \{1 - F(x)\}\,\mathrm{d}x - \int_{-\infty}^{0} F(x)\,\mathrm{d}x, \tag{2.4}$$

and the first two moments yield the *variance* of X: $\mathbb{V}\mathrm{ar}\,X = \mathbb{E}X^2 - (\mathbb{E}X)^2$. If X has a finite variance, then by *Chebyshev's inequality* we have

$$\mathbb{P}\big(|X - \mathbb{E}X| \ge a\big) \le (\mathbb{V}\mathrm{ar}\,X)/a^2 \qquad [a > 0]. \tag{2.5}$$

Further, if g is a convex function on an open interval I and if X is an I-valued random variable with finite mean, then $\mathbb{E}\,g(X)$ exists and

$$\mathbb{E}\,g(X) \ge g(\mathbb{E}X); \tag{2.6}$$

this basic result is known as *Jensen's inequality*. It can be used to show that

$$\|\,X\,\|_r := \big(\mathbb{E}\,|X|^r\big)^{1/r} \text{ is nondecreasing in } r \ge 1, \tag{2.7}$$

and also to prove *Hölder's inequality* which for random variables reads as follows:

$$(2.8)\qquad \| XY \|_1 \le \| X \|_r \, \| Y \|_s \qquad [\, r, s \ge 1 \text{ with } 1/r + 1/s = 1\,];$$

taking $r = s = 2$ yields a result that is known as *Schwarz's inequality.* In turn, 'Hölder' can be used to prove *Minkowski's inequality*; for random variables this triangle inequality reads as follows:

$$(2.9)\qquad \| X + Y \|_r \le \| X \|_r + \| Y \|_r \qquad [\, r \ge 1\,].$$

Finally, we note that for independent random variables X and Y, both nonnegative or both integrable, the expectation of XY exists with $\mathbb{E}(XY) = (\mathbb{E}X)(\mathbb{E}Y)$.

Convergence in distribution, weak convergence. For random variables $X, X_1, X_2, \ldots$, possibly on different probability spaces, we say that the sequence (X_n) *converges in distribution to* X as $n \to \infty$ if

$$\lim_{n\to\infty} \mathbb{E}\, g(X_n) = \mathbb{E}\, g(X) \text{ for all continuous bounded } g;$$

notation: $X_n \xrightarrow{\mathrm{d}} X$ as $n \to \infty$. According to *Helly's theorem* this is equivalent to saying that

$$(2.10)\qquad \lim_{n\to\infty} F_n(x) = F(x) \text{ for all continuity points } x \text{ of } F,$$

where $F_n := F_{X_n}$ and $F := F_X$; in this case (F_n) is said to *converge weakly to* F as $n \to \infty$. *Helly's selection theorem* states that any sequence (F_n) of distribution functions contains a subsequence (F_{n_k}) satisfying (2.10) with F a *sub*-distribution function, i.e., a right-continuous nondecreasing function on $\mathbb{R}$ with $0 \le F(x) \le 1$ for all x. This result can be used to show that (F_n) converges weakly iff the set $\{m_{F_n} : n \in \mathbb{N}\}$ is *tight*, i.e.,

$$(2.11)\qquad \lim_{t\to\infty} \sup_n m_{F_n}\big(\mathbb{R}\backslash(-t, t\,]\big) = 0,$$

and all weakly convergent subsequences of (F_n) have the same limit. Instances of convergence in distribution are given by the *Central limit theorem* (see Theorem I.5.1) and by the following important result on maxima.

Theorem 2.2. *Let $X_1, X_2, \ldots$ be independent, identically distributed random variables, and set $M_n := \max\{X_1, \ldots, X_n\}$ for $n \in \mathbb{N}$. Suppose there exist $a_n \in \mathbb{R}$ and $b_n > 0$ such that for some non-degenerate random variable V:*

$$\frac{1}{b_n}\,(M_n - a_n) \xrightarrow{\mathrm{d}} V \qquad [\, n \to \infty\,].$$

Then there exist $a, b \in \mathbb{R}$, $b \neq 0$ such that the distribution function F of $(V-a)/b$ is given by one of the following functions, with $\alpha > 0$:

$$(2.12) \qquad F(x) = \big(1 - \exp[-x^{\alpha}]\big)\, 1_{\mathbb{R}_+}(x), \;\; \exp[-x^{-\alpha}]\, 1_{\mathbb{R}_+}(x), \;\; \exp[-e^{-x}].$$

Characteristic function and FSt: basic results. A useful tool in probability theory is the *characteristic function* of a random variable X (and of $\mathbb{P}_X$ and of F_X), this is the function $\phi_X : \mathbb{R} \to \mathbb{C}$ with

$$\phi_X(u) := \mathbb{E}\, e^{iuX} = \int_{\mathbb{R}} e^{iux}\, \mathbb{P}_X(dx) = \int_{\mathbb{R}} e^{iux}\, dF_X(x) =: \widetilde{F}_X(u).$$

Also for general *bounded* right-continuous nondecreasing functions G on $\mathbb{R}$ we denote the *Fourier-Stieltjes transform* (FSt) of G by $\widetilde{G}$; if G is absolutely continuous with density g, then $\widetilde{G}$ is the (ordinary) *Fourier transform* (Ft) of g. The characteristic function ϕ_X of X determines the distribution of X, so we have the following *uniqueness theorem*:

$$(2.13) \qquad \phi_X = \phi_Y \iff X \overset{d}{=} Y.$$

There are several *inversion theorems*; we only mention two special cases. Let ϕ be the characteristic function of X; then the atoms of $\mathbb{P}_X$ can be obtained from

$$(2.14) \qquad \mathbb{P}(X = x) = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} e^{-iux}\, \phi(u)\, du \qquad [x \in \mathbb{R}],$$

and if ϕ satisfies $\int_{\mathbb{R}} |\phi(u)|\, du < \infty$, then X has an absolutely continuous distribution with (continuous bounded) density f given by

$$(2.15) \qquad f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux}\, \phi(u)\, du \qquad [x \in \mathbb{R}].$$

The characteristic function ϕ of a random variable X satisfies $|\phi(u)| \leq 1$ for all u and $\phi(0) = 1$. We have $|\phi(u)| = 1$ for some $u \neq 0$ iff there exist $c \in \mathbb{R}$ and $d > 0$ such that X is $(c + d\,\mathbb{Z})$-valued. On the other hand, if X has an absolutely continuous distribution, then not only $|\phi(u)| < 1$ for all $u \neq 0$, but also $\phi(u) \to 0$ as $u \to \infty$ or $u \to -\infty$. A characteristic function ϕ is continuous. According to *Bochner's theorem* a continuous function $\phi : \mathbb{R} \to \mathbb{C}$ with $\phi(0) = 1$ is a characteristic function iff ϕ is *nonnegative definite*, i.e., for all $n \in \mathbb{N}$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \phi(u_j - u_k)\, z_j \overline{z}_k \geq 0 \qquad [u_1, \ldots, u_n \in \mathbb{R},\; z_1, \ldots, z_n \in \mathbb{C}];$$

this result is, however, not very useful in practice. Of more importance is the fact that characteristic functions transform convolutions into products. In fact, we have

$$(2.16) \qquad X \text{ and } Y \text{ independent } \implies \phi_{X+Y} = \phi_X \, \phi_Y;$$

the converse is not true. On the other hand we do have: $F = G \star H$ iff $\widetilde{F} = \widetilde{G}\,\widetilde{H}$, for bounded right-continuous nondecreasing functions F, G and H on $\mathbb{R}$. The characteristic function of $-X$ is given by the complex conjugate $\overline{\phi}_X$ of ϕ_X, so X has a *symmetric* distribution iff ϕ_X is real or, equivalently, ϕ_X is even. If ϕ is a characteristic function, then so is $|\phi|^2$, because $|\phi|^2 = \phi\,\overline{\phi}$. Besides products and mixtures also limits of characteristic functions are characteristic functions again, provided they are continuous at zero; this is part of *Lévy-Cramér's continuity theorem*, which can be stated as follows.

Theorem 2.3 (Continuity theorem). *For $n \in \mathbb{N}$ let X_n be a random variable with characteristic function ϕ_n.*

(i) *If $X_n \xrightarrow{\text{d}} X$ as $n \to \infty$, then $\lim_{n\to\infty} \phi_n(u) = \phi_X(u)$ for $u \in \mathbb{R}$.*

(ii) *If $\lim_{n\to\infty} \phi_n(u) = \phi(u)$ for $u \in \mathbb{R}$ with ϕ a function that is continuous at zero, then ϕ is the characteristic function of a random variable X and $X_n \xrightarrow{\text{d}} X$ as $n \to \infty$.*

A sufficient condition for a given function to be a characteristic function, is given by *Pólya's criterion*; we state it together with a periodic variant, which yields distributions on $\mathbb{Z}$.

Theorem 2.4. *Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be an even continuous function satisfying $\phi(0) = 1$.*

(i) *If ϕ is nonincreasing and convex on $(0, \infty)$, then ϕ is the characteristic function of a symmetric distribution on $\mathbb{R}$.*

(ii) *If ϕ is nonincreasing and convex on $(0, \pi)$ and 2π-periodic, then ϕ is the characteristic function of a symmetric distribution on $\mathbb{Z}$.*

Poisson's summation formula. We return to the inversion formula (2.15) which gives a continuous density f corresponding to a characteristic function ϕ with $\int_{\mathbb{R}} \big|\phi(u)\big| \, du < \infty$.

Theorem 2.5. *Let ϕ be an absolutely integrable characteristic function, and let f be the corresponding continuous density. If one of the following conditions is satisfied:*

— *the series $\sum_{k\in\mathbb{Z}} \phi(\cdot + 2k\pi)$ converges to a continuous function on $\mathbb{R}$;*

— *the series $\sum_{k\in\mathbb{Z}} f(k)$ converges,*

then so is the other, and ϕ and f are related by

$$\sum_{k\in\mathbb{Z}} \phi(u+2k\pi) = \sum_{k\in\mathbb{Z}} f(k)\, e^{iuk} \qquad [u\in\mathbb{R}]. \tag{2.17}$$

An important consequence of the identity (2.17) is the fact that its left-hand side, when normalized, can be viewed as the characteristic function of a distribution $(p_k)_{k\in\mathbb{Z}}$ on $\mathbb{Z}$ which is proportional to $\big(f(k)\big)$.

Corollary 2.6. *Let ϕ be a nonnegative integrable characteristic function that is nonincreasing on $(0,\infty)$, and let f be the corresponding continuous density. Then (2.17) holds.*

Proof. Note that ϕ is real and hence even. So, by the monotonicity of ϕ,

$$\sum_{k\in\mathbb{Z}} \phi(2k\pi) = \phi(0) + 2\sum_{k=1}^{\infty} \phi(2k\pi) \le 1 + \frac{1}{\pi}\int_{\mathbb{R}_+} \phi(u)\,\mathrm{d}u < \infty.$$

Using the monotonicity of ϕ once more, we see that for $u\in(0,\pi)$ and $n\in\mathbb{N}$

$$\sum_{|k|\ge n} \phi(u+2k\pi) \le 2\sum_{k=n-1}^{\infty} \phi(2k\pi).$$

It follows that the series $\sum_{k\in\mathbb{Z}} \phi(\cdot + 2k\pi)$, which is 2π-periodic, converges uniformly on $\mathbb{R}$ and hence converges to a continuous function. Now apply the theorem. □

Characteristic function and moments. Let ϕ be the characteristic function of a random variable X. If, for some $n\in\mathbb{N}$, X has a finite moment μ_n of order n, then ϕ can be differentiated n times with $\phi^{(n)}$ continuous, μ_n can be obtained from $\phi^{(n)}(0) = i^n\,\mu_n$, and ϕ satisfies

$$\phi(u) = \sum_{j=0}^{n} \mu_j\,\frac{(iu)^j}{j!} + o(u^n) \qquad [u\to 0]. \tag{2.18}$$

There is a partial converse; if, for some *even* $n\in\mathbb{N}$, ϕ has a finite derivative $\phi^{(n)}(0)$ at zero, then X has a finite moment of order n. If X has finite moments of all orders, then on some neighbourhood $(-\varepsilon, \varepsilon)$ of $u = 0$ ϕ can be represented in terms of the moment sequence (μ_j) as

$$\phi(u) = \sum_{j=0}^{\infty} \mu_j \frac{(iu)^j}{j!} \qquad \big[\,|u| < \varepsilon\,\big] \tag{2.19}$$

iff $\limsup_{j\to\infty} |\mu_j|^{1/j}/j$ is finite; in this case there is no other probability distribution with the same moment sequence. The cumulants of X, as far as they exist, can be obtained by differentiating the function $\psi := \log \phi$ which is well defined near $u = 0$; if X has a finite moment of order n, then the constant κ_n satisfying $\psi^{(n)}(0) = i^n \kappa_n$ is called the *cumulant of order n* of X. Note that the n-th order cumulant of a sum $X_1 + \cdots + X_m$ of independent random variables is equal to the sum of the n-th order cumulants of $X_1, \ldots, X_m$. For $n \in \{1, 2\}$ this also follows from the fact that $\kappa_1 = \mathbb{E}X$ and $\kappa_2 = \mathbb{V}\mathrm{ar}\, X$. More generally, by differentiating n times the relation $\phi' = \phi\, \psi'$ we see that the moments and cumulants of order $\leq n+1$ are related by

$$\mu_{n+1} = \sum_{j=0}^{n} \binom{n}{j} \mu_j \kappa_{n-j+1}. \tag{2.20}$$

The function ψ above also plays a role in the following criterion for finiteness of *logarithmic moments.*

Proposition 2.7. *Let X be a random variable with characteristic function ϕ and $\psi := \log \phi$. Then the following four quantities are either all finite or all infinite (with $\varepsilon > 0$ sufficiently small):*

$$\mathbb{E} \log \big(1+|X|\big), \quad \mathbb{E} \log^+ |X|, \quad \int_0^\varepsilon \frac{\big|1-\phi(u)\big|}{u}\, du, \quad \int_0^\varepsilon \frac{\big|\psi(u)\big|}{u}\, du.$$

PROOF. Similar to the proof of the $\mathbb{R}_+$-valued analogue given by Proposition 3.2. ◫

Logarithm of a nonvanishing characteristic function. Let ϕ be a characteristic function that vanishes nowhere on $\mathbb{R}$. Then the *logarithm* $\log \phi$ of ϕ can be defined, as usual, by

$$\log \phi(u) := \log \big|\phi(u)\big| + i \arg \phi(u) \qquad [\,u \in \mathbb{R}\,], \tag{2.21}$$

where one has to prescribe an interval of the form $[\,b, b+2\pi)$ for the values of $\arg \phi$. One often takes $b = -\pi$, and then obtains the so-called *principal value* of $\log \phi$; it might be, however, discontinuous at those $u \in \mathbb{R}$ for which $\phi(u) < 0$. We do not want this; we would like to have, for instance, $\log e^{iu} = iu$ for $u \in \mathbb{R}$. Now, one can show that, because of the continuity of ϕ, $\arg \phi$ can be chosen such that it is continuous on $\mathbb{R}$; moreover, there is only one such choice with $\arg \phi(0) = 0$. This unique function $\arg \phi$ will further be used to define $\log \phi$ by (2.21); $\log \phi$ is then *continuous* on $\mathbb{R}$ with $\log \phi(0) = 0$. For $t > 0$ the function ϕ^t is now defined by

$$(2.22) \qquad \phi^t(u) := \exp\big[\, t \log \phi(u) \,\big] \qquad [\, u \in \mathbb{R} \,].$$

Analytic extension of a characteristic function. Sometimes we want to consider a characteristic function ϕ for *complex* values of its argument. Any ϕ has two abscissas of convergence; these are the largest numbers u_ϕ and v_ϕ in $[\,0, \infty\,]$ such that $\phi(z) := \mathbb{E}\, e^{izX}$ is well defined for all $z \in \mathbb{C}$ with $-u_\phi < \operatorname{Im} z < v_\phi$. The set of these values of z (if non-empty) is called the *strip of analyticity* of ϕ, because ϕ can be shown to be analytic on this set. Moreover, u_ϕ and v_ϕ are determined by the tails of the corresponding distribution function F in the following way:

$$(2.23) \qquad u_\phi = \liminf_{x \to \infty} \frac{-\log\{1 - F(x)\}}{x}, \quad v_\phi = \liminf_{x \to \infty} \frac{-\log F(-x)}{x}.$$

Information about the strip of analyticity can be found from the following lemma; it yields bounds for u_ϕ and v_ϕ.

Lemma 2.8. *Let X be a random variable with characteristic function ϕ satisfying*

$$\phi(u) = A(u) \qquad [\, -\varepsilon < u < \varepsilon, \ \textit{some } \varepsilon > 0 \,],$$

where A is a function analytic on the disk $|z| < \rho$ in $\mathbb{C}$ with $\rho \geq \varepsilon$. Then $\phi(z) := \mathbb{E}\, e^{izX}$ is well defined for all $z \in \mathbb{C}$ with $|z| < \rho$, and

$$\phi(z) = A(z) \qquad [\, z \in \mathbb{C} \ \textit{ with } |z| < \rho \,].$$

If ϕ has no zeroes in its strip of analyticity, then, as before for ϕ on $\mathbb{R}$, one can show that $\log \phi$, and hence ϕ^t with $t > 0$, can be defined on this strip such that it is continuous there with $\log \phi(0) = 0$.

Unimodal distributions and log-concave densities. A distribution on $\mathbb{R}$ and its distribution function F are said to be *unimodal* if there exists $x_1 \in \mathbb{R}$ such that F is *convex* on $(-\infty, x_1)$ and *concave* on (x_1, ∞);

the number x_1 is called a *mode* of F. In this case, if F is continuous at x_1, then F is absolutely continuous having a density f that is nondecreasing on $(-\infty, x_1)$ and nonincreasing on (x_1, ∞); f is called *unimodal* as well. Clearly, the class of unimodal distributions with a fixed mode is closed under mixing. This proves the converse part in the following characterization of unimodality about zero.

Theorem 2.9. *A random variable X has a unimodal distribution with mode at zero iff a random variable Z exists such that $X \stackrel{\mathrm{d}}{=} UZ$, where U is uniform on $(0,1)$ and independent of Z.*

Since unimodality is defined in terms of weak inequalities, it is preserved under weak convergence. It is, however, *not* preserved under convolution, except in some special situations.

Theorem 2.10. *The convolution $F \star G$ of two unimodal distribution functions F and G is again unimodal in each of the following cases: (i) Both F and G are symmetric; (ii) $G = F_{-X}$ if $F = F_X$.*

A second positive result concerns *log-concave* densities, i.e., densities f for which

$$\left\{f\left(\tfrac{1}{2}\,(x+y)\right)\right\}^2 \geq f(x)\, f(y) \qquad [\,x, y \in \mathbb{R}\,];$$

for positive f this means that $\log f$ is concave. Now, a distribution with a log-concave density is easily seen to be unimodal; it turns out to be even *strongly unimodal*, i.e., its convolution with any unimodal distribution is again unimodal. Moreover, there is a converse.

Theorem 2.11. *A non-degenerate distribution on $\mathbb{R}$ has a log-concave density iff it is strongly unimodal.*

In view of examples it is worthwile to note that a distribution with a log-concave density has all its moments finite.

3. Distributions on the nonnegative reals

Introductory remarks. In this section we restrict ourselves to random variables X that are $\mathbb{R}_+$-*valued*. There are several equivalent ways to express this starting-point: $\mathbb{P}_X$ is a distribution *on* $\mathbb{R}_+$ or $\mathbb{P}_X(\mathbb{R}_+) = 1$;

F_X is a distribution function *on* $\mathbb{R}_+$ or $F_X(0-) = 0$; the support of F_X satisfies $S(F_X) \subset \mathbb{R}_+$; the left extremity of X satisfies $\ell_X \geq 0$. Note that an $\mathbb{R}$-valued random variable X satisfying $\ell_X > -\infty$ can be shifted so as to become $\mathbb{R}_+$-valued, so often it is no essential restriction to assume that $\ell_X \geq 0$; for the same reason we may often assume that even $\ell_X = 0$. Also, results for $\mathbb{R}_+$-valued random variables can be used to obtain results for $\mathbb{R}$-valued random variables X with $r_X < \infty$, because then $\ell_{-X} > -\infty$. Without further comment a distribution function F on $\mathbb{R}_+$ and a density f of F (in case of absolute continuity) are sometimes restricted to $\mathbb{R}_+$ and to $(0, \infty)$, respectively; we can then speak, for instance, of *concavity* of F and of *monotonicity* of f. We will also do so for the more general functions G and their densities g which will be considered in a moment. Of course, all of Section 2 applies to the present situation. But for distributions on $\mathbb{R}_+$ there are sometimes more detailed results available or more convenient tools to use; we will discuss these briefly in the sequel.

Convolution. We consider, more general than distribution functions on $\mathbb{R}_+$, right-continuous nondecreasing functions on $\mathbb{R}$ that vanish everywhere on $(-\infty, 0)$. If G and H are functions of this type, then so is their convolution, and the domain of integration in its definition can be restricted:

$$(G \star H)(x) = \int_{[0,x]} G(x-y)\, \mathrm{d}H(y) \qquad [\, x \geq 0\,];$$

a similar remark holds for the convolution $g * h$ of two densities g and h on $(0, \infty)$. The support of $G \star H$ is equal to the direct sum of the supports:

$$S(G \star H) = S(G) \oplus S(H); \tag{3.1}$$

this immediately follows from Proposition 2.1 and the fact that the direct sum of two closed sets that are bounded below, is closed.

PLSt: basic results. Rather than the characteristic function, for an $\mathbb{R}_+$-valued random variable X we use the *probability Laplace-Stieltjes transform* (pLSt) of X (and of $\mathbb{P}_X$ and of F_X), this is the following function $\pi_X : \mathbb{R}_+ \to (0, \infty)$:

$$\pi_X(s) := \mathbb{E}\, \mathrm{e}^{-sX} = \int_{\mathbb{R}_+} \mathrm{e}^{-sx}\, \mathbb{P}_X(\mathrm{d}x) = \int_{\mathbb{R}_+} \mathrm{e}^{-sx}\, \mathrm{d}F_X(x) =: \widehat{F}_X(s).$$

If F_X is absolutely continuous with density f, then $\widehat{F}_X$ is the *probability Laplace transform* (pLt) of f. By Fubini's theorem $\pi_X(s)$ with $s > 0$ can be written as an ordinary Lebesgue integral:

$$\pi_X(s) = s \int_0^\infty \mathrm{e}^{-sx} F_X(x)\,\mathrm{d}x \qquad [\,s > 0\,]. \tag{3.2}$$

Of course, we have counterparts to (2.13) and (2.16); the *uniqueness theorem* says that $\pi_X = \pi_Y$ iff $X \stackrel{\mathrm{d}}{=} Y$, and if X and Y are independent, then $\pi_{X+Y} = \pi_X\, \pi_Y$. The analogue of Theorem 2.3 can be stated as follows.

Theorem 3.1 (Continuity theorem). *For $n \in \mathbb{N}$ let X_n be a random variable with pLSt π_n.*

(i) *If $X_n \xrightarrow{\mathrm{d}} X$ as $n \to \infty$, then $\lim_{n\to\infty} \pi_n(s) = \pi_X(s)$ for $s \geq 0$.*

(ii) *If $\lim_{n\to\infty} \pi_n(s) = \pi(s)$ for $s \geq 0$ with π a function that is (right-) continuous at zero, then π is the pLSt of a random variable X and $X_n \xrightarrow{\mathrm{d}} X$ as $n \to \infty$.*

PLSt, cgf and moments. The pLSt π of a random variable X is a continuous function on $\mathbb{R}_+$ with $\pi(0) = 1$, and possesses derivatives of all orders on $(0, \infty)$ with

$$\pi^{(n)}(s) = (-1)^n \int_{\mathbb{R}_+} \mathrm{e}^{-sx}\, x^n\, \mathrm{d}F_X(x) \qquad [\,n \in \mathbb{N};\ s > 0\,]. \tag{3.3}$$

By letting $s \downarrow 0$ it follows that for $n \in \mathbb{N}$ the right-hand limit of $\pi^{(n)}$ at zero exists with $\pi^{(n)}(0+) = (-1)^n \mu_n$, where μ_n is the n-th moment of X, possibly infinite. So $-\pi'(0+) = \mu_1$. On the other hand, rewriting (3.2) as

$$\frac{1 - \pi(s)}{s} = \int_0^\infty \mathrm{e}^{-sx} \{1 - F_X(x)\}\,\mathrm{d}x \qquad [\,s > 0\,], \tag{3.4}$$

and using (2.4) one sees that the (right-hand) derivative of π at zero exists and satisfies $-\pi'(0) = \mu_1$. Hence $\pi'(0+) = \pi'(0)$, so if $\mu_1 < \infty$, then π', like π, is continuous on $\mathbb{R}_+$. Moreover, by the mean value theorem we can write $\big(1 - \pi(s)\big)/s = -\pi'(\sigma_s)$ for $s > 0$ and some $\sigma_s \in (0, s)$; since $-\pi'$ is nonincreasing, it follows that

$$0 \leq -\pi'(s) \leq \frac{1 - \pi(s)}{s} \ \text{ for } s > 0, \ \text{ and hence } \lim_{s \downarrow 0} s\,\pi'(s) = 0. \tag{3.5}$$

Actually, (3.5) holds for all functions π on $(0, \infty)$ with $\pi(0+) = 1$ that are positive, nonincreasing and convex, and have a continuous derivative.

If $\mu_n < \infty$, then the cumulant κ_n of order n of X can be obtained from $C^{(n)}(0+) = (-1)^{n-1}\kappa_n$, where C is the *cumulant generating function* (cgf) of X (and of $\mathbb{P}_X$ and of F_X):

$$C(s) := -\log \pi(s) \qquad [\,s \geq 0\,].$$

By Schwarz's inequality we have $\pi\big(\frac{1}{2}\,(s+t)\big)^2 \leq \pi(s)\,\pi(t)$ for $s, t > 0$, so π is not only convex, but even *log-convex*; see also the end of this section. It follows that the cgf C is *concave*, and hence nondecreasing because C is nonnegative. The behaviour of C at zero is also related to *logarithmic moments* of X.

Proposition 3.2. *Let X be $\mathbb{R}_+$-valued with pLSt π and cgf $C = -\log \pi$. Then the following four quantities are either all finite or all infinite:*

$$\mathbb{E}\,\log(1+X),\;\; \mathbb{E}\,\log^+ X,\;\; \int_0^1 \frac{1-\pi(s)}{s}\,ds,\;\; \int_0^1 \frac{C(s)}{s}\,ds.$$

Proof. The first two quantities are simultaneously finite because one has $\log(1+x) \sim \log x$ as $x \to \infty$, and the last two because $C(s) \sim 1 - \pi(s)$ as $s \downarrow 0$. Next, observe that by Fubini's theorem

$$\mathbb{E}\,\log^+ X = \int_1^\infty \frac{1}{x}\,\{1 - F_X(x)\}\,\mathrm{d}x,$$

and, because of (3.4),

$$\int_0^1 \frac{1-\pi(s)}{s}\,\mathrm{d}s = \int_0^\infty \frac{1}{x}\,(1-\mathrm{e}^{-x})\,\{1 - F_X(x)\}\,\mathrm{d}x.$$

Now, noting that $1 - \mathrm{e}^{-x} \sim x$ as $x \downarrow 0$ and ~ 1 as $x \to \infty$ completes the proof. ◻

PLSt: inversion results. An $\mathbb{R}_+$-valued random variable X has several characteristics that can easily be obtained from its pLSt π. First of all, π is a nonincreasing function with limit

(3.6) $$\lim_{s\to\infty} \pi(s) = \mathbb{P}(X=0), \;\text{ so }\; \lim_{s\to\infty} \pi(s)\,\mathrm{e}^{s\ell_X} = \mathbb{P}(X = \ell_X);$$

the second statement follows by applying the first one to $X - \ell_X$. The left extremity ℓ_X itself can be computed from π in the following way.

Proposition 3.3. *Let X be $\mathbb{R}_+$-valued with pLSt π and cgf $C = -\log \pi$, and define on $(0, \infty)$ the functions ρ and ρ_a with $a > 0$ by*

$$\rho(s) := C'(s) = \frac{-\pi'(s)}{\pi(s)},$$
$$\rho_a(s) := \frac{C(s+a) - C(s)}{a} = \frac{1}{a} \log \frac{\pi(s)}{\pi(s+a)}.$$

Then these functions are nonincreasing and their limits as $s \to \infty$ are given by the left extremity of X:

(3.7) $$\lim_{s\to\infty} \rho(s) = \ell_X, \quad \lim_{s\to\infty} \rho_a(s) = \ell_X \ \text{ for } a > 0.$$

PROOF. The assertion on monotonicity immediately follows from the concavity of C. By considering $X - \ell_X$ one sees that it is sufficient to prove (3.7) when $\ell_X = 0$. Also, the second equality follows from the first one by noting that by the mean value theorem we can write $\rho_a(s) = \rho(s + \theta_s a)$ for $s > 0$ and some $\theta_s \in (0, 1)$. So, assume $\ell_X = 0$, and take $\varepsilon > 0$; then $F := F_X$ satisfies $F(\varepsilon) > 0$. Now, using (3.2) we can estimate $\rho(s)$ in the following way:

$$\rho(s) \le \frac{\int_0^\infty x\, e^{-sx}\, F(x)\, dx}{\int_0^\infty e^{-sx}\, F(x)\, dx} \le$$
$$\le 2\varepsilon + \frac{\int_{2\varepsilon}^\infty x\, e^{-sx}\, dx}{F(\varepsilon) \int_\varepsilon^\infty e^{-sx}\, dx} = 2\varepsilon + \frac{2\varepsilon + 1/s}{F(\varepsilon)}\, e^{-s\varepsilon},$$

which is less than 3ε for s sufficiently large. This proves the first equality in (3.7). ◻

The last inversion result we mention, concerns densities.

Proposition 3.4. *Let X be $\mathbb{R}_+$-valued with pLSt π, and suppose that X has an absolutely continuous distribution with density f for which $f(0+)$ exists (possibly infinite). Then*

(3.8) $$\lim_{s\to\infty} s\, \pi(s) = f(0+).$$

PROOF. First, let $f(0+) = \infty$. Then by Fatou's lemma we can estimate as follows:

$$\liminf_{s\to\infty} s\, \pi(s) = \liminf_{s\to\infty} \int_0^\infty e^{-y}\, f(y/s)\, dy \ge \int_0^\infty e^{-y}\, f(0+)\, dy,$$

which is infinite. Next, let $f(0+) < \infty$. Then we can take $\varepsilon > 0$ such that $f(x) \le f(0+) + 1$ for all $x \in (0, \varepsilon)$. Now, for $s > 0$ write $s\,\pi(s)$ as

$$s\,\pi(s) = \int_0^\infty \mathrm{e}^{-y}\, f(y/s)\, 1_{(0,s\varepsilon)}(y)\,\mathrm{d}y + \int_\varepsilon^\infty s\,\mathrm{e}^{-sx}\, f(x)\,\mathrm{d}x,$$

and apply the dominated convergence theorem to both terms; with respect to the second term note that $s\,\mathrm{e}^{-sx} \le 1/(\mathrm{e}\,\varepsilon)$ for $s > 0$ and $x > \varepsilon$. □

Analytic extension of a pLSt. Sometimes we want to consider the pLSt π of a random variable X for *complex* values of its argument. Any π has an abscissa of convergence; this is the smallest $s_\pi \in [-\infty, 0]$ such that $\pi(z) := \mathbb{E}\,\mathrm{e}^{-zX}$ is well defined for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z > s_\pi$. The set of these values of z is called the *halfplane of analyticity* of π, because π can be shown to be analytic on this set. In fact, we have $s_\pi = -u_\phi$ with u_ϕ one of the abscissas of convergence of the corresponding characteristic function ϕ; cf. Section 2. If π has no zeroes in its halfplane of analyticity, then, as before for ϕ, one can show that $\log\,\pi$, and hence π^t with $t > 0$, can be defined on this halfplane such that it is continuous there with $\log\,\pi(0) = 0$.

LSt. Consider, more general than distribution functions on $\mathbb{R}_+$, *LSt-able* functions G, i.e., G is a right-continuous nondecreasing function on $\mathbb{R}$ with $G(x) = 0$ for $x < 0$ and such that $\widehat{G}(s)$ is finite for all $s > 0$. Here $\widehat{G}$ is the *Laplace-Stieltjes transform* (LSt) of G:

$$\widehat{G}(s) = \int_{\mathbb{R}_+} \mathrm{e}^{-sx}\,\mathrm{d}G(x) \qquad [\,s > 0\,];$$

if G is absolutely continuous with density g, then $\widehat{G}$ is the *Laplace transform* (Lt) of g. Several results for pLSt's above have obvious analogues for LSt's. The uniqueness theorem says for LSt-able functions G and H that $G = H$ iff $\widehat{G} = \widehat{H}$ $\big($on $(0, \infty)$ or, also sufficient, on (s_0, ∞) for some $s_0 > 0\big)$. So, if $G \not\equiv 0$, then $\widehat{G}$ is positive on $(0, \infty)$; mostly we silently exclude the case where $G \equiv 0$. Further, for LSt-able functions F, G and H we have

$$F = G \star H \iff \widehat{F} = \widehat{G}\,\widehat{H}. \tag{3.9}$$

The continuity theorem, as given in Theorem 3.1 and reformulated by use of Helly's theorem $\big($cf. (2.10)$\big)$, can be extended as follows; the need for the boundedness condition in (i) is shown, e.g., by the special case where $G_n = \mathrm{e}^{n^2} 1_{[n,\infty)}$.

Theorem 3.5. *For $n \in \mathbb{N}$ let G_n be an LSt-able function with LSt ρ_n.*

(i) *If $\lim_{n\to\infty} G_n(x) = G(x)$ for all continuity points x of G and the sequence $\big(\rho_n(s)\big)_{n\in\mathbb{N}}$ is bounded for every $s > 0$, then G is an LSt-able function with LSt ρ, say, and $\lim_{n\to\infty} \rho_n(s) = \rho(s)$ for $s > 0$.*

(ii) *If $\lim_{n\to\infty} \rho_n(s) = \rho(s)$ for $s > 0$, then ρ is the LSt of an LSt-able function G, say, and $\lim_{n\to\infty} G_n(x) = G(x)$ for all continuity points x of G.*

Finally, we mention some inversion results for an LSt-able function G. We have

$$(3.10) \qquad \widehat{G}(0+) = \lim_{x\to\infty} G(x) \ (\le \infty), \qquad \lim_{s\to\infty} \widehat{G}(s) = G(0),$$

and the left extremity ℓ_G of G and the limit $g(0+)$ for a density g of G can be obtained as in Propositions 3.3 and 3.4.

LSt and complete monotonicity. From (3.3) it immediately follows that the pLSt $\widehat{F}$ of a distribution function F on $\mathbb{R}_+$ is completely monotone. Here a real-valued function ρ on $(0,\infty)$ is said to be *completely monotone* if ρ is nonnegative and has derivatives of all orders, alternating in sign:

$$(-1)^n \rho^{(n)}(s) \ge 0 \qquad [\, n \in \mathbb{Z}_+;\ s > 0\,].$$

Similarly and more general, the LSt $\widehat{G}$ of an LSt-able function G is completely monotone. There is an important converse of this result.

Theorem 3.6 (Bernstein). *A real-valued function ρ on $(0,\infty)$ is completely monotone iff there exists an LSt-able function G such that $\rho = \widehat{G}$, i.e., such that*

$$\rho(s) = \int_{\mathbb{R}_+} e^{-sx}\, dG(x) \qquad [\, s > 0\,].$$

In the proposition below we give a list of useful properties of completely monotone functions. Most of them are easily proved. For instance, property (iv) and the second part of (iii) follow from Theorem 3.5 (ii) and (3.9), respectively, by use of Bernstein's theorem. The latter case can also be proved just from the definition; use Leibniz's rule or induction. A combination of these two methods proves the important result (vi). For absolute monotonicity see Section 4.

Proposition 3.7. *Consider real-valued functions ρ, σ, ρ_n on $(0,\infty)$.*

(i) *ρ is completely monotone iff $\lim_{s\to\infty}\rho(s)\geq 0$ and $-\rho'$ is completely monotone.*

(ii) *If ρ is completely monotone, then so are the functions $a\rho$, $\rho(a\,\cdot)$, $\rho(\cdot+a)$ and $\rho-\rho(\cdot+a)$ for every $a>0$.*

(iii) *If ρ and σ are completely monotone, then so are $\rho+\sigma$ and $\rho\,\sigma$.*

(iv) *If ρ_n is completely monotone for all $n\in\mathbb{N}$ and $\rho:=\lim_{n\to\infty}\rho_n$ exists and is finite, then ρ is completely monotone.*

(v) *If Q is absolutely monotone on $[0,a)$ for some $a\in(0,\infty]$ and ρ is completely monotone with $\rho<a$ on $(0,\infty)$, then $Q\circ\rho$ is completely monotone. For instance, if ρ is completely monotone, then so are $\exp[\rho]$ and $(1-\rho)^{-1}$, in the latter case if $\rho<1$ on $(0,\infty)$.*

(vi) *If ρ and σ' are completely monotone with $\sigma(0+)\geq 0$, then $\rho\circ\sigma$ is completely monotone. For instance, if σ' is completely monotone with $\sigma(0+)\geq 0$, then $\exp[-\sigma]$ and $(1+\sigma)^{-1}$ are completely monotone.*

Proof. We only prove (vi) in the way indicated above. Let D^n denote 'differentiation n times', set $(-D)^n:=(-1)^nD^n$, and reformulate the assertion as follows: For every $n\in\mathbb{Z}_+$, for every completely monotone ρ, for every completely monotone σ' with $\sigma(0+)\geq 0$: $(-D)^n(\rho\circ\sigma)\geq 0$. Now, observe that by Leibniz's rule we have

$$(-D)^{n+1}(\rho\circ\sigma)=\sum_{k=0}^{n}\binom{n}{k}(-D)^k\big((-\rho')\circ\sigma\big)\,(-D)^{n-k}\sigma',$$

and apply induction with respect to n. ◻◻

Complete monotonicity and log-convexity. A completely monotone function ρ is nonnegative satisfying either $\rho=0$ on $(0,\infty)$ or $\rho>0$ on $(0,\infty)$. Moreover, it is nonincreasing and convex; here ρ is said to be *convex* (cf. Section 5) if

$$\rho\big(\tfrac12(s+t)\big)\leq\tfrac12\,\{\rho(s)+\rho(t)\}\qquad[\,s,t>0\,].$$

Clearly, the set of convex functions is closed under addition. Also, by using the inequality $\sqrt{ab}\leq\frac12\,(a+b)$ for $a,b\geq 0$, one proves the following closure property:

(3.11) $\qquad \rho$ convex $\implies$ $\exp[\rho]$ convex.

One can 'interpolate' between complete monotonicity and convexity by means of log-convexity. A *nonnegative* function ρ on $(0,\infty)$ is said to be *log-convex* if

$$\bigl\{\rho\bigl(\tfrac{1}{2}(s+t)\bigr)\bigr\}^2 \le \rho(s)\,\rho(t) \qquad [\,s,t>0\,].$$

Indeed, combining Bernstein's theorem and Schwarz's inequality one obtains the following result.

Proposition 3.8. *A completely monotone function is log-convex.*

PROOF. Let ρ be completely monotone. We restrict ourselves to the special case where $\rho(0+)=1$; then ρ is a pLSt, so $\rho(s)=\mathbb{E}\,\mathrm{e}^{-sX}$ for some $\mathbb{R}_+$-valued random variable X. It follows that for $s,t>0$

$$\bigl\{\rho\bigl(\tfrac{1}{2}(s+t)\bigr)\bigr\}^2 = (\mathbb{E}\,\mathrm{e}^{-\frac{1}{2}sX}\mathrm{e}^{-\frac{1}{2}tX})^2 \le \mathbb{E}\,\mathrm{e}^{-sX}\,\mathbb{E}\,\mathrm{e}^{-tX} = \rho(s)\,\rho(t),$$

which shows that ρ is log-convex. ◻

On the other hand, from (3.11) it follows that for a nonnegative function ρ on $(0,\infty)$ one has

(3.12) $\qquad \rho$ log-convex $\implies$ ρ convex;

just note that a log-convex function ρ satisfies either $\rho=0$ on $(0,\infty)$ or $\rho>0$ on $(0,\infty)$, and that in the latter case log-convexity is equivalent to convexity of the function $\log\rho$. For functions that are positive and twice differentiable, log-convexity can also be characterized in terms of convexity as in part (ii) of the following proposition.

Proposition 3.9. *Let ρ be a positive, twice differentiable function on $(0,\infty)$. Then log-convexity of ρ is equivalent to each of the following properties of ρ:*

(i) $\bigl\{\rho'(s)\bigr\}^2 \le \rho(s)\,\rho''(s)$ *for all* $s>0$.

(ii) *The function* ρ_λ *on* $(0,\infty)$ *with* $\rho_\lambda(s) := e^{\lambda s}\rho(s)$ *is convex for all* $\lambda\in\mathbb{R}$.

PROOF. The characterization in part (i) follows from the fact that the function $\psi := \log\rho$ is convex iff $\psi''\ge 0$. Turning to part (ii) we use the same criterion for convexity of ρ_λ; then we see that (ii) is equivalent to

$$\bigl\{\lambda^2\rho(s)+2\lambda\rho'(s)+\rho''(s)\bigr\}\,\mathrm{e}^{\lambda s}\ge 0 \qquad [\,s>0;\ \lambda\in\mathbb{R}\,],$$

which means that the quadratic form in λ above must have a non-positive discriminant for $s > 0$. But this yields precisely the inequalities in (i). ◻

An immediate consequence of this characterization is the not so obvious property that for positive, twice differentiable functions on $(0, \infty)$ log-convexity, like complete monotonicity and convexity, is preserved under addition. Now, any log-convex function can be written as the pointwise limit of a sequence of log-convex functions that are positive and twice differentiable. Also, log-convexity is preserved under pointwise limits, since it is defined in terms of weak inequalities. Hence we may conclude that the set of *all* log-convex functions is closed under addition. We formally state this together with closedness under multiplication, which is trivial.

Proposition 3.10. *If ρ and σ are log-convex, then so are $\rho + \sigma$ and $\rho\,\sigma$.*

Monotone densities. So far, we considered complete monotonicity and log-convexity mainly because *LSt's* of LSt-able functions have these properties. But *densities* of (absolutely continuous) LSt-able functions with these properties are also of interest. If a density is completely monotone, then it is *monotone*; a log-convex density need not be monotone. We will further restrict ourselves to *probability* densities f on $(0, \infty)$; now f is monotone if it is log-convex. Complete monotonicity can be characterized as follows.

Proposition 3.11. *A probability density f on $(0, \infty)$ is completely monotone iff it is a mixture of exponential densities, i.e., there exists a distribution function G on $(0, \infty)$ such that*

$$f(x) = \int_{(0,\infty)} \lambda\, e^{-\lambda x}\, dG(\lambda) \qquad [\, x > 0\,]. \tag{3.13}$$

PROOF. Let f be completely monotone. Then by Bernstein's theorem there exists an LSt-able function H such that $f(x) = \widehat{H}(x)$ for $x > 0$. Since f is a probability density, the function H necessarily satisfies $H(0) = 0$ and $\int_{(0,\infty)} (1/y)\, \mathrm{d}H(y) = 1$. Now, set

$$G(\lambda) := \int_{(0,\lambda]} \frac{1}{y}\, \mathrm{d}H(y) \qquad [\, \lambda > 0\,];$$

then G is a distribution function on $(0, \infty)$ and f can be written as in (3.13). The converse is trivial. ◻

We next characterize the monotone probability densities on $(0, \infty)$ or, slightly more general, the distribution functions on $\mathbb{R}_+$ that are *concave*; cf. Theorem 2.9. To this end we first prove the following result; here, and in the subsequent theorem, U is a random variable with a uniform distribution on $(0, 1)$.

Proposition 3.12. *Let X and Z be positive random variables with distribution functions F and G, respectively. Then $X \stackrel{d}{=} UZ$ with U and Z independent iff X has a density f such that F and G are related by*

$$(3.14) \qquad F(x) = G(x) + x\, f(x) \qquad [x > 0],$$

or, equivalently, such that f is given by

$$(3.15) \qquad f(x) = \int_{(x,\infty)} \frac{1}{y}\, dG(y) \qquad [x > 0].$$

PROOF. Integration by parts, or Fubini's theorem, shows that for any distribution function H:

$$(3.16) \qquad \int_{(x,\infty)} \frac{1}{y}\, \mathrm{d}H(y) = \int_x^\infty H(y) \frac{1}{y^2}\, \mathrm{d}y - \frac{1}{x} H(x) \qquad [x > 0].$$

By calculating F_{UZ} and then using (3.16) with $H = G$, one obtains

$$(3.17) \qquad F_{UZ}(x) = x \int_x^\infty G(y) \frac{1}{y^2}\, \mathrm{d}y = G(x) + x\, f(x) \qquad [x > 0],$$

with f given by (3.15). To show that this function f is indeed a density of UZ we use Fubini's theorem to rewrite the right-hand side of (3.17), and obtain in this way

$$F_{UZ}(x) = \int_{(0,x]} \int_0^y \mathrm{d}t\, \frac{\mathrm{d}G(y)}{y} + \int_{(x,\infty)} \int_0^x \mathrm{d}t\, \frac{\mathrm{d}G(y)}{y} = \int_0^x f(t)\, \mathrm{d}t.$$

Next, let X have a density f such that (3.14) holds. Applying (3.16) twice (with $H = G$ and $H = F$) then shows that f is necessarily given by (3.15). Finally, if X has f in (3.15) as a density, then $X \stackrel{d}{=} UZ$ because, as we saw above, f is also a density of UZ. ◻◻

Theorem 3.13. *An $\mathbb{R}_+$-valued random variable X has a concave distribution function iff an $\mathbb{R}_+$-valued random variable Z exists such that $X \stackrel{d}{=} UZ$ with U and Z independent.*

PROOF. First, let X be positive. If $X \overset{d}{=} UZ$ for some $\mathbb{R}_+$-valued random variable Z independent of U, then also Z is positive and by Proposition 3.12 X has a concave distribution function F. Conversely, let F be concave, so X has a monotone density f, which, of course, may be taken continuous from the right. In view of Proposition 3.12 we define the function G by (3.14); then we only have to show that G is a distribution function. Now, for $x > 0$ and $h > 0$ we can write

$$G(x+h) - G(x) = \int_x^{x+h} f(y)\,\mathrm{d}y - (x+h)\,f(x+h) + x\,f(x),$$

which because of the monotonicity of f is nonnegative. Hence G is nondecreasing. Since $F(x) \geq x\,f(x)$ for all $x > 0$, we also have

$$0 \leq G(x) \leq F(x) \qquad [\,x > 0\,].$$

It follows that the limit $p := \lim_{x\to\infty} G(x)$ exists with $p \leq 1$, and hence that $\lim_{x\to\infty} x\,f(x) = 1 - p$, which implies that $p = 1$.
Next, assume that $\mathbb{P}(X = 0) > 0$. Then X can be uniquely represented as $X \overset{d}{=} A\widetilde{X}$ where A and $\widetilde{X}$ are independent, A is $\{0,1\}$-valued, and $\widetilde{X}$ is positive; in fact, we have $\mathbb{P}(A = 0) = \mathbb{P}(X = 0)$ and $\widetilde{X} \overset{d}{=} (X \,|\, X > 0)$. If Z is an $\mathbb{R}_+$-valued random variable with $\mathbb{P}(Z = 0) > 0$ that is represented similarly as $Z \overset{d}{=} B\widetilde{Z}$, then we have $X \overset{d}{=} UZ$ iff $\widetilde{X} \overset{d}{=} U\widetilde{Z}$ and $A \overset{d}{=} B$. Now use the fact that the theorem has already been proved for $\widetilde{X}$. ◻

We finally state an immediate consequence of Proposition 3.12 and Theorem 3.13 that can be viewed as a representation theorem for monotone densities. The second part follows from the first one by putting: $H(x) := \mu \int_{(0,x]} (1/y)\,\mathrm{d}G(y)$ for $x > 0$; it establishes a connection with renewal theory.

Theorem 3.14. *A positive random variable X has a concave distribution function iff a distribution function G on $(0,\infty)$ exists such that X has a density f given by*

$$f(x) = \int_{(x,\infty)} \frac{1}{y}\, dG(y) \qquad [\,x > 0\,]. \tag{3.18}$$

In this case, $f(0+) < \infty$ iff a distribution function H on $(0,\infty)$ exists with finite mean μ such that

$$f(x) = \frac{1}{\mu}\{1 - H(x)\} \qquad [\,x > 0\,]. \tag{3.19}$$

4. Distributions on the nonnegative integers

Introductory remarks. In this section we restrict ourselves to random variables X that are $\mathbb{Z}_+$-*valued*. The distribution $\mathbb{P}_X$ of X is then discrete and its density vanishes everywhere outside $\mathbb{Z}_+$. So, $\mathbb{P}_X$ is determined by the sequence $p = (p_k)_{k\in\mathbb{Z}_+}$ with $p_k := \mathbb{P}(X = k)$; this sequence p will also be called the (probability) *distribution* of X. Note that results for $\mathbb{Z}_+$-valued random variables can be used to obtain results for $\mathbb{R}$-valued random variables X with values in the *lattice* $h\,\mathbb{Z}_+$ with $h > 0$, because then X/h is $\mathbb{Z}_+$-valued. Of course, the general results of Sections 2 and 3 apply to the present situation, but for distributions on $\mathbb{Z}_+$ they are not always convenient; sometimes more detailed results are available, which can often be obtained by more appropriate 'discrete' tools. We will discuss some of these briefly in what follows.

Convolution and support. Consider, more general than probability distributions $p = (p_k)_{k\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$, sequences $q = (q_k)_{k\in\mathbb{Z}_+}$ of real numbers. If $r = (r_k)_{k\in\mathbb{Z}_+}$ is another such sequence, then for the convolution $q * r$ of q and r we have

$$(q * r)_k = \sum_{i=0}^{k} q_i\, r_{k-i} = \sum_{j=0}^{k} q_{k-j}\, r_j \qquad [\,k \in \mathbb{Z}_+\,].$$

Of course (cf. Section 2), the *support* $S(q)$ of a sequence $q = (q_k)_{k\in\mathbb{Z}_+}$ of nonnegative numbers is defined as $S(q) := \{k \in \mathbb{Z}_+ : q_k > 0\}$, so $S(q)$ and the set of zeroes of q are complementary sets (with respect to $\mathbb{Z}_+$). Clearly, the support of the convolution of two sequences of nonnegative numbers is equal to the direct sum of the supports:

$$S(q * r) = S(q) \oplus S(r). \tag{4.1}$$

Expectation. For the expectation of a function g of a $\mathbb{Z}_+$-valued random variable X with distribution (p_k) we have

$$\mathbb{E}\, g(X) = \sum_{k=0}^{\infty} g(k)\, p_k,$$

provided that this sum exists (possibly ∞ or $-\infty$). The mean of X can be obtained by summing tail probabilities:

$$\mathbb{E}X = \sum_{n=0}^{\infty} \mathbb{P}(X > n). \tag{4.2}$$

Convergence in distribution. Let $X, X_1, X_2, \ldots$ be $\mathbb{Z}_+$-valued random variables, let X have distribution $(p_k)_{k\in\mathbb{Z}_+}$ and let X_n have distribution $(p_k^{(n)})_{k\in\mathbb{Z}_+}$. Then convergence in distribution of the sequence (X_n) to X can be characterized as follows:

$$(4.3) \qquad X_n \xrightarrow{\text{d}} X \iff \lim_{n\to\infty} p_k^{(n)} = p_k \text{ for all } k \in \mathbb{Z}_+.$$

Pgf: basic results. Rather than the characteristic function or the pLSt, for a $\mathbb{Z}_+$-valued random variable X with distribution (p_k) we use the *probability generating function* (pgf) of X (and of (p_k)), this is the function $P_X : [0,1] \to [0,1]$ with

$$P_X(z) := \mathbb{E}\, z^X = \sum_{k=0}^{\infty} p_k\, z^k \qquad [0 \le z \le 1].$$

Again, we have a *uniqueness theorem*, which says that $P_X = P_Y$ iff $X \stackrel{\text{d}}{=} Y$; in fact, the distribution (p_k) can be obtained from P_X by differentiation as follows:

$$(4.4) \qquad p_k = \frac{1}{k!}\, P_X^{(k)}(0) \qquad [k \in \mathbb{Z}_+].$$

Further, if X and Y are independent, then $P_{X+Y} = P_X\, P_Y$. The analogue of Theorems 2.3 and 3.1 can be stated as follows.

Theorem 4.1 (Continuity theorem). *For $n \in \mathbb{N}$ let X_n be a random variable with pgf P_n.*

(i) *If $X_n \xrightarrow{\text{d}} X$ as $n \to \infty$, then $\lim_{n\to\infty} P_n(z) = P_X(z)$ for $0 \le z \le 1$.*

(ii) *If $\lim_{n\to\infty} P_n(z) = P(z)$ for $0 \le z \le 1$ with P a function that is (left-) continuous at one, then P is the pgf of a random variable X and $X_n \xrightarrow{\text{d}} X$ as $n \to \infty$.*

Pgf and moments. The pgf P of a random variable X with distribution (p_k) is a continuous function on $[0,1]$ with $P(1) = 1$, and possesses derivatives of all orders on $[0,1)$ with

$$(4.5) \qquad P^{(n)}(z) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\, p_k\, z^{k-n} \quad [n \in \mathbb{N};\ 0 \le z < 1].$$

By letting $z \uparrow 1$ it follows that for $n \in \mathbb{N}$ the left-hand limit of $P^{(n)}$ at one exists with $P^{(n)}(1-) = \mathbb{E}\, X(X-1)\cdots(X-n+1)$, possibly infinite;

hence the n-th moment μ_n of X is finite iff $P^{(n)}(1-) < \infty$. Taking $n = 1$ we get $P'(1-) = \mu_1$. On the other hand, by Fubini's theorem we have

$$(4.6)\qquad \frac{1 - P(z)}{1 - z} = \sum_{n=0}^{\infty} \mathbb{P}(X > n)\, z^n \qquad [\,0 \le z < 1\,],$$

so using (4.2) one sees that the (left-hand) derivative of P at one exists and satisfies $P'(1) = \mu_1$. Hence $P'(1-) = P'(1)$, so if $\mu_1 < \infty$, then P', like P, is continuous on $[\,0,1\,]$. Moreover, by the mean value theorem we can write $\big(1 - P(z)\big)/(1 - z) = P'(\theta_z)$ for $0 \le z < 1$ and some $\theta_z \in (z,1)$; since P' is nondecreasing, it follows that

$$(4.7)\qquad 0 \le P'(z) \le \frac{1 - P(z)}{1 - z} \ \text{ for } 0 \le z < 1, \ \text{ so } \lim_{z \uparrow 1}\, (1 - z)\, P'(z) = 0,$$

also when $\mu_1 = \infty$. In fact, (4.7) holds for all functions P on $[\,0,1)$ with $P(1-) = 1$ that are positive, nondecreasing and convex, and have a continuous derivative. The behaviour of $\log P$ at one is related to *logarithmic moments* of X.

Proposition 4.2. *Let X be $\mathbb{Z}_+$-valued with pgf P. Then the following four quantities are either all finite or all infinite:*

$$\mathbb{E}\,\log\,(1 + X),\ \ \mathbb{E}\,\log^+ X,\ \ \int_0^1 \frac{1 - P(z)}{1 - z}\, dz,\ \ \int_0^1 \frac{-\log\, P(z)}{1 - z}\, dz.$$

Proof. The first two quantities are simultaneously finite because one has $\log\,(1 + k) \sim \log k$ as $k \to \infty$, and the last two because $-\log\, P(z) \sim 1 - P(z)$ as $z \uparrow 1$. Next, observe that by (2.4) and (4.6)

$$\mathbb{E}\,\log^+ X = \int_1^{\infty} \frac{1}{x}\, \mathbb{P}(X > x)\, \mathrm{d}x,$$

$$\int_0^1 \frac{1 - P(z)}{1 - z}\, \mathrm{d}z = \sum_{n=0}^{\infty} \frac{1}{n+1}\, \mathbb{P}(X > n).$$

Now, an application of the integral test for series completes the proof. ◻

Analytic extension of a pgf. Sometimes we want to consider the pgf P of a random variable X for *complex* values of its argument. Any P has a radius of convergence; this is the smallest $\rho \in [\,1,\infty]$ such that $P(z) := \mathbb{E}\, z^X$ is well-defined for all $z \in \mathbb{C}$ with $|z| < \rho$. The set of these values of z is called the *disk of analyticity* of P, because P can be shown to be analytic on this set. If P has no zeroes in its disk of analyticity, then, as before for characteristic functions and pLSt's, one can show that $\log P$, and hence P^t with $t > 0$, can be defined on this disk such that it is continuous there with $\log P(1) = 0$.

Gf and absolute monotonicity. From (4.5) it immediately follows that the pgf P of a distribution (p_k) on $\mathbb{Z}_+$ is absolutely monotone. Here a real-valued function R on $[0,1)$ is said to be *absolutely monotone* if R is nonnegative and possesses nonnegative derivatives of all orders:

$$R^{(n)}(z) \geq 0 \qquad [n \in \mathbb{Z}_+;\ 0 \leq z < 1].$$

Occasionally, we need a somewhat more general concept: For $a > 0$ a function R on $[0,a)$ is said to be *absolutely monotone on* $[0,a)$ if $R^{(n)}(z) \geq 0$ for all $n \in \mathbb{Z}_+$ and $0 \leq z < a$. For the same reason as for P above the *generating function* (gf) R of a sequence $r = (r_k)_{k \in \mathbb{Z}_+}$ of nonnegative numbers is absolutely monotone (provided R is finite on $[0,1)$). There is a converse.

Theorem 4.3. *A real-valued function R on $[0,1)$ is absolutely monotone iff there exists a sequence $r = (r_k)_{k \in \mathbb{Z}_+}$ of nonnegative numbers such that R is the gf of r, i.e., such that*

$$R(z) = \sum_{k=0}^{\infty} r_k\, z^k \qquad [0 \leq z < 1].$$

Of course, also for gf's there is a uniqueness and a convolution theorem; we do not make them explicit. The following proposition contains a list of useful properties of absolutely monotone functions; they are easily proved.

Proposition 4.4. *Consider real-valued functions R, S, R_n on $[0,1)$.*

(i) *R is absolutely monotone iff $z \mapsto (d/dz)\big[\, z\, R(z)\big]$ is absolutely monotone.*

(ii) *R is absolutely monotone iff R' is absolutely monotone and $R(0) \geq 0$.*

(iii) *If R is absolutely monotone, then so are aR for every $a > 0$, and $R(1-\alpha+\alpha\,\cdot)$, $R(\alpha\,\cdot)$ and $R - R(\alpha\,\cdot)$ for every $\alpha \in (0,1)$.*

(iv) *If R and S are absolutely monotone, then so are $R+S$ and $R\,S$.*

(v) *If R_n is absolutely monotone for all $n \in \mathbb{N}$ and $R := \lim_{n\to\infty} R_n$ exists and is finite, then R is absolutely monotone.*

(vi) *If Q is absolutely monotone on $[0, a)$ for some $a \in (0, \infty]$ and R is absolutely monotone with $R < a$ on $[0, 1)$, then $Q \circ R$ is absolutely monotone. For instance, if R is absolutely monotone, then so are* $\exp[R]$ *and* $(1 - R)^{-1}$, *in the latter case if $R < 1$ on $[0, 1)$.*

Completely monotone and log-convex sequences. The concepts of complete monotonicity and log-convexity for functions on $(0, \infty)$, as introduced in Section 3, have discrete counterparts. A sequence $r = (r_k)_{k \in \mathbb{Z}_+}$ in $\mathbb{R}$ is said to be *completely monotone* if

$$(-1)^n \Delta^n r_k \geq 0 \qquad [n \in \mathbb{Z}_+;\ k \in \mathbb{Z}_+],$$

where $\Delta^0 r_k := r_k$, $\Delta r_k := r_{k+1} - r_k$ and $\Delta^n := \Delta \circ \Delta^{n-1}$. If X is a $[0, 1]$-valued random variable and, for $k \in \mathbb{Z}_+$, $\mu_k := \mathbb{E}\, X^k$ is the k-th moment of X, then one can show that

$$(-1)^n \Delta^n \mu_k = \mathbb{E}\, X^k (1 - X)^n \qquad [n \in \mathbb{Z}_+;\ k \in \mathbb{Z}_+],$$

so the moment sequence $(\mu_k)_{k \in \mathbb{Z}_+}$ is completely monotone. There is an important converse of this.

Theorem 4.5 (Hausdorff). *A sequence $r = (r_k)_{k \in \mathbb{Z}_+}$ in $\mathbb{R}$ is completely monotone iff it can be represented as*

$$r_k = \int_{[0,1]} x^k \, \nu(dx) \qquad [k \in \mathbb{Z}_+],$$

where ν is a (necessarily finite) measure on $[0, 1]$. Equivalently, (r_k) is completely monotone iff it can be written as

$$r_k = r_0\, \mathbb{E} X^k \qquad [k \in \mathbb{Z}_+],$$

where X is a random variable with values in $[0, 1]$.

Using this theorem one easily shows that the set of completely monotone sequences is closed under addition and multiplication.

Proposition 4.6. *If (r_k) and (s_k) are completely monotone, then so are $(r_k + s_k)$ and $(r_k s_k)$.*

Note that a completely monotone sequence (r_k) is nonnegative satisfying either $r_k = 0$ for all $k \in \mathbb{N}$ or $r_k > 0$ for all $k \in \mathbb{Z}_+$. Moreover, it is nonincreasing and convex; here (r_k) is said to be *convex* if

$$r_k \leq \tfrac{1}{2}\,(r_{k-1} + r_{k+1}) \qquad [\,k \in \mathbb{N}\,].$$

Clearly, the set of convex sequences is closed under addition. Moreover, by using the inequality $\sqrt{ab} \leq \frac{1}{2}\,(a+b)$ for $a, b \geq 0$, one proves the following closure property:

$$(4.8) \qquad (r_k) \text{ convex} \implies \big(\exp[r_k]\big) \text{ convex.}$$

One can 'interpolate' between complete monotonicity and convexity by means of log-convexity. A sequence $r = (r_k)_{k \in \mathbb{Z}_+}$ of *nonnegative* numbers is said to be *log-convex* if

$$r_k^2 \leq r_{k-1}\, r_{k+1} \qquad [\,k \in \mathbb{N}\,].$$

Indeed, combining Hausdorff's theorem and Schwarz's inequality one obtains the following result.

Proposition 4.7. *A completely monotone sequence in $\mathbb{R}$ is log-convex.*

Proof. Let (r_k) be completely monotone; it is no essential restriction to assume that $r_0 = 1$. Then $r_k = \mathbb{E}X^k$ for all k, for some $[\,0,1\,]$-valued random variable X, and hence for $k \in \mathbb{N}$

$$r_k^2 = \big(\mathbb{E}\, X^{(k-1)/2} X^{(k+1)/2}\big)^2 \leq (\mathbb{E}\, X^{k-1})\,(\mathbb{E}\, X^{k+1}) = r_{k-1}\, r_{k+1},$$

which shows that (r_k) is log-convex. □

On the other hand, from (4.8) it follows that for a sequence $(r_k)_{k \in \mathbb{Z}_+}$ of nonnegative numbers one has

$$(4.9) \qquad (r_k) \text{ log-convex} \implies (r_k) \text{ convex;}$$

just note that a log-convex sequence (r_k) satisfies either $r_k = 0$ for all $k \in \mathbb{N}$ or $r_k > 0$ for all $k \in \mathbb{Z}_+$, and that in the latter case log-convexity is equivalent to convexity of the sequence $(\log r_k)$. Log-convexity can also be characterized in terms of convexity in the following way.

Proposition 4.8. *A sequence $r = (r_k)_{k \in \mathbb{Z}_+}$ of nonnegative numbers is log-convex iff the sequence $(a^k r_k)$ is convex for every $a > 0$.*

Proof. The convexity of $(a^k r_k)$ for every $a > 0$ is equivalent to saying that

$$\{a^2 r_{k+1} - 2a r_k + r_{k-1}\}\, a^{k-1} \geq 0 \qquad [\, k \in \mathbb{N};\ a > 0\,].$$

This means that in the left-hand side the quadratic form in a, which is nonnegative for $a \leq 0$, must have a non-positive discriminant for all $k \in \mathbb{N}$; this yields precisely the inequalities defining the log-convexity of (r_k). □

An immediate consequence of this characterization is the not so obvious property that the set of log-convex sequences, like that of completely monotone sequences and that of convex sequences, is closed under addition; closedness under multiplication is trivial.

Proposition 4.9. *If (r_k) and (s_k) are log-convex, then so are $(r_k + s_k)$ and $(r_k s_k)$.*

$\mathbb{Z}_+$-valued fractions of $\mathbb{Z}_+$-valued random variables. Let X be a $\mathbb{Z}_+$-valued random variable, and let $\alpha \in (0,1)$. Extend the underlying probability space, if necessary, so that it supports independent random variables $Z_1, Z_2, \ldots$, independent of X, all having a standard Bernoulli (α) distribution: $\mathbb{P}(Z_i = 0) = 1 - \alpha$, $\mathbb{P}(Z_i = 1) = \alpha$, for all i. Then the *α-fraction* $\alpha \odot X$ of X is defined by

$$\alpha \odot X := Z_1 + \cdots + Z_X,$$

where an empty sum is zero. The random variable $\alpha \odot X$ is again $\mathbb{Z}_+$-valued and its distribution can be expressed in that of X as follows:

$$\text{(4.10)} \qquad \mathbb{P}(\alpha \odot X = k) = \sum_{n=k}^{\infty} \mathbb{P}(\alpha \odot n = k)\, \mathbb{P}(X = n) \qquad [\, k \in \mathbb{Z}_+\,].$$

Note that the α-fraction of a non-zero constant n is *not* a constant: $\alpha \odot n$ is *binomially* (n, α) distributed; therefore, the operation $\odot$ can be interpreted as *binomial thinning*. It has several properties in common with ordinary multiplication; one easily shows, for instance, that

$$\text{(4.11)} \qquad \alpha \odot (\beta \odot X) \stackrel{\mathrm{d}}{=} (\alpha\beta) \odot X \qquad [\, 0 < \alpha, \beta < 1\,].$$

We refer to Section V.8 for other properties and some more background information; here we only mention two uniqueness results:

$$\text{(4.12)} \qquad \begin{cases} \alpha \odot X \stackrel{\mathrm{d}}{=} \beta \odot X \text{ for some } X \not\equiv 0 \implies \alpha = \beta, \\ \alpha \odot X \stackrel{\mathrm{d}}{=} \alpha \odot Y \text{ for some } \alpha \in (0,1) \implies X \stackrel{\mathrm{d}}{=} Y. \end{cases}$$

It is convenient to define $\alpha \odot X$ also when $\alpha \in \{0,1\}$; we set $0 \odot X :\equiv 0$, $1 \odot X := X$. In terms of pgf's (4.10) takes the following compact form:

$$P_{\alpha \odot X}(z) = P_X(1 - \alpha + \alpha z). \tag{4.13}$$

Often we only need $\alpha \odot X$ *in distribution*. The same holds for $V \odot X$ with V a $[0,1]$-valued random variable; by definition its distribution is obtained as follows:

$$\mathbb{P}(V \odot X = k) = \int_{[0,1]} \mathbb{P}(\alpha \odot X = k)\, \mathrm{d}F_V(\alpha) \qquad [k \in \mathbb{Z}_+].$$

In the special case where X is *Poisson*(λ) distributed, we can allow here arbitrary $\alpha \geq 0$ and $V \geq 0$; then $\alpha \odot X$ is Poisson$(\alpha\lambda)$ distributed and $V \odot X$ satisfies

$$P_{V \odot X}(z) = \int_{\mathbb{R}_+} \exp\left[-\alpha\lambda\,(1-z)\right] \mathrm{d}F_V(\alpha) = \widehat{F}_V\big(\lambda\,\{1-z\}\big). \tag{4.14}$$

Monotone distributions. A completely monotone sequence is *monotone*; a log-convex sequence need not be monotone. We will further restrict ourselves to *probability distributions* $p = (p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$; now p is monotone if it is log-convex. Complete monotonicity can be characterized as follows; note that in (4.15) a mixing with the degenerate distribution at zero is possibly included.

Proposition 4.10. *A probability distribution $p = (p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ is completely monotone iff it is a mixture of geometric distributions, i.e., there exists a distribution function H on $[0,1)$ such that*

$$p_k = \int_{[0,1)} (1-p)\, p^k\, dH(p) \qquad [k \in \mathbb{Z}_+]. \tag{4.15}$$

PROOF. Let p be completely monotone. Then by Hausdorff's theorem there exists a finite measure ν on $[0,1]$ such that $p_k = \int_{[0,1]} x^k\, \nu(\mathrm{d}x)$ for $k \in \mathbb{Z}_+$. Since p is a probability distribution, the measure ν necessarily satisfies $\nu(\{1\}) = 0$ and $\int_{[0,1)} 1/(1-x)\, \nu(\mathrm{d}x) = 1$. Now, set

$$H(p) := \int_{[0,p]} \frac{1}{1-x}\, \nu(\mathrm{d}x) \qquad [0 \leq p < 1];$$

then H is a distribution function on $[0,1)$ and p can be written as in (4.15). The converse is trivial. ◻

We next characterize the monotone distributions on $\mathbb{Z}_+$, and give discrete counterparts of Theorems 3.13 and 3.14. To this end, let U be uniformly distributed on $(0,1)$, and let Z be $\mathbb{Z}_+$-valued. Then one easily verifies that the distribution of $U \odot Z$, as defined above, is given by

$$(4.16) \qquad \mathbb{P}(U \odot Z = k) = \sum_{n=k}^{\infty} \frac{1}{n+1} \mathbb{P}(Z = n) \qquad [\, k \in \mathbb{Z}_+ \,].$$

Letting $\lfloor a \rfloor$ denote the largest integer not exceeding a, and taking U and Z independent, one may also express this relation in the following way:

$$(4.17) \qquad U \odot Z \stackrel{\mathrm{d}}{=} \big\lfloor U(1+Z) \big\rfloor.$$

The distribution of Z is uniquely determined by that of $X := U \odot Z$; from (4.16) it follows that

$$(4.18) \qquad \mathbb{P}(Z = n) = (n+1)\big\{\mathbb{P}(X = n) - \mathbb{P}(X = n+1)\big\} \qquad [\, n \in \mathbb{Z}_+ \,].$$

Theorem 4.11. *A $\mathbb{Z}_+$-valued random variable X has a monotone distribution iff a $\mathbb{Z}_+$-valued random variable Z exists such that $X \stackrel{\mathrm{d}}{=} U \odot Z$.*

Proof. If $X \stackrel{\mathrm{d}}{=} U \odot Z$ for some Z, then by (4.16) X has a monotone distribution $p = (p_k)$. Conversely, let p be monotone. In view of (4.18) we consider the sequence $q = (q_n)_{n \in \mathbb{Z}_+}$ with $q_n := (n+1)\,\{p_n - p_{n+1}\}$. Then $q_n \geq 0$ for all n, and by Fubini's theorem we have

$$\sum_{n=0}^{m} q_n = \sum_{k=0}^{m} p_k - (m+1)\, p_{m+1} \qquad [\, m \in \mathbb{Z}_+ \,].$$

It follows that both $(m+1)\, p_{m+1}$ and $\sum_{n=0}^{m} q_n$ have finite limits as $m \to \infty$; in fact, these limits are necessarily given by 0 and 1, respectively. Thus q is a probability distribution, and we can take a random variable Z with distribution q. By (4.16) we then have $X \stackrel{\mathrm{d}}{=} U \odot Z$. ◻◻

By putting $h_n := (\mu/n)\, \mathbb{P}(Z = n-1)$ for $n \in \mathbb{N}$ one is led to the following representation of monotone distributions on $\mathbb{Z}_+$; it establishes a connection with renewal theory.

Theorem 4.12. *A $\mathbb{Z}_+$-valued random variable X has a monotone distribution iff a probability distribution $(h_n)_{n \in \mathbb{N}}$ on $\mathbb{N}$ exists with finite mean μ such that*

$$(4.19) \qquad \mathbb{P}(X = k) = \frac{1}{\mu} \sum_{n=k+1}^{\infty} h_n \qquad [\, k \in \mathbb{Z}_+ \,].$$

5. Other auxiliaries

Convex functions. Let I be an open interval in $\mathbb{R}$, bounded or not. A function $f : I \to \mathbb{R}$ is said to be *convex* if

$$f\big(\alpha\, x + (1-\alpha)\, y\big) \le \alpha\, f(x) + (1-\alpha)\, f(y) \quad \big[\, x, y \in I;\ \alpha \in (0,1)\,\big].$$

It is easily shown that a convex function is continuous. On the other hand, a continuous function is convex if the inequality above only holds for $x, y \in I$ and $\alpha = \frac{1}{2}$; this property is sometimes called *J-convex* (for Jensen) or *mid-convex*. Though not-continuous mid-convex functions do exist, only very little is needed to make them continuous, and hence convex. Two sufficient conditions are (a) measurability; (b) boundedness from above in a neighbourhood of a (single) point of I. Therefore, since we only use measurable functions, we will identify convexity and mid-convexity.

Karamata's inequality. Let the function $f : \mathbb{R} \to \mathbb{R}$ be convex, and let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be finite nondecreasing sequences in $\mathbb{R}$ with the property that $\sum_{j=1}^m a_j \le \sum_{j=1}^m b_j$ for $m = 1, \ldots, n-1$. If, in addition, $\sum_{j=1}^n a_j = \sum_{j=1}^n b_j$, or f is nondecreasing and $\sum_{j=1}^n a_j \le \sum_{j=1}^n b_j$, or f is nonincreasing and $\sum_{j=1}^n a_j \ge \sum_{j=1}^n b_j$, then

$$\sum_{j=1}^{n} f(-a_j) \ge \sum_{j=1}^{n} f(-b_j).$$

Cramer's rule. For $n \in \mathbb{N}$ let $A = (a_{ij})_{i,j=1,..,n}$ be a real $(n \times n)$-matrix with $\det A \neq 0$, and let $b = (b_j)_{j=1,..,n} \in \mathbb{R}^n$. Then the equation

$$A\, x = b \qquad [\, x \in \mathbb{R}^n\,]$$

has exactly one solution $x = (x_j)_{j=1,..,n}$; it is given by

$$x_j = (\det A_b^{(j)})/(\det A) \qquad [\, j = 1, \ldots, n\,],$$

where $A_b^{(j)}$ is the matrix that is obtained from A by replacing $a_{1j}, \ldots, a_{nj}$ by $b_1, \ldots, b_n$, respectively.

Bürmann-Lagrange theorem. Let G be a function of a complex variable w, which is regular on $|w| \le 1$ and has no zeroes there. Further, suppose that $\big|G(w)\big| \le 1$ for $|w| = 1$. Then for every $z \in \mathbb{C}$ with $|z| < 1$ the equation

$$z\,G(w) = w$$

has exactly one solution $w = w(z)$ with $|w| < 1$, which can be expanded as

$$w(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \Big(\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \{G(x)\}^n \Big) \Big|_{x=0} \qquad \big[\, |z| < 1 \,\big];$$

more generally, for every function H that is regular on $|w| < 1$:

$$H\big(w(z)\big) = H(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \Big(\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} H'(x) \, \{G(x)\}^n \Big) \Big|_{x=0}.$$

Euler's constant. This is the constant γ defined by

$$\gamma := \lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \log n \Big) \quad [\approx 0.5772\,].$$

It can also be obtained as $\gamma = -\Gamma'(1)$, with Γ the gamma function, and as

$$\gamma = \lim_{t\to\infty} \Big(\int_0^t \frac{1 - \mathrm{e}^{-x}}{x} \,\mathrm{d}x - \log t \Big),$$

$$\gamma = \int_0^\infty \Big(\frac{1}{1 - \mathrm{e}^{-x}} - \frac{1}{x} \Big) \mathrm{e}^{-x} \,\mathrm{d}x,$$

$$\gamma = 1 - \int_0^\infty \Big(\sin x - \frac{x}{1 + x^2} \Big) \frac{1}{x^2} \,\mathrm{d}x.$$

Gamma function. This is the function $\Gamma \colon (0, \infty) \to (0, \infty)$ with

$$\Gamma(r) := \int_0^\infty x^{r-1} \,\mathrm{e}^{-x} \,\mathrm{d}x \qquad [\, r > 0\,].$$

The gamma function is log-convex and satisfies $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$ and $\Gamma(r+1) = r\,\Gamma(r)$ for $r > 0$, so $\Gamma(n+1) = n!$ for $n \in \mathbb{Z}_+$. According to *Stirling's formula* we have

$$\Gamma(r) \sim (\sqrt{2\pi})\, r^{r-\frac{1}{2}} \,\mathrm{e}^{-r} \qquad [\, r \to \infty\,];$$

here $a(r) \sim b(r)$ as $r \to \infty$ means that $a(r)/b(r) \to 1$ as $r \to \infty$. The gamma function is differentiable; its derivative is given by

$$\Gamma'(r) = \int_0^\infty x^{r-1} \,\mathrm{e}^{-x} \log x \,\mathrm{d}x \qquad [\, r > 0\,],$$

with $\Gamma'(1) = -\gamma$, where γ is Euler's constant. For $r \in (0,1)$ the integral $\Gamma(r)$ can be obtained as a complex integral in the following way:

$$\Gamma(r) = -(1-r)\,\mathrm{e}^{\frac{1}{2}(1-r)\pi i} \int_0^\infty \frac{\mathrm{e}^{it}-1}{t^{2-r}}\,\mathrm{d}t =$$
$$= i\,(2-r)\,(1-r)\,\mathrm{e}^{\frac{1}{2}(1-r)\pi i} \int_0^\infty \frac{\mathrm{e}^{it}-1-it}{t^{3-r}}\,\mathrm{d}t.$$

The domain of Γ can be extended. For $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ one defines $\Gamma(z)$ as above; it can be obtained as

$$\Gamma(z) = \exp\Big[\int_0^\infty \Big((z-1)\,\mathrm{e}^{-x} - \frac{\mathrm{e}^{-x}-\mathrm{e}^{-zx}}{1-\mathrm{e}^{-x}}\Big)\,\frac{1}{x}\,\mathrm{d}x\Big],$$

and hence

$$\Gamma(1-iu) = \exp\Big[-\int_0^\infty \Big(iu + \frac{1-\mathrm{e}^{iux}}{1-\mathrm{e}^{-x}}\Big)\,\frac{1}{x}\,\mathrm{e}^{-x}\,\mathrm{d}x\Big] \qquad [\,u \in \mathbb{R}\,].$$

An extension to $\mathbb{C} \setminus \mathbb{Z}_-$ is obtained by using *Euler's formula*:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z\,(z+1)\,\cdots\,(z+n)} \qquad [\,z \in \mathbb{C} \setminus \mathbb{Z}_-\,];$$

it has the property that for $z \in \mathbb{C} \setminus \mathbb{Z}$ and for $u \in \mathbb{R} \setminus \{0\}$, respectively:

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \qquad \Gamma(iu)\,\Gamma(1-iu) = \frac{-\pi i}{\sinh \pi u}.$$

Beta function. This is the function $B : (0,\infty)^2 \to (0,\infty)$ defined by

$$B(r,s) := \int_0^1 x^{r-1}\,(1-x)^{s-1}\,\mathrm{d}x \qquad [\,r > 0,\ s > 0\,].$$

The beta function can be expressed in terms of the gamma function; according to *Binet's formula* we have

$$B(r,s) = \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)} \qquad [\,r > 0,\ s > 0\,].$$

For $z, w \in \mathbb{C}$ with $\operatorname{Re} z > 0$, $\operatorname{Re} w > 0$ one defines $B(z,w)$ as above.

Bessel functions. The Bessel function of the first kind of order $r \geq 0$ is the function $J_r : (0,\infty) \to \mathbb{R}$ defined by

$$J_r(x) := \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(k+r+1)}\,(\tfrac{1}{2}x)^{2k+r} \qquad [\,x > 0\,].$$

The modified Bessel function of the first kind of order $r \geq 0$ is the function $I_r : (0, \infty) \to \mathbb{R}$ defined by

$$I_r(x) := \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+r+1)} \left(\tfrac{1}{2}x\right)^{2k+r} \qquad [\, x > 0\,].$$

Random walks. The following inequalities on symmetric random walks seem not to be very well known; we therefore give proofs.

Proposition 5.1. *Let $(S_n)_{n\in\mathbb{Z}_+}$ be a random walk on $\mathbb{R}$, started at zero, with step-size distribution that is symmetric and has no mass at zero. Then*

$$\mathbb{P}(S_{2k+1} = 0) \leq \mathbb{P}(S_{2k} = 0) \qquad [\, k \in \mathbb{Z}_+\,],$$

and if $(T_n)_{n\in\mathbb{Z}_+}$ is the symmetric Bernoulli walk started at zero, then

$$\mathbb{P}(S_{2k} = 0) \leq \mathbb{P}(T_{2k} = 0) \qquad [\, k \in \mathbb{Z}_+\,].$$

PROOF. Let Y be a random variable with $Y \stackrel{\mathrm{d}}{=} S_1$; since Y has a symmetric distribution, the characteristic function of Y can be written as $\phi_Y(u) = \mathbb{E} \cos uY$. Hence by the inversion formula (A.2.14) we have

$$\mathbb{P}(S_{2k} = 0) = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} (\mathbb{E} \cos uY)^{2k}\, \mathrm{d}u,$$

and similarly for $\mathbb{P}(S_{2k+1} = 0)$. The first inequality now immediately follows. Turning to the second one, we apply Jensen's inequality for the convex function $z \mapsto z^{2k}$ to obtain

$$\mathbb{P}(S_{2k} = 0) \leq \limsup_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} \mathbb{E}\,(\cos uY)^{2k}\, \mathrm{d}u.$$

Next, using Fubini's theorem and Fatou's lemma, we can estimate further:

$$\mathbb{P}(S_{2k} = 0) \leq \mathbb{E}\left(\limsup_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} (\cos uY)^{2k}\, \mathrm{d}u\right) =$$

$$= \mathbb{P}(Y = 0) + \mathbb{E}\left(1_{\{Y\neq 0\}} \limsup_{t\to\infty} \frac{1}{2tY} \int_{-tY}^{tY} (\cos v)^{2k}\, \mathrm{d}v\right) =$$

$$= \mathbb{P}(Y = 0) + \mathbb{P}(Y \neq 0) \limsup_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} (\cos v)^{2k}\, \mathrm{d}v.$$

Finally, use the formula for $\mathbb{P}(S_{2k} = 0)$ above with Y replaced by Y_0 having a symmetric distribution on $\{-1, 1\}$. Since $\mathbb{P}(Y = 0)$ is supposed to be zero, the desired inequality follows. ◻

6. Notes

There are numerous books presenting the basic concepts and results of probability theory; we name the following classical ones: Breiman (1968), Loève (1977, 1978), Billingsley (1979). Distributions on $\mathbb{Z}_+$ are treated in detail in Feller (1968). For information on characteristic functions we refer to Lukacs (1970, 1983), and for properties of Laplace-Stieltjes transforms and complete monotonicity to Feller (1971) and Widder (1972). Unimodality and strong unimodality are treated in Dharmadhikari and Joag-Dev (1988), (log-) convexity and its ramifications, such as Karamata's inequality, in Pečarić et al. (1992), and in Roberts and Varberg (1973). The Bürmann-Lagrange theorem is taken from Whittaker and Watson (1996); see also Pólya and Szegö (1970). The analytic auxiliaries, such as integrals and special functions, can be found in Abramowitz and Stegun (1992), or in Gradshteyn and Ryzhik (1980); in case of an emergency one can use Maple or Mathematica. The result on random walks in Proposition 5.1 is due to Huff (1974).

Appendix B

Selected well-known distributions

1. Introduction

The sections below serve two purposes. On the one hand, we want to establish terminology and notation concerning a number of well-known distributions, and list some of their elementary properties to be used in the main text. On the other hand, for well-known distributions we want to enable the reader to quickly decide whether or not they are infinitely divisible or, more particularly, self-decomposable or stable; this information is based on results from the main text. We note that sometimes our terminology differs slightly from the usual one. As in Appendix A, Sections 2, 3 and 4 concern distributions on $\mathbb{R}$, on $\mathbb{R}_+$ and on $\mathbb{Z}_+$, respectively. There is a final section with some bibliographical information.

2. Distributions on the real line

In this section we present a number of probability distributions on $\mathbb{R}$ with support not contained in $\mathbb{R}_-$ or in $\mathbb{R}_+$. All of them are absolutely continuous and are specified by their densities f and/or characteristic functions ϕ; in a few cases the distribution function F is given. The notations F, f and ϕ are used without further comment, and X will always be a generic random variable. The formulas given for $F(x)$ and $f(x)$ hold for all $x \in \mathbb{R}$.

The **uniform** distribution on $(-1,1)$:

$$f(x) = \tfrac{1}{2}\, 1_{(-1,1)}(x), \quad \phi(u) = \frac{\sin u}{u}.$$

The uniform distribution on $(-1,1)$ is *not* infinitely divisible; in fact, it is not n-divisible for any $n \geq 2$.

The **normal** (μ, σ^2) distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big[-\tfrac{1}{2}\big(\frac{x-\mu}{\sigma}\big)^2\Big], \quad \phi(u) = \exp\big[iu\mu - \tfrac{1}{2}\sigma^2 u^2\big].$$

The normal (μ, σ^2) distribution is *weakly stable* with exponent $\gamma = 2$, and hence *self-decomposable* and *infinitely divisible*, for all values of the parameters. *Standard normal* means normal $(0,1)$, in which case F is denoted by Φ and $\mathbb{P}(X > x) \sim \big(\exp[-\frac{1}{2}x^2]\big)/\big(x\sqrt{2\pi}\big)$ as $x \to \infty$.

The **Cauchy** (λ) distribution with parameter $\lambda > 0$:

$$F(x) = \tfrac{1}{2} + \frac{1}{\pi}\arctan\frac{x}{\lambda}, \quad f(x) = \frac{1}{\pi}\,\frac{\lambda}{\lambda^2 + x^2}, \quad \phi(u) = \mathrm{e}^{-\lambda|u|}.$$

The Cauchy (λ) distribution is (symmetric) *stable* with exponent $\gamma = 1$, and hence *self-decomposable* and *infinitely divisible*, for all values of the parameter. *Standard Cauchy* means Cauchy (1); note that in this case $X \overset{\mathrm{d}}{=} \tan\frac{1}{2}\pi Y$ with Y uniform on $(-1,1)$, and also $X \overset{\mathrm{d}}{=} Y/Z$ with Y and Z independent and standard normal.

The **Laplace** (λ) distribution with parameter $\lambda > 0$:

$$f(x) = \tfrac{1}{2}\lambda\,\mathrm{e}^{-\lambda|x|}, \quad \phi(u) = \frac{\lambda^2}{\lambda^2 + u^2}.$$

The Laplace (λ) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter. Note that $X \overset{\mathrm{d}}{=} Y - Z$ with Y and Z independent and exponential (λ). *Standard Laplace* means Laplace (1).

The **sym-gamma** (r, λ) distribution with parameters $r > 0$ and $\lambda > 0$:

$$\phi(u) = \Big(\frac{\lambda^2}{\lambda^2 + u^2}\Big)^r;$$

here 'sym-gamma' stands for 'symmetrized-gamma'. The sym-gamma (r, λ) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameters. Note that $X \overset{\mathrm{d}}{=} Y - Z$ with Y and Z independent and gamma (r, λ). *Sym-gamma* (r) means sym-gamma (r, λ) with some $\lambda > 0$; *standard sym-gamma* (r) means sym-gamma $(r, 1)$. Note that in the standard case with $r = \frac{1}{2}$ one has $X \overset{\mathrm{d}}{=} YZ$ with Y and Z independent and standard normal. For $r = 1$ one obtains the Laplace distribution.

The **double-gamma** (r, λ) distribution with parameters $r > 0$ and $\lambda > 0$:

$$f(x) = \tfrac{1}{2}\frac{\lambda^r}{\Gamma(r)}\,|x|^{r-1}\,\mathrm{e}^{-\lambda|x|}, \quad \phi(u) = \Big(\frac{\lambda^2}{\lambda^2 + u^2}\Big)^{\frac{1}{2}r}\cos\big(r\arctan\frac{u}{\lambda}\big).$$

The double-gamma (r, λ) distribution is *infinitely divisible* when $r \leq 1$, and *not* infinitely divisible when $r > 1$, for all values of λ. Note that $X \stackrel{\text{d}}{=} AY$ with A and Y independent, A Bernoulli $(\frac{1}{2})$ on $\{-1, 1\}$ and Y gamma (r, λ). For $r = 1$ one obtains the Laplace distribution.

The **Gumbel** distribution:

$$F(x) = \exp\left[-\mathrm{e}^{-x}\right], \quad f(x) = \exp\left[-(x+\mathrm{e}^{-x})\right], \quad \phi(u) = \Gamma(1-iu).$$

The Gumbel distribution is *self-decomposable*, and hence *infinitely divisible*. Note that $X \stackrel{\text{d}}{=} -\log Y$ with Y standard exponential.

The **logistic** distribution:

$$F(x) = \frac{1}{1+\mathrm{e}^{-x}}, \qquad f(x) = \tfrac{1}{4}\,\frac{1}{\cosh^2 \frac{1}{2}x}, \qquad \phi(u) = \frac{\pi u}{\sinh \pi u}.$$

The logistic distribution is *self-decomposable*, and hence *infinitely divisible*. Note that $X \stackrel{\text{d}}{=} U - V$ with U and V independent and having a Gumbel distribution; hence $X \stackrel{\text{d}}{=} \log(Y/Z)$ with Y and Z independent and standard exponential.

The **hyperbolic-cosine** distribution:

$$f(x) = \frac{1}{2\pi}\,\frac{1}{\cosh \frac{1}{2}x}, \qquad \phi(u) = \frac{1}{\cosh \pi u}.$$

The hyperbolic-cosine distribution is *self-decomposable*, and hence *infinitely divisible*. Note that $X \stackrel{\text{d}}{=} \log(Y/Z)$ with Y and Z independent and standard gamma $(\frac{1}{2})$.

The **hyperbolic-sine** distribution:

$$f(x) = \frac{1}{2\pi^2}\,\frac{x}{\sinh \frac{1}{2}x}, \qquad \phi(u) = \frac{1}{\cosh^2 \pi u}.$$

The hyperbolic-sine distribution is *self-decomposable*, and hence *infinitely divisible*. Note that $X \stackrel{\text{d}}{=} Y + Z$ with Y and Z independent and having a hyperbolic-cosine distribution.

The **student** (r) distribution with parameter $r > 0$:

$$f(x) = \frac{1}{B(r, \frac{1}{2})}\left(\frac{1}{1+x^2}\right)^{r+\frac{1}{2}}.$$

The student (r) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter. Note that $X \stackrel{\mathrm{d}}{=} Y/\sqrt{Z}$ with Y and Z independent, Y standard normal and Z gamma $(r, \frac{1}{2})$. When $r = \frac{1}{2}k$ with $k \in \mathbb{N}$, the distribution of $\sqrt{k}\, X$ is also called the *student* distribution *with k degrees of freedom*; for $k = 1$ one obtains the Cauchy distribution.

The **double-Pareto** (r) distribution with parameter $r > 1$:

$$f(x) = \tfrac{1}{2}(r-1)\left(\frac{1}{1+|x|}\right)^r.$$

The double-Pareto (r) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter.

3. Distributions on the nonnegative reals

All distributions in this section are on $\mathbb{R}_+$, are absolutely continuous and are specified by their densities f and/or pLSt's π; in a few cases the distribution function F is given. The notations F, f and π are used without further comment, and X will always be a generic random variable. Also, the formulas given for $F(x)$ and $f(x)$ only hold for $x > 0$.

The **uniform** distribution on $(0, 1)$:

$$f(x) = 1_{(0,1)}(x), \quad \pi(s) = \frac{1}{s}\,(1 - \mathrm{e}^{-s}).$$

The uniform distribution on $(0, 1)$ is *not* infinitely divisible; in fact, it is not n-divisible for any $n \geq 2$.

The **beta** (r, s) distribution with parameters $r > 0$ and $s > 0$:

$$f(x) = \frac{1}{B(r, s)}\, x^{r-1}\,(1 - x)^{s-1}\, 1_{(0,1)}(x).$$

The beta (r, s) distribution is *not* infinitely divisible for any value of the parameter pair. For $r = s = 1$ we get the uniform distribution on $(0, 1)$; the beta $(\frac{1}{2}, \frac{1}{2})$ distribution, which is also called the *arcsine* distribution, is even *indecomposable*. When $s = 1 - r$ with $r < 1$, we have $\mathbb{E}X^n = \binom{n-r}{n}$ for all $n \in \mathbb{Z}_+$.

The **exponential** (λ) distribution with parameter $\lambda > 0$:

$$F(x) = 1 - \mathrm{e}^{-\lambda x}, \quad f(x) = \lambda\,\mathrm{e}^{-\lambda x}, \quad \pi(s) = \frac{\lambda}{\lambda + s}.$$

The exponential (λ) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter. *Standard exponential* means exponential (1); note that in this case $X \stackrel{\rm d}{=} -\log Y$ with Y uniform on $(0,1)$.

The **gamma** (r,λ) distribution with parameters $r>0$ and $\lambda>0$:

$$f(x)=\frac{\lambda^r}{\Gamma(r)}\,x^{r-1}\,\mathrm{e}^{-\lambda x},\qquad \pi(s)=\Big(\frac{\lambda}{\lambda+s}\Big)^r.$$

The gamma (r,λ) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameters. *Gamma* (r) means gamma (r,λ) with some $\lambda>0$; *standard gamma* (r) means gamma $(r,1)$. For $r=1$ we get the exponential (λ) distribution. The gamma $(\frac12 k,\frac12)$ distribution with $k\in\mathbb{N}$ is also called the *chi-square* distribution *with k degrees of freedom*; note that in this case $X\stackrel{\rm d}{=}Y_1^2+\cdots+Y_k^2$ with $Y_1,\ldots,Y_k$ independent and standard normal.

The **inverse-gamma** (r) distribution with parameter $r>0$:

$$f(x)=\frac{1}{\Gamma(r)}\,x^{-r-1}\,\mathrm{e}^{-1/x}.$$

The inverse-gamma (r) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter. The inverse-gamma $(\frac12)$ distribution is *stable* with exponent $\gamma=\frac12$. Note that $X\stackrel{\rm d}{=}1/Y$ with Y standard gamma (r).

The **Weibull** (α) distribution with parameter $\alpha>0$:

$$F(x)=1-\exp\big[-x^\alpha\big],\qquad f(x)=\alpha\,x^{\alpha-1}\exp\big[-x^\alpha\big].$$

The Weibull (α) distribution is *self-decomposable*, and hence *infinitely divisible*, when $\alpha\le 1$; it is *not* infinitely divisible when $\alpha>1$. Note that $X\stackrel{\rm d}{=}Y^{1/\alpha}$ with Y standard exponential.

The **generalized-gamma** (r,α) distribution with parameters $r>0$ and $\alpha\in\mathbb{R}\setminus\{0\}$:

$$f(x)=\frac{|\alpha|}{\Gamma(r)}\,x^{\alpha r-1}\exp\big[-x^\alpha\big].$$

The generalized-gamma (r,α) distribution is *self-decomposable*, and hence *infinitely divisible*, when $r>0$ and $|\alpha|\le 1$; it is *not* infinitely divisible when $r>0$ and $\alpha>1$; the case where $r>0$ and $\alpha<-1$, is open. Note that

$X \overset{d}{=} Y^{1/\alpha}$ with Y standard gamma (r). For $\alpha = 1$ we get the gamma (r) distribution and for $\alpha = -1$ the inverse-gamma (r) distribution; when $\alpha > 0$ and $r = 1$ the Weibull (α) distribution is obtained.

The **generalized-inverse-Gaussian (β, a)** distribution with parameters $\beta \in \mathbb{R}$ and $a > 0$ (and $c_{\beta,a} > 0$ a norming constant):

$$f(x) = c_{\beta,a}\, x^{\beta-1}\, \mathrm{e}^{-(ax+1/x)}.$$

The generalized-inverse-Gaussian (β, a) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameters. The distribution with $\beta = -\frac{1}{2}$ is just called *inverse-Gaussian* or *Wald* distribution; when $\beta = 0$ one speaks of the *hyperbolic* distribution. For $\beta < 0$ we get the inverse-gamma $(r = -\beta)$ distribution, exponentially tilted.

The **Pareto (r)** distribution with parameter $r > 1$:

$$F(x) = 1 - \Big(\frac{1}{1+x}\Big)^{r-1}, \quad f(x) = (r-1)\Big(\frac{1}{1+x}\Big)^{r}.$$

The Pareto (r) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter.

The **beta (r, s)** distribution **of the second kind** with parameters $r > 0$ and $s > 0$:

$$f(x) = \frac{1}{B(r,s)}\, x^{r-1} \Big(\frac{1}{1+x}\Big)^{r+s}.$$

The beta (r, s) distribution of the second kind is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameters. Note that one has $X \overset{d}{=} 1/Y - 1$ with Y beta (r, s), and also $X \overset{d}{=} Y/Z$ with Y and Z independent, Y standard gamma (r) and Z standard gamma (s). For $r = 1$ we get the Pareto distribution. When $r = \frac{1}{2}k$ and $s = \frac{1}{2}\ell$ with $k, \ell \in \mathbb{N}$, the distribution of $(\ell/k)\, X$ is also called the *Snedecor* or F-distribution *with k and ℓ degrees of freedom*.

The **log-normal** distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}}\, \frac{1}{x}\, \exp\big[-(\log x)^2\big].$$

The log-normal distribution is *self-decomposable*, and hence *infinitely divisible*. Note that $X \overset{d}{=} \mathrm{e}^{Y}$ with Y standard normal.

The **half-normal** distribution:

$$f(x) = \frac{2}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}x^2\right].$$

The half-normal distribution is *not* infinitely divisible. Note that $X \stackrel{d}{=} |Y|$ with Y standard normal.

The **half-Cauchy** distribution:

$$f(x) = \frac{2}{\pi} \frac{1}{1+x^2}.$$

The half-Cauchy distribution is *self-decomposable*, and hence *infinitely divisible*. Note that $X \stackrel{d}{=} |Y|$ with Y standard Cauchy.

The **half-Gumbel** distribution:

$$f(x) = \frac{\mathrm{e}}{\mathrm{e}-1} \exp\left[-(x + \mathrm{e}^{-x})\right].$$

The half-Gumbel distribution is *infinitely divisible*. Note that here we have $X \stackrel{d}{=} (Y \mid Y > 0)$ with Y Gumbel.

4. Distributions on the nonnegative integers

In this section we present a number of probability distributions $p = (p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ together with their pgf's P. The notations p_k and P are used without further comment, and X will always be a generic random variable. Also, the formulas given for p_k hold for all $k \in \mathbb{Z}_+$.

The **uniform** distribution on $\{0, \ldots, n\}$ with parameter $n \in \mathbb{Z}_+$:

$$p_k = \frac{1}{n+1} 1_{\{0,\ldots,n\}}(k), \quad P(z) = \frac{1}{n+1} \frac{1 - z^{n+1}}{1-z}.$$

The uniform distribution on $\{0, \ldots, n\}$ is *not* infinitely divisible when $n \in \mathbb{N}$; for $n = 0$ we get the *degenerate* distribution at zero, which is *infinitely divisible.*

The **Bernoulli** (p) distribution on $\{0, 1\}$ with parameter $p \in (0, 1)$:

$$p_k = p^k (1-p)^{1-k} 1_{\{0,1\}}(k), \quad P(z) = 1 - p + p\, z.$$

The Bernoulli (p) distribution on $\{0, 1\}$ is *not* infinitely divisible for any value of the parameter. Sometimes we need the *Bernoulli* (p) distribution with support $\{x_0, x_1\}$ different from $\{0, 1\}$, for which

$$\mathbb{P}(X = x_1) = p = 1 - \mathbb{P}(X = x_0) \;\text{ if }\; x_0 < x_1.$$

The **binomial** (n,p) distribution with parameters $n \in \mathbb{N}$ and $p \in (0,1)$:

$$p_k = \binom{n}{k} p^k (1-p)^{n-k}, \quad P(z) = (1-p+p\,z)^n.$$

The binomial (n,p) distribution is *not* infinitely divisible for any value of the parameter pair. For $n=1$ we get the Bernoulli (p) distribution on $\{0,1\}$.

The **Poisson** (λ) distribution with parameter $\lambda > 0$:

$$p_k = \frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda}, \quad P(z) = \exp\big[-\lambda\,(1-z)\big].$$

The Poisson (λ) distribution is *stable* with exponent $\gamma = 1$, and hence *self-decomposable* and *infinitely divisible*, for all values of the parameter. For the tail probability $\overline{P}_k := \mathbb{P}(X > k)$ we have $\overline{P}_k = \mathbb{P}(S_{k+1} \leq \lambda)$ with S_{k+1} standard gamma $(k+1)$ distributed, so $\overline{P}_k$, at a fixed $k \in \mathbb{Z}_+$, increases with λ.

The **geometric** (p) distribution with parameter $p \in (0,1)$:

$$p_k = (1-p)\,p^k, \quad P(z) = \frac{1-p}{1-p\,z}.$$

The geometric (p) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter. Note that $X \stackrel{\mathrm{d}}{=} N(T)$, where $N(\cdot)$ is a unit-rate Poisson process independent of T which is exponential.

The **negative-binomial** (r,p) distribution with parameters $r > 0$ and $p \in (0,1)$:

$$p_k = \binom{r+k-1}{k} p^k (1-p)^r, \quad P(z) = \Big(\frac{1-p}{1-p\,z}\Big)^r.$$

The negative-binomial (r,p) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameters. Note that $X \stackrel{\mathrm{d}}{=} N(T)$, where $N(\cdot)$ is a unit-rate Poisson process independent of T which is gamma (r). *Negative-binomial* (r) means negative-binomial (r,p) with some $p \in (0,1)$. For $r=1$ we get the geometric (p) distribution.

The **logarithmic-series** (p) distribution with parameter $p \in (0,1)$:

$$p_k = c_p\,\frac{1}{k+1}\,p^{k+1}, \quad P(z) = c_p\,\frac{-\log(1-p\,z)}{z},$$

where $c_p := 1/\{-\log(1-p)\}$. The logarithmic-series (p) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter.

The **discrete Pareto** (r) distribution with parameter $r > 1$:

$$p_k = c_r \, \frac{1}{(k+1)^r},$$

where $c_r := 1/\zeta(r)$ and ζ is the Riemann zeta function: $\zeta(r) := \sum_{n=1}^{\infty} 1/n^r$. The discrete Pareto (r) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter.

The **Borel** (λ) distribution with parameter $\lambda \in (0, 1\,]$:

$$p_k = \mathrm{e}^{-\lambda}\,(\lambda\,\mathrm{e}^{-\lambda})^k \, \frac{(k+1)^k}{(k+1)!}.$$

The Borel (λ) distribution is *self-decomposable*, and hence *infinitely divisible*, for all values of the parameter.

5. Notes

The absolutely continuous distributions listed in Sections 2 and 3 can be found (together with many others) in Johnson et al. (1994, 1995), the discrete distributions of Section 4 in Johnson et al. (1992). Here many properties are given, but infinite divisibility is not always indicated. The indecomposability of the arcsine distribution is proved by Kudina (1972). Many of the known infinitely divisible distributions turn out to be self-decomposable; cf. Jurek (1997). Some more, somewhat less-known infinitely divisible distributions can be found in Bondesson (1992) and, of course, in the Examples sections of Chapters II through VII.

Bibliography

ABRAMOWITZ, M. AND STEGUN, I.A. (1992): *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Dover Publications, New York.

ALAMATSAZ, M.H. (1985): A note on an article by Artikis. *Acta Math. Hung.* **45**, 159–162.

ALF, C. AND O'CONNOR, T.A. (1977): Unimodality of the Lévy spectral function. *Pacific J. Math.* **69**, 285–290.

ALZAID, A.A. AND PROSCHAN, F. (1994): Max-infinite divisibility and multivariate total positivity. *J. Appl. Probab.* **31**, 721–730.

ARRATIA, R. AND GOLDSTEIN, L. (1998): Size biasing: when is the increment independent? Preprint.

ARTIKIS, T. (1982): A note on certain power mixtures. *Acta Math. Hung.* **39**, 69–72.

ATHREYA, K.B. AND NEY, P.E. (1972): *Branching processes.* Die Grundlehren der mathematischen Wissenschaften **196**, Springer, Berlin.

BALKEMA, A.A. AND RESNICK, S.I. (1977): Max-infinite divisibility. *J. Appl. Probab.* **14**, 309–319.

BARENDREGT, L. (1979): Solution to problem 73 *b*. *Statist. Neerl.* **33**, 163–164.

BARNDORFF-NIELSEN, O., BLAESILD, P. AND HALGREEN, C. (1978): First hitting time models for the generalized-inverse-Gaussian distribution. *Stoch. Proc. Appl.* **7**, 49–54.

BARNDORFF-NIELSEN, O. AND HALGREEN, C. (1977): Infinite divisibility of the hyperbolic and generalized-inverse-Gaussian distributions. *Z. Wahrsch. verw. Gebiete* **38**, 309–311.

Barndorff-Nielsen, O. and Shephard, N. (2001): Modelling by Lévy processes for financial econometrics. In: *Lévy processes*, 283–318. Birkhäuser, Boston.

Baxter, G. and Shapiro, J.M. (1960): On bounded infinitely divisible random variables. *Sankhyà* **22**, 253–260.

Berg, C. and Forst, G. (1983): Multiply self-decomposable probability measures on $\mathbb{R}_+$ and $\mathbb{Z}_+$. *Z. Wahrsch. verw. Gebiete* **62**, 147–163.

Bertoin, J. (1996): *Lévy processes.* Cambridge Tracts in Mathematics **121**, Cambridge University Press, Cambridge.

Bertoin, J., van Harn, K. and Steutel, F.W. (1999): Renewal theory and level passage by subordinators. *Statist. Probab. Lett.* **45**, 65–69.

Biggins, J.D. and Shanbhag, D.N. (1981): Some divisibility problems in branching processes. *Math. Proc. Cambr. Phil. Soc.* **90**, 321–330.

Billingsley, P. (1979): *Probability and measure.* Wiley, New York.

Blum, J.R. and Rosenblatt, M. (1959): On the structure of infinitely divisible distributions. *Pacific J. Math.* **9**, 1–7.

Bondesson, L. (1979): On the infinite divisibility of products of powers of gamma variables. *Z. Wahrsch. verw. Gebiete* **49**, 171–175.

Bondesson, L. (1981): Classes of infinitely divisible distributions and densities. *Z. Wahrsch. verw. Gebiete* **57**, 39–71 and **59**, 277.

Bondesson, L. (1987): On the infinite divisibility of the half-Cauchy and other decreasing densities and probability functions on the nonnegative line. *Scand. Actuar. J.*, 225–247.

Bondesson, L. (1991): A characterization of first-passage time distributions for random walks. *Stoch. Proc. Appl.* **39**, 81–88.

Bondesson, L. (1992): *Generalized gamma convolutions and related classes of distributions and densities.* Lecture Notes in Statistics **76**, Springer, Berlin.

Bondesson, L., Kristiansen, G.K. and Steutel, F.W. (1996): Infinite divisibility of random variables and their integer parts. *Statist. Probab. Lett.* **28**, 271–278.

Bondesson, L. and Steutel, F.W. (2002): A class of infinitely divisible distributions connected to branching processes and random walks. SPOR-Report 2002-3, Dept. of Math. and Comp. Sc., Eindhoven University of Technology, Eindhoven.

Bouwkamp, C.J. (1986): Solution to problem 85-16. *SIAM Rev.* **28**, 568–569.

Breiman, L. (1968): *Probability.* Addison-Wesley, Reading.

Brown, G. (1976): Infinitely divisible distributions and support properties. *Proc. Roy. Irish Acad. Sect. A* **22**, 227–234.

Bunge, J. (1997): Nested classes of C-decomposable laws. *Ann. Probab.* **25**, 215–229.

Byczkowski, T., Rajput, B.S. and Zak, T. (1996): Zeros of the densities of infinitely divisible measures on $\mathbb{R}^n$. *J. Theoret. Probab.* **9**, 533–540.

Carasso, A.S. (1987): Infinitely divisible pulses, continuous deconvolution, and the characterization of linear time invariant systems. *SIAM J. Appl. Math.* **47**, 892–927.

Chang, D.K. (1989): On infinitely divisible discrete distributions. *Utilitas Math.* **36**, 215–217.

Chung, K.L. (1960): *Markov chains with stationary transition probabilities.* Die Grundlehren der mathematischen Wissenschaften **104**, Springer, Berlin.

Cohen, J.W. (1982): *The single server queue*, 2-nd ed. North-Holland Series in Applied Mathematics and Mechanics **8**, North-Holland, Amsterdam.

Consul, P.C. and Famoye, F. (1986a): On the unimodality of generalized Poisson distribution. *Statist. Neerl.* **40**, 117–122.

CONSUL, P.C. AND FAMOYE, F. (1986b): On the unimodality of the generalized negative-binomial distribution. *Statist. Neerl.* **40**, 141–144.

CSŐRGÖ, S. (1989): An extreme-sum approximation to infinitely divisible laws without a normal component. In: *Probability theory on vector spaces* IV, 47–58. Lecture Notes in Mathematics **1391**, Springer, Berlin.

CSŐRGÖ, S., HÄUSLER, E. AND MASON, D.M. (1988): A probabilistic approach to the asymptotic distribution of sums of independent identically distributed random variables. *Adv. Appl. Math.* **9**, 259–333.

CSŐRGÖ, S. AND MASON, D.M. (1991): A probabilistic approach to the tails of infinitely divisible laws. In: *Sums, trimmed sums and extremes*, 317–335. Progress in Probability **23**, Birkhäuser, Boston.

DALEY, D.J. (1965): On a class of renewal functions. *Proc. Cambr. Phil. Soc.* **61**, 519–526.

DANIAL, E.J. (1988): Generalization to the sufficient conditions for a discrete random variable to be infinitely divisible. *Statist. Probab. Lett.* **6**, 379–382.

DIÉDHIOU, A. (1998): On the self-decomposability of the half-Cauchy distribution. *J. Math. Anal. Appl.* **220**, 42–64.

DHARMADHIKARI, S. AND JOAG-DEV, K. (1988): *Unimodality, convexity, and applications.* Academic Press, Boston.

DOOB, J.L. (1953): *Stochastic processes.* Wiley, New York; Chapman & Hall, London.

DWASS, M. (1967): Poisson recurrence times. *J. Appl. Probab.* **4**, 605–608.

DWASS, M. (1968): A theorem about infinitely divisible characteristic functions. *Z. Wahrsch. verw. Gebiete* **9**, 287–289.

EMBRECHTS, P. AND GOLDIE, C.M. (1981): Comparing the tail of an infinitely divisible distribution with integrals of its Lévy measure. *Ann. Probab.* **9**, 468–481.

Embrechts, P., Goldie, C.M. and Veraverbeke, N. (1979): Subexponentiality and infinite divisibility. *Z. Wahrsch. verw. Gebiete* **49**, 335–347.

Embrechts, P. and Hawkes, J. (1982): A limit theorem for the tails of discrete infinitely divisible laws with applications to fluctuation theory. *J. Austral. Math. Soc. Ser. A* **32**, 412–422.

Epstein, B. (1977): Infinite divisibility of student's t-distribution. *Sankhyà Ser. B* **39**, 103–120.

Esseen, C. (1965): On infinitely divisible one-sided distributions. *Math. Scand.* **17**, 65–76.

Feller, W. (1939): Neuer Beweis für die Kolmogoroff-P. Lévysche Charakterisierung der unbeschränkt teilbaren Verteilungsfunktionen. *Bull. Int. Acad. Yougoslave Cl. Sci. Math. Nat.* **32**, 1–8.

Feller, W. (1968): *An introduction to probability theory and its applications*, vol. 1, 3-rd ed. Wiley, New York.

Feller, W. (1971): *An introduction to probability theory and its applications*, vol. 2, 2-nd ed. Wiley, New York.

de Finetti, B. (1929): Sulle funzioni a incremento aleatorio. *Rend. Acad. Lincei (6)* **10**, 163–168, 325–329, 548–553.

Fisz, M. (1962): Infinitely divisible distributions: recent results and applications. *Ann. Math. Statist.* **33**, 68–84.

Fisz, M. and Varadarajan, V.S. (1963): A condition for absolute continuity of infinitely divisible distribution functions. *Z. Wahrsch. verw. Gebiete* **1**, 335–339.

Forst, G. (1979): A characterization of self-decomposable probabilities on the half-line. *Z. Wahrsch. verw. Gebiete* **49**, 349–352.

Forst, G. (1981): Subordinates of the Poisson semigroup. *Z. Wahrsch. verw. Gebiete* **55**, 35–40.

Forst, G. (1984): Self-decomposability on $\mathbb{R}$ and $\mathbb{Z}$. In: *Probability measures on groups* VII, 99–115. Lecture Notes in Mathematics **1064**, Springer, Berlin.

FREEDMAN, D. (1972): *Approximating countable Markov chains.* Holden-Day, San Francisco.

GAWRONSKI, W. (1984): On the bell-shape of stable densities. *Ann. Probab.* **12**, 230–242.

GELUK, J.L. AND DE HAAN, L. (2000): Stable probability distributions and their domains of attraction: a direct approach. *Probab. Math. Statist.* **20**, 169–188.

GERBER, H.U. (1992): On the probability of ruin for infinitely divisible claim amount distributions. *Insurance Math. Econom.* **11**, 163–166.

GNEDENKO, B.V. (1991): *Einführung in die Wahrscheinlichkeitstheorie.* Mathematische Lehrbücher und Monographien, I. Abteilung **39**, Akademie, Berlin.

GNEDENKO, B.V. AND KOLMOGOROV, A.N. (1968): *Limit distributions for sums of independent random variables*, 2-nd ed. Addison-Wesley, Reading.

GOLDIE, C.M. (1967): A class of infinitely divisible random variables. *Proc. Cambr. Phil. Soc.* **63**, 1141–1143.

GOOVAERTS, M.J., D'HOOGE, L. AND DE PRIL, N. (1977): On the infinite divisibility of the product of two Γ-distributed stochastical variables. *Appl. Math. Comp.* **3**, 127–135.

GOOVAERTS, M.J., D'HOOGE, L. AND DE PRIL, N. (1978): On the infinite divisibility of the ratio of two gamma distributed variables. *Stoch. Proc. Appl.* **7**, 291–297.

GRADSHTEYN, I.S. AND RYZHIK, I.M. (1980): *Table of integrals, series, and products.* Academic Press, New York.

GRANDELL, J. (1997): *Mixed Poisson processes.* Monographs on Statistics and Applied Probability **77**, Chapman & Hall, London.

GROSSWALD, E. (1976): The student t-distribution of any degree of freedom is infinitely divisible. *Z. Wahrsch. verw. Gebiete* **36**, 103–109.

GUPTA, A.K., MÓRI, T.F. AND SZÉKELY, G.J. (1994): Testing for Poissonity-normality vs other infinite divisibility. *Statist. Probab. Lett.* **19**, 245–248.

HALGREEN, C. (1979): Self-decomposability of the generalized-inverse-Gaussian and hyperbolic distributions. *Z. Wahrsch. verw. Gebiete* **47**, 13–17.

HALL, P. (1981): A comedy of errors: the canonical form for a stable characteristic function. *Bull. London Math. Soc.* **13**, 23–27.

HALL, P. (1984): On unimodality and rates of convergence for stable laws. *J. London Math. Soc. (2)* **30**, 371–384.

HANSEN, B.G. (1988a): *Monotonicity properties of infinitely divisible distributions.* Dissertation, Technische Universiteit, Eindhoven.

HANSEN, B.G. (1988b): On log-concave and log-convex infinitely divisible sequences and densities. *Ann. Probab.* **16**, 1832–1839.

HANSEN, B.G. (1996): Stability and self-decomposability of semi-group valued random variables. *Statist. Neerl.* **50**, 295–305.

HANSEN, B.G. AND STEUTEL, F.W. (1988): On moment sequences and infinitely divisible sequences. *J. Math. Anal. Appl.* **136**, 304–313.

HANSEN, B.G. AND WILLEKENS, E. (1990): The generalized logarithmic-series distribution. *Statist. Probab. Lett.* **9**, 311–316.

VAN HARN, K. (1978): *Classifying infinitely divisible distributions by functional equations.* Mathematical Centre Tracts **103**, Mathematisch Centrum, Amsterdam.

VAN HARN, K. AND STEUTEL, F.W. (1993): Stability equations for processes with stationary independent increments using branching processes and Poisson mixtures. *Stoch. Proc. Appl.* **45**, 209–230.

VAN HARN, K. AND STEUTEL, F.W. (1995): Infinite divisibility and the waiting-time paradox. *Stoch. Models* **11**, 527–540.

VAN HARN, K. AND STEUTEL, F.W. (2001): Stationarity of delayed subordinators. *Stoch. Models* **17**, 369–374.

van Harn, K., Steutel, F.W. and Vervaat, W. (1982): Self-decomposable discrete distributions and branching processes. *Z. Wahrsch. verw. Gebiete* **61**, 97–118.

Hartman, P. and Wintner, A. (1942): On the infinitesimal generators of integral convolutions. *Amer. J. Math.* **64**, 273–298.

Hawkes, J. and Jenkins, J.D. (1978): Infinitely divisible sequences. *Scand. Actuar. J.*, 65–76.

Heinrich, L. (1985): An elementary proof of Spitzer's identity. *Statistics* **16**, 249–252.

Hirsch, F. (1975): Familles d'opérateurs potentiels. *Ann. Inst. Fourier* **25**, 263–288.

Hoffmann-Jørgensen, J. (1993): Stable densities. *Th. Probab. Appl.* **38**, 350–355.

Horn, R.A. (1967): On infinitely divisible matrices, kernels, and functions. *Z. Wahrsch. verw. Gebiete* **8**, 219–230.

Horn, R.A. (1969): The theory of infinitely divisible matrices and kernels. *Trans. Amer. Math. Soc.* **136**, 269–286.

Horn, R.A. (1970): On certain power series and sequences. *J. London Math. Soc.* **2**, 160–162.

Horn, R.A. (1972): On necessary and sufficient conditions for an infinitely divisible distribution to be normal or degenerate. *Z. Wahrsch. verw. Gebiete* **21**, 179–187.

Horn, R.A. and Steutel, F.W. (1978): On multivariate infinitely divisible distributions. *Stoch. Proc. Appl.* **6**, 139–151.

Hudson, W.N. and Tucker, H.G. (1975a): Equivalence of infinitely divisible distributions. *Ann. Probab.* **3**, 70–79.

Hudson, W.N. and Tucker, H.G. (1975b): On admissible translates of infinitely divisible distributions. *Z. Wahrsch. verw. Gebiete* **32**, 65–72.

Huff, B.W. (1972): On the continuity of infinitely divisible distributions. *Sankhyà Ser. A* **34**, 443–446.

Huff, B.W. (1974): Random walks and the continuity of infinitely divisible distributions. *SIAM J. Appl. Math.* **26**, 372–375.

Huff, B.W. (1978): Some comments on mixtures and infinite divisibility. *SIAM J. Appl. Math.* **35**, 17–20.

Ismail, M.E.H. (1977): Bessel functions and the infinite divisibility of the student t-distribution. *Ann. Probab.* **5**, 582–585.

Ismail, M.E.H. and Kelker, D.H. (1979): Special functions, Stieltjes transforms and infinite divisibility. *SIAM J. Math. Anal.* **10**, 884–901.

Ismail, M.E.H. and May, C.P. (1979): Special functions, infinite divisibility and transcendental equations. *Math. Proc. Cambr. Phil. Soc.* **85**, 453–464.

Ismail, M.E.H. and Miller, K.S. (1982): An infinitely divisible distribution involving modified Bessel functions. *Proc. Amer. Math. Soc.* **85**, 233–238.

Itô, K. (1942): On stochastic processes I (Infinitely divisible laws of probability). *Jap. J. Math.* **18**, 261–301.

Johansen, S. (1966): An application of extreme point methods to the representation of infinitely divisible distributions. *Z. Wahrsch. verw. Gebiete* **5**, 304–316.

Johnson, N.L., Kotz, S. and Balakrishnan, N. (1994): *Continuous univariate distributions*, vol. 1, 2-nd ed. Wiley, New York.

Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995): *Continuous univariate distributions*, vol. 2, 2-nd ed. Wiley, New York.

Johnson, N.L., Kotz, S. and Kemp, A.W. (1992): *Univariate discrete distributions*, 2-nd ed. Wiley, New York.

Jurek, Z.J. (1982): Structure of a class of operator-self-decomposable probability measures. *Ann. Probab.* **10**, 849–856.

JUREK, Z.J. (1983): The classes $L_m(Q)$ of probability measures on Banach spaces. *Bull. Polish Acad. Sci. Math.* **31**, 51–62.

JUREK, Z.J. (1985): Relations between the s-self-decomposable and self-decomposable measures. *Ann. Probab.* **13**, 592–608.

JUREK, Z.J. (1997): Self-decomposability: an exception or a rule? *Ann. Univ. Mariae Curie-Sklodowska Sect. A* **51**, 93–107.

JUREK, Z.J. AND MASON, J.D. (1993): *Operator-limit distributions in probability theory.* Wiley, New York.

JUREK, Z.J. AND URBANIK, K. (1978): Remarks on stable measures on Banach spaces. *Colloq. Math.* **38**, 269–276.

JUREK, Z.J. AND VERVAAT, W. (1983): An integral representation for self-decomposable Banach space valued random variables. *Z. Wahrsch. verw. Gebiete* **62**, 247–262.

KALUZA, T. (1928): Ueber die Koeffizienten reziproker Potenzreihen. *Math. Zeitschrift* **28**, 161–170.

KATTI, S.K. (1967): Infinite divisibility of integer-valued random variables. *Ann. Math. Statist.* **38**, 1306–1308.

KATTI, S.K. (1979): Infinite divisibility of discrete distributions III. In: *Analytic function methods in probability theory*, 165–171. Colloq. Math. Soc. Janos Bolyai **21**, North-Holland, Amsterdam.

KAWATA, T. (1972): *Fourier analysis in probability theory.* Probability and Mathematical Statistics **15**, Academic Press, New York.

KAZAKYAVICHYUS, V. (1996): Limit distributions of finite Markov chains, and semigroups of stochastic matrices. *Lithuanian Math. J.* **36**, 45–50.

KEILSON, J. (1979): *Markov chain models — rarity and exponentiality.* Applied Mathematical Sciences **28**, Springer, Berlin.

KEILSON, J. AND STEUTEL, F.W. (1972): Families of infinitely divisible distributions closed under mixing and convolution. *Ann. Math. Statist.* **43**, 242–250.

KEILSON, J. AND STEUTEL, F.W. (1974): Mixtures of distributions, moment inequalities and measures of exponentiality and normality. *Ann. Probab.* **2**, 112–130.

KEILSON, J. AND SUMITA, U. (1983): A decomposition of the beta distribution, related order and asymptotic behavior. *Ann. Inst. Statist. Math.* **35**, 243–253.

KELKER, D.H. (1971): Infinite divisibility and variance mixtures of the normal distribution. *Ann. Math. Statist.* **42**, 802–808.

KENDALL, D.G. AND HARDING, E.F. (1973): *Stochastic analysis. A tribute to the memory of Rollo Davidson.* Wiley, New York.

KENT, J.T. (1978): Some probabilistic properties of Bessel functions. *Ann. Probab.* **6**, 760–770.

KENT, J.T. (1982): The spectral decomposition of a diffusion hitting time. *Ann. Probab.* **10**, 207–219.

KENT, J.T. AND LONGFORD, N.T. (1983): An eigenvalue decomposition for first hitting times in random walks. *Z. Wahrsch. verw. Gebiete* **63**, 71–84.

KERSTAN, J., MATTHES, K. AND MECKE, J. (1978): *Infinitely divisible point processes.* Wiley, New York.

KHINTCHINE, A.Y. (1937a): Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze. *Rec. Math. Moscou* **2**, 79–117.

KHINTCHINE, A.Y. (1937b): Déduction nouvelle d'une formule de M. Paul Lévy. *Bull. Univ. Etat Moscou, Sér. Int. Sec. A* **1**, 1–5.

KHINTCHINE, A.Y. (1938): *Limit theorems for sums of independent random variables.* Gonti Publ., Moscow.

KHINTCHINE, A.Y. AND LÉVY, P. (1936): Sur les lois stables. *C. R. Acad. Sci. Paris* **202**, 374–376.

KINGMAN, J.F.C. (1965): The heavy traffic approximation in the theory of queues. In: *Proceedings Symposium Congestion Theory*, 137–169. Univ. North Carolina Press, Chapel Hill.

Klaassen, C.A.J. (1981): Solution to problem 106. *Statist. Neerl.* **35**, 231–232.

Klebanov, L.B., Maniya, G.M. and Melamed, I.A. (1984): A problem of V. M. Zolotarev and analogues of infinitely divisible and stable distributions in a scheme for summation of a random number of random variables. *Th. Probab. Appl.* **29**, 791–794.

Kleinrock, L. (1975): *Queueing systems*, vol. I: Theory. Wiley, New York.

Kolmogorov, A.N. (1932): Sulla forma generale di un processo stocastico omogeneo. *Rend. Acad. Lincei (6)* **15**, 805–808, 866–869.

Kristiansen, G.K. (1994): A proof of Steutel's conjecture. *Ann. Probab.* **22**, 442–452.

Kristiansen, G.K. (1995): A counter-example to a conjecture concerning infinitely divisible distributions. *Scand. J. Statist.* **22**, 139–141.

Kruglov, V.M. (1970): A note on infinitely divisible distributions. *Th. Probab. Appl.* **15**, 319–324.

Kruglov, V.M. and Antonov, S.N. (1984): Once more on the asymptotic behavior of infinitely divisible distributions in a Banach space. *Th. Probab. Appl.* **29**, 766–775.

Kuczma, M. (1968): *Functional equations in a single variable.* Monografie Matematyczne **46**, Państwowe Wydawnictwo Naukowe, Warsaw.

Kudina, L.S. (1972): Indecomposability of the arcsine law. *Soviet Math. Dokl.* **13**, 581–583.

Kumar, A. and Schreiber, B.M. (1975): Self-decomposable probability measures on Banach spaces. *Studia Math.* **53**, 55–71.

Kumar, A. and Schreiber, B.M. (1978): Characterization of subclasses of class L probability distributions. *Ann. Probab.* **6**, 279–293.

Kumar, A. and Schreiber, B.M. (1979): Representation of certain infinitely divisible probability measures on Banach spaces. *J. Multivariate Anal.* **9**, 288–303.

LAHA, R.G. AND ROHATGI, V.K. (1979): *Probability theory.* Wiley, New York.

LAMPERTI, J.W. (1996): *Probability; a survey of the mathematical theory*, 2-nd ed. Wiley, New York.

LÉVY, P. (1923): Sur les lois stables en calcul des probabilités. *C.R. Acad. Sci. Paris* **176**, 1284–1286.

LÉVY, P. (1934): Sur les intégrales dont les élements sont des variables aléatoires indépendantes. *Ann. Ec. Norm. Pisa (2)* **3**, 337–366.

LÉVY, P. (1937): *Théorie de l'addition des variables aléatoires.* Gauthier-Villars, Paris.

LIN, G.D. AND HU, C. (2001): The Riemann zeta distribution. *Bernoulli* **7**, 817–828.

LINNIK, Y.V. (1964): *Decomposition of probability distributions.* Oliver and Boyd, London.

LOÈVE, M. (1973): Paul Lévy, 1886–1971. *Ann. Probab.* **1**, 1–18.

LOÈVE, M. (1977): *Probability theory*, part I, 4-th ed. Graduate Texts in Mathematics **45**, Springer, Berlin.

LOÈVE, M. (1978): *Probability theory*, part II, 4-th ed. Graduate Texts in Mathematics **46**, Springer, Berlin.

LOOIJENGA, L.M. (1990): Solution to problem 219. *Statist. Neerl.* **44**, 275–276.

LUKACS, E. (1970): *Characteristic functions*, 2-nd ed. Hafner, New York.

LUKACS, E. (1983): *Developments in characteristic function theory.* Macmillan, New York.

MASUDA, Y. (1988): First-passage times of birth-death processes and simple random walks. *Stoch. Proc. Appl.* **29**, 51–63.

MCCRUDDEN, M. AND WALKER, S. (1999): Infinitely divisible probabilities on linear p-adic groups. *Proc. Indian Acad. Sci. Math.* **109**, 299–302.

McCrudden, M. and Walker, S. (2000): Embedding infinitely divisible probabilities on subsemigroups of Lie groups. In: *Probability on algebraic structures*, 43–57. Contemporary Mathematics **261**. Amer. Math. Soc., Providence.

McKenzie, E. (1982): Product autoregression: a time-series characterization of the gamma distribution. *J. Appl. Probab.* **19**, 463–468.

McKenzie, E. (1986): Autoregressive moving-average processes with negative-binomial and geometric marginal distributions. *Adv. Appl. Probab.* **18**, 679–705.

McKenzie, E. (1988): Some ARMA models for dependent sequences of Poisson counts. *Adv. Appl. Probab.* **20**, 822–835.

Mejzler, D. (1973): On a certain class of infinitely divisible distributions. *Israel J. Math.* **16**, 1–19.

Millar, P.W. (1995): On a theorem of Hartman and Wintner. *Proc. Amer. Math. Soc.* **123**, 1893–1896.

Miller, H.D. (1967): A note on passage times and infinitely divisible distributions. *J. Appl. Probab.* **4**, 402–405.

Miller, H.D. (1979): Infinite divisibility in stochastic processes. *Ann. Probab.* **7**, 406–417.

Nolan, J.P. (1997): Numerical calculation of stable densities and distribution functions. Heavy tails and highly volatile phenomena. *Stoch. Models* **13**, 759–774.

O'Brien, G.L. and Steutel, F.W. (1981): Divisibility properties of Lebesgue measure. *Indag. Math.* **43**, 393–398.

O'Connor, T.A. (1979a): Infinitely divisible distributions with unimodal Lévy spectral functions. *Ann. Probab.* **7**, 494–499.

O'Connor, T.A. (1979b): Infinitely divisible distributions similar to class L distributions. *Z. Wahrsch. verw. Gebiete* **50**, 265–271.

O'Connor, T.A. (1981): Some classes of limit laws containing the stable distributions. *Z. Wahrsch. verw. Gebiete* **55**, 25–33.

Ohkubo, H. (1979): On the asymptotic tail behaviors of infinitely divisible distributions. *Yokohama Math. J.* **27**, 77–89.

Oluyede, B.O. (1999): On inequalities and selection of experiments for length-biased distributions. *Probab. Engrg. Inform. Sci.* **13**, 169–185.

Orey, S. (1968): On continuity properties of infinitely divisible distribution functions. *Ann. Math. Statist.* **39**, 936–937.

Pakes, A.G. (1995): Characterization of discrete laws via mixed sums and Markov branching processes. *Stoch. Proc. Appl.* **55**, 285–300.

Pakes, A.G., Sapatinas, T. and Fosam, E.B. (1996): Characterizations, length-biasing, and infinite divisibility. *Statist. Papers* **37**, 53–69.

Parzen, E. (1962): *Stochastic processes.* Holden-Day, San Francisco.

Pečarić, J.E., Proschan, F. and Tong, Y.L. (1992): *Convex functions, partial orderings, and statistical applications.* Mathematics in Science and Engineering **187**, Academic Press, Boston.

Petrov, V.V. (1975): *Sums of independent random variables.* Ergebnisse der Mathematik und ihrer Grenzgebiete **82**, Springer, Berlin.

Petrov, V.V. (1995): *Limit theorems of probability theory. Sequences of independent random variables.* Oxford Studies in Probability **4**, Oxford University Press, New York.

Pitman, J. and Yor, M. (1981): Bessel processes and infinitely divisible laws. In: *Stochastic integrals*, 285–370. Lecture Notes in Mathematics **851**, Springer, Berlin.

Pólya, G. and Szegö, G. (1970): *Aufgaben und Lehrsätze aus der Analysis* I, 4-rd ed. Heidelberger Taschenbücher **73**, Springer, Berlin.

Puri, P.S. and Goldie, C.M. (1979): Poisson mixtures and quasi-infinite divisibility of distributions. *J. Appl. Probab.* **16**, 138–153.

Ramachandran, B. (1969): On characteristic functions and moments. *Sankhyà Ser. A* **31**, 1–12.

RESNICK, S.I. (1987): *Extreme values, regular variation, and point processes.* Applied Probability **4**, Springer, Berlin.

RIEDEL, M. (1975): On the one-sided tails of infinitely divisible distributions. *Math. Nachr.* **70**, 155–163.

ROBERTS, A.W. AND VARBERG, D.E. (1973): *Convex functions.* Pure and Applied Mathematics **57**, Academic Press, New York.

ROHATGI, V.K., STEUTEL, F.W. AND SZÉKELY, G.J. (1990): Infinite divisibility of products and quotients of i.i.d. random variables. *Math. Sci.* **15**, 53–59.

ROSENBLATT, M. (1962): *Random processes.* Oxford University Press, New York.

ROSSBERG, H.-J., JESIAK, B. AND SIEGEL, G. (1985): *Analytic methods of probability theory.* Mathematical Monographs **67**, Akademie, Berlin.

RUBIN, H. (1967): Supports of convolutions of identical distributions. In: *Proc. 5-th Berkeley Symp. Math. Statist. Probab.*, part I, 415–422, Univ. Calif.

RUEGG, A.F. (1970): A characterization of certain infinitely divisible laws. *Ann. Math. Statist.* **41**, 1354–1356.

RUEGG, A.F. (1971): A necessary condition on the infinite divisibility of probability distributions. *Ann. Math. Statist.* **42**, 1681–1685.

RUNNENBURG, J. TH. (1960): Probabilistic interpretation of some formulae in queueing theory. *Bull. Inst. Internat. Statist.* **37**, 405–414.

SAMORODNITSKY, G. AND TAQQU, M.S. (1994): *Stable non-Gaussian random processes.* Chapman & Hall, New York.

SATO, K. (1973): A note on infinitely divisible distributions and their Lévy measures. *Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A* **12**, 101–109.

SATO, K. (1994): Multimodal convolutions of unimodal infinitely divisible distributions. *Th. Probab. Appl.* **39**, 336–347.

SATO, K. (1999): *Lévy processes and infinitely divisible distributions.* Cambridge Studies in Advanced Mathematics **68**, Cambridge University Press, Cambridge.

SATO, K. (2001): Subordination and self-decomposability. *Statist. Probab. Lett.* **54**, 317–324.

SATO, K. AND STEUTEL, F.W. (1998): Note on the continuation of infinitely divisible distributions and canonical measures. *Statistics* **31**, 347–357.

SATO, K. AND YAMAZATO, M. (1978): On distribution functions of class *L*. *Z. Wahrsch. verw. Gebiete* **43**, 273–308.

SATO, K. AND YAMAZATO, M. (1981): On higher derivatives of distribution functions of class *L*. *J. Math. Kyoto Univ.* **21**, 575–591.

SCHOUTENS, W. AND TEUGELS, J.L. (1998): Lévy processes, polynomials and martingales. *Stoch. Models* **14**, 335–349.

SHANBHAG, D.N., PESTANA, D. AND SREEHARI, M. (1977): Some further results in infinite divisibility. *Math. Proc. Cambr. Phil. Soc.* **82**, 289–295.

SHANBHAG, D.N. AND SREEHARI, M. (1977): On certain self-decomposable distributions. *Z. Wahrsch. verw. Gebiete* **38**, 217–222.

SHANBHAG, D.N. AND SREEHARI, M. (1979): An extension of Goldie's result and further results in infinite divisibility. *Z. Wahrsch. verw. Gebiete* **47**, 19–25.

SHAPIRO, J.M. (1956): A condition for existence of moments of infinitely divisible distributions. *Canad. J. Math.* **8**, 69–71.

SHARPE, M.J. (1969a): Zeroes of infinitely divisible densities. *Ann. Math. Statist.* **40**, 1503–1505.

SHARPE, M.J. (1969b): Operator-stable probability distributions on vector groups. *Trans. Amer. Math. Soc.* **136**, 51–65.

SHARPE, M.J. (1995): Supports of convolution semigroups and densities. In: *Probability measures on groups and related structures* XI, 364–369. World Sci. Publ., River Edge.

Shkol'nik, S.M. (1992): Modal properties of stable, nearly symmetric distributions. Stability problems for stochastic models. *J. Soviet Math.* **59**, 1008–1010.

Steutel, F.W. (1967): Note on the infinite divisibility of exponential mixtures. *Ann. Math. Statist.* **38**, 1303–1305.

Steutel, F.W. (1970): *Preservation of infinite divisibility under mixing and related topics.* Mathematical Centre Tracts **33**, Mathematisch Centrum, Amsterdam.

Steutel, F.W. (1973): Some recent results in infinite divisibility. *Stoch. Proc. Appl.* **1**, 125–143.

Steutel, F.W. (1974): On the tails of infinitely divisible distributions. *Z. Wahrsch. verw. Gebiete* **28**, 273–276.

Steutel, F.W. (1979): Infinite divisibility in theory and practice. *Scand. J. Statist.* **6**, 57–64.

Steutel, F.W. and van Eenige, M.J. (1997): Note on the approximation of distributions on $\mathbb{Z}_+$ by mixtures of negative-binomial distributions. *Stoch. Models* **13**, 271–274.

Steutel, F.W. and van Harn, K. (1979): Discrete analogues of self-decomposability and stability. *Ann. Probab.* **7**, 893–899.

Steutel, F.W., Vervaat, W. and Wolfe, S.J. (1983): Integer-valued branching processes with immigration. *Adv. Appl. Probab.* **15**, 713–725.

Steutel, F.W. and Wolfe, S.J. (1977): On the asymptotic behaviour of moments of infinitely divisible distributions. Memorandum COSOR 77-6, Dept. of Math. and Comp. Sc., Eindhoven University of Technology, Eindhoven.

Stroock, D.W. (1993): *Probability theory, an analytic view.* Cambridge University Press, Cambridge.

Sumita, U. and Masuda, Y. (1985): On first-passage time structure of random walks. *Stoch. Proc. Appl.* **20**, 133–147.

TAKANO, K. (2003): On infinite divisibility of normed product of Cauchy densities. *J. Comp. Appl. Math.* **150**, 253–263.

THORIN, O. (1977a): On the infinite divisibility of the Pareto distribution. *Scand. Actuar. J.*, 31–40.

THORIN, O. (1977b): On the infinite divisibility of the log-normal distribution. *Scand. Actuar. J.*, 121–148.

THORIN, O. (1978a): An extension of the notion of a generalized Γ-convolution. *Scand. Actuar. J.*, 141–149.

THORIN, O. (1978b): Proof of a conjecture of L. Bondesson concerning infinite divisibility of powers of a gamma variable. *Scand. Actuar. J.*, 151–164.

TORTRAT, A. (1988): Le support des lois indéfiniment divisibles dans un groupe abélien localement compact. *Math. Z.* **197**, 231–250.

TUCKER, H.G. (1961): Best one-sided bounds of infinitely divisible random variables. *Sankhyà Ser. A* **24**, 387–396.

TUCKER, H.G. (1962): Absolute continuity of infinitely divisible distributions. *Pacific J. Math.* **12**, 1125–1129.

TUCKER, H.G. (1964): On continuous singular infinitely divisible distribution functions. *Ann. Math. Statist.* **35**, 330–335.

TUCKER, H.G. (1965): On a necessary and sufficient condition that an infinitely divisible distribution be absolutely continuous. *Trans. Amer. Math. Soc.* **118**, 316–330.

TUCKER, H.G. (1967): *A graduate course in probability.* Academic Press, New York.

TUCKER, H.G. (1975): The supports of infinitely divisible distribution functions. *Proc. Amer. Math. Soc.* **49**, 436–440.

URBANIK, K. (1968): A representation of self-decomposable distributions. *Bull. Acad. Polon. Sci. Math.* **16**, 209–214.

URBANIK, K. (1972): Lévy's probability measures on Euclidean spaces. *Studia Math.* **44**, 119–148.

URBANIK, K. (1978): Lévy's probability measures on Banach spaces. *Studia Math.* **63**, 283–308.

URBANIK, K. (1983): Multiplicative properties of infinitely divisible random variables. *Bull. Polish Acad. Sci. Math.* **31**, 63–69.

VERVAAT, W. (1979): On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Adv. Appl. Probab.* **11**, 750–783.

VOUDOURI, A. (1995): An infinitely divisible distribution in financial modelling. *Bull. Greek Math. Soc.* **37**, 113–122.

WARDE, W.D. AND KATTI, S.K. (1971): Infinite divisibility of discrete distributions II. *Ann. Math. Statist.* **42**, 1088–1090.

WHITTAKER, E.T. AND WATSON, G.N. (1996): *A course of modern analysis.* Cambridge University Press, Cambridge.

WIDDER, D.V. (1972): *The Laplace Transform.* Princeton Mathematical Series **6**, Princeton University Press, Princeton.

WILMS, R.J.G. (1994): *Fractional parts of random variables: limit theorems and infinite divisibility.* Dissertation, Technische Universiteit, Eindhoven.

WOLFE, S.J. (1971a): On the unimodality of L functions. *Ann. Math. Statist.* **42**, 912–918.

WOLFE, S.J. (1971b): On moments of infinitely divisible distribution functions. *Ann. Math. Statist.* **42**, 2036–2043.

WOLFE, S.J. (1971c): On the continuity properties of L functions. *Ann. Math. Statist.* **42**, 2064–2073.

WOLFE, S.J. (1978a): On the unimodality of infinitely divisible distribution functions. *Z. Wahrsch. verw. Gebiete* **45**, 329–335.

WOLFE, S.J. (1978b): On the infinite divisibility of variance mixtures of normal distribution functions. *Indag. Math.* **40**, 154–156.

WOLFE, S.J. (1982): On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$. *Stoch. Proc. Appl.* **12**, 301–312.

YAMAZATO, M. (1975): Some results on infinitely divisible distributions of class L with applications to branching processes. *Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A* **13**, 133–139.

YAMAZATO, M. (1978): Unimodality of infinitely divisible distribution functions of class L. *Ann. Probab.* **6**, 523–531.

YAMAZATO, M. (1982): On strongly unimodal infinitely divisible distributions. *Ann. Probab.* **10**, 589–601.

YAMAZATO, M. (1995): On the strong unimodality of infinitely divisible distributions of class CME. *Th. Probab. Appl.* **40**, 518–532.

YASUDA, K. (2000): On infinitely divisible distributions on locally compact abelian groups. *J. Theoret. Probab.* **13**, 635–657.

ZOLOTAREV, V.M. (1965): Asymptotic behavior of distributions of processes with independent increments. *Th. Probab. Appl.* **10**, 28–44.

ZOLOTAREV, V.M. (1986): *One-dimensional stable distributions.* Translations of Mathematical Monographs **65**, American Mathematical Society, Providence.